MATH 526 Homework 8. due Thursday, 4/9

Hand in the solutions of Problems 1,2,3,5,6.

Problem 1. Consider the simple population model with $\lambda_i = i\lambda$ and $\mu_i = i\mu$. Assume that $X_0 = 1$. Find the probability distribution of the number of population at the time of the first death.

Problem 2. Consider an M/M/1 queueing model for the number of customers in a post office with $\lambda_i = \lambda$, $\mu_i = \mu$. Assume that $\lambda < \mu$ so that the chain is positive recurrent. Suppose that you entered the post office a long time after the post office opened. What is the approximate probability that you will be served immediately? What is the approximate expected value of the time you need to spend in the post office?

Problem 3. Consider a simple population model X_t with immigration: $\lambda_i = i\lambda + \nu$, $\mu_i = i\mu$. Suppose that $X_0 = J$. Set $M(t) = \mathbb{E}[X_t]$. Show that

$$\frac{d}{dt}M(t) = \nu + (\lambda - \mu)M(t).$$

Then check that the solution (recall the initial state) is given by

$$M(t) = \frac{\nu}{\lambda - \mu}\left(e^{(\lambda-\mu)t} - 1\right) + Je^{(\lambda-\mu)t} \qquad \text{if } \lambda \neq \mu,$$

and

$$M(t) = \nu t + J, \qquad \text{if } \lambda = \mu.$$

(It is enough to plug this formula into the differential equation and check that the differential equation is satisfied, and to check that the initial condition $M(0) = J$ is also satisfied. This remark also applies to all other questions.)

Problem 4. Given a non-negative-integer-valued random variable X, define its generating function

$$G(s) = \mathbb{E}[s^X] = \sum_{k=0}^{\infty} s^k \mathbb{P}\{X = k\}.$$

(a) Show that $\mathbb{E}[X] = G'(1)$, $\mathbb{P}\{X = 0\} = \lim_{s\downarrow 0} G(s)$, $\mathbb{P}\{X = 1\} = \lim_{s\downarrow 0} G'(s)$ and $Var(X) = G''(1) + G'(1) - (G'(1))^2$.

(b) Compute the generating function explicitly for Poisson, geometric and binomial random variables.

Problem 5. Consider a *continuous-time birth-disaster process*: X_t is a continuous-time Markov process defined as follows. In each short time interval of duration Δt, each person of the population can give birth of one new person with probability $\lambda\Delta t + o(\Delta t)$, and during this interval, the whole population can disappear due to a disaster that occurs with probability $\delta\Delta t + o(\Delta t)$.

(a) Show that

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \delta & -(\delta+\lambda) & \lambda & 0 & 0 & \cdots \\ \delta & 0 & -(\delta+2\lambda) & 2\lambda & 0 & \cdots \\ \delta & 0 & 0 & -(\delta+3\lambda) & 3\lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}$$

(b) Let $M(t) = \mathbb{E}[X_t]$. Find a differential equation for $M(t)$.

(c) Suppose that $X_0 = J > 0$. Show that

$$\mathbb{E}[X_t] = Je^{(\lambda-\delta)t}.$$

What is the limit of $\mathbb{E}[X_t]$ as $t \to \infty$?

Problem 6.

(a) Consider a general birth-and-death process. Let E_i be the expected value of the time it takes for the population, starting at i, becomes $i-1$ for the first time. Show that

$$E_i = \frac{\lambda_i}{\mu_i} E_{i+1} + \frac{1}{\mu_i}, \qquad i = 1, 2, 3, \cdots.$$

(b) Consider a M/M/1 queue with $\lambda < \mu$. Then the chain is positive recurrent. From (a), show that

$$E_1 = \rho^i E_{i+1} + \frac{1}{\mu} \cdot \frac{1-\rho^i}{1-\rho}, \qquad \rho \equiv \frac{\lambda}{\mu}$$

for any $i = 1, 2, 3, \cdots$. From this conclude that if there is currently 1 customer in the queue, the expected time until the queue becomes empty for the first time is given by

$$\frac{1}{\mu - \lambda}.$$

(Hint: Recall that $1 + x + x^2 + \cdots + x^{N-1} = \frac{1-x^N}{1-x}$.)

(c) Consider a simple population model, $\lambda_i = i\lambda$ and $\mu_i = i\mu$. Assume that $\lambda < \mu$. We will see in the next homework set that the population becomes extinct with probability 1. Suppose that $X_0 = 1$. Using (a), show that the expected time until the extinction is given by

$$\frac{1}{\lambda} \log\left(\frac{\mu}{\mu - \lambda}\right).$$

(Hint: Recall that $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$.)