MATH 526 Homework 6. Due Tuesday 3/17

Hand in the solutions of all questions

Problem 1. Consider the following queueing model in Homework set 5, Problem 6: X_n denotes the number of customers waiting in line for some service, and the transition probabilities are given by

$$p(i,i-1) = q(1-p), \quad p(i,i) = qp + (1-q)(1-p), \quad p(i,i+1) = (1-q)p$$

for $i = 1, 2, 3, \cdots$, and

$$p(0,0) = 1-p, \qquad p(0,1) = p.$$

(a) Determine for which values of p and q the chain is positive recurrent. In this case, compute the expected value of the length of the queue in the limit $n \to \infty$.

(b) For the transient case, compute the probability that the length of the queue never becomes 0, assuming that initially there were m customers in the queue, $m = 1, 2, 3, \cdots$.

Problem 2. Let $a_1, a_2, a_3, \cdots$ be positive numbers satisfying $\sum_{i=1}^{\infty} a_i = 1$. Consider the following Markov chain with state space $S = \{0, 1, 2, \cdots\}$. Whenever the chain reaches state 0 it chooses a new state i according to the probability a_i. Whenever the chain is at a state other than 0 it proceeds deterministically, one step at a time, toward 0. Hence the transition probabilities are

$$p(0,i) = a_i, \qquad i = 1, 2, 3, \cdots,$$

and

$$p(i,i-1) = 1. \qquad i = 1, 2, 3, \cdots.$$

This is clearly irreducible and recurrent (why?). Under what conditions on a_i's is the chain positive recurrent? In this case compute the probability that the chain is at state i after long time.

(Hint: If you want, you can compute π_0 from $\mathbb{E}[T_0|X_0 = 0]$ directly, where T_0 is the time of first return to 0.)

Problem 3. The lifetime T of a light bulb, measured in discrete units, is distributed as a geometric random variable with parameter p: $\mathbb{P}\{T = k\} = (1-p)^{k-1}p$, $k = 1, 2, 3, \cdots$. Suppose we start with a new light bulb and a bulb is replaced by a new one immediately upon failure. Let X_n be the age of the bulb in service at discrete time n. Then X_n is a Markov chain with state space $S = \{0, 1, 2, \cdots\}$.

(a) Find the transition probabilities.

(b) Find an approximate value of $\mathbb{P}\{X_{1000} = 5\}$.

(c) Find the expected age of the bulb in service after long long time.

Problem 4. Consider the following queueing model which is different from Problem 1. Customers arrive for service and form a queue. During each period of discrete time, one customer is served and leaves the queue (if at least one customer is present). Also during each period of time, a random number of new customers arrive and join the queue. Assume that the number ξ of new arrivals in each period is iid and has the distribution $\mathbb{P}\{\xi = j\} = a_j$, $j = 0, 1, 2, 3, \cdots$. (You may imagine a taxi stand in an airport where a new taxicab arrives as soon as a first person in the queue gets on a taxicab.)Let X_n be the number of customers in the queue. This is a Markov chain with $S = \{0, 1, 2, \cdots\}$ and the transition probability given by

$$\mathbf{P} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Assume that $a_j = (1-p)^j p$. Set

$$\mu = \mathbb{E}[\xi] = \frac{1}{p} - 1 = \frac{1-p}{p}.$$

(a) In order to determine for which value of p the chain is transient, solve the "y-equations" as follows. Take the special state $k = 0$. Then check that a solution to the equations except for the first equation is given by

$$y_i = C_1 + C_2 \left(\frac{1}{\mu}\right)^i, \qquad i = 1, 2, 3, \cdots$$

for some constants C_1 and C_2. One can show that this is the only solution, but you don't need to show this. Then use the first equation (given by $y_1 = a_1 y_1 + a_2 y_2 + a_3 y_3 + \cdots$) to obtain $C_1 + C_2 = 0$. Hence the solution is given by

$$y_i = C_1 \left(1 - \left(\frac{1}{\mu}\right)^i\right), \qquad i = 1, 2, 3, \cdots$$

and conclude that the chain is transience if and only if $\mu > 1$.

(b) In order to determine for which value of p the chain is positive recurrent, we solve the "π-equations". Show that

$$\pi_i = C\mu^i, \qquad i = 0, 1, 2, 3, \cdots$$

solves the π-equation. One can show that this is the only solution, but you don't need to show this. Conclude that the chain is positive recurrent if and only if $\mu < 1$. In this case, compute

$$\lim_{n \to \infty} \mathbb{E}[X_n].$$

Problem 5. Let a_i, $i = 01, 2, 3, \cdots$ be numbers such that

$$a_i > 0, \qquad \sum_{i=0}^{\infty} a_i = 1.$$

Consider the discrete time Markov chain with $S = \{0, 1, 2, 3, \cdots\}$ and

$$\mathbf{P} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Determine for which values of a_i's the chain is positive recurrent, and in this case compute the invariant distribution.