MATH 526 Homework 7. Do not hand in the solutions

Problem 1. Suppose that the number of calls per hour arriving at an answering service follows a Poisson process with $\lambda = 4$.

(a) What is the probability that fewer than 2 calls come in the first two hours?

(b) What is the probability that fewer than 2 calls come in the next two hours?

(c) Suppose that exactly k calls arrived in the first two hours. What is the probability that exactly j of them arrives in the first hour?

Problem 2. Let X_t and Y_t be two independent Poisson processes with rate λ_1 and λ_2, respectively, measuring the number of calls arriving at two different phones. Let $Z_t = X_t + Y_t$.

(a) Show that Z_t is a Poisson process. What is the rate?

(b) What is the probability that the first call comes on the first phone?

(c) Let T denote the first time that at least one call has come from each of the two phones. Find the density of T.

Problem 3. Let X_t and Y_t be two independent Poisson processes with rate λ_1 and λ_2, respectively, measuring the number of customers arriving in stores 1 and 2, respectively.

(a) What is the probability that a customer arrives in store 1 before any customers arrive in store 2?

(b) Suppose that at time t we know that total 4 customers have arrived at two stores together. What is the probability that 1 of them went to store 1? Observe that the answer does not depend on t.

(c) Let T denote the time of arrival of the first customer at store 2. Then X_T is the number of customers in store 1 at the (random) time of the first customer arrival at store 2. Find the probability distribution of X_T.

Problem 4. There are two transatlantic cables each of which handle one telegraph message at a time. The time-to-breakdown for each has the same exponential random distribution with parameter λ. The time to repair for each cable has the same exponential random distribution with parameter μ.

(a) Let X_t denote the number of operating cables at time t. This is a Markov chain. Find the infinitesimal generator A.

(Hint: Recall that $e^{ax} = 1 + ax + o(x)$ as $x \to 0$.)

(b) Let $A = QDQ^{-1}$ be the diagonalization of the matrix A. Find the diagonal matrix $D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$ where $d_1 > d_2 > d_3$. One can check (say, using Maple) that

$$Q = \begin{pmatrix} 1 & 2\mu & \mu^2 \\ 1 & \mu - \lambda & -\lambda\mu \\ 1 & -2\lambda & \lambda^2 \end{pmatrix}, \qquad Q^{-1} = \frac{1}{(\lambda+\mu)^2} \begin{pmatrix} \lambda^2 & 2\lambda\mu & \mu^2 \\ \lambda & \mu - \lambda & -\mu \\ 1 & -2 & 1 \end{pmatrix}.$$

Given that at tiem 0 both cables are in working condition, find the probability that at time t both cables are operative.

Problem 5. Let $X_1, X_2, \cdots, X_N$ be independent random variables and $X_k \sim Exp(b_k)$. Define $X = \min\{X_1, \cdots, X_N\}$. Show that $X \sim Exp(b_1 + \cdots + b_N)$ and $\mathbb{P}\{X = X_k\} = \frac{b_k}{b_1 + \cdots + b_N}$.

(Hint: Compute $\mathbb{P}\{X > t\}$.)

Problem 6. Consider the continuous-time Markov chain with $s = \{1, 2, 3, 4\}$

and infinitesimal generator

$$\mathbf{A} = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -3 & 2 & 1 \\ 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

(a) Find the invariant distribution.

(b) Suppose the chain starts in state 1. What is the expected amount of time until it changes state for the first time?

(c) Assume that the chain starts in state 1. What is the expected amount of time until the chain reaches the state 4?