

MATH 526 Homework 9. due Thursday, 4/16

Hand in the solutions of all the problems

Problem 1. Consider a *continuous-time birth-disaster process*: X_t is a continuous-time Markov process defined as follows. In each short time interval of duration Δt , each person of the population can give birth of one new person with probability $\lambda\Delta t + o(\Delta t)$, and during this interval, the whole population can disappear due to a disaster that occurs with probability $\delta\Delta t + o(\Delta t)$.

- (a) Let $G(s, t) = \sum_{j=0}^{\infty} s^j p_j(t) = \mathbb{E}[s^{X_t}]$ denote the generating function, of the chain X_t . Show that $G(s, t)$ satisfies the first-order partial differential equation

$$\frac{\partial G}{\partial t} + \lambda s(1-s) \frac{\partial G}{\partial s} + \delta G = \delta.$$

- (b) Assume that $X(0) = J > 0$. Show that the solution to the differential equation in (b) satisfying the proper ‘initial condition’ $G(s, 0) = s^J$ is

$$G(s, t) = 1 + e^{-\delta t} \left[\left(\frac{se^{-\lambda t}}{1-s+se^{-\lambda t}} \right)^J - 1 \right].$$

- (c) Compute $\mathbb{P}\{X_t = 0\}$ and $\mathbb{E}[X_t]$ for each time $t > 0$.
- (d) Now let us perform a simplified analysis of the process X_t . Assume that we are not interested in the exact size of the population, but only whether it is zero (extinct) or non-zero (not extinct). Let \tilde{X}_t be a continuous-time Markov chain with two states $\{0, 1\}$, defined by

$$\tilde{X}(t) = \begin{cases} 0 & \text{if } X(t) = 0 \\ 1 & \text{if } X(t) \geq 1. \end{cases}$$

Show that the generator of \tilde{X}_t is

$$\tilde{A} = \begin{pmatrix} 0 & 0 \\ \delta & -\delta \end{pmatrix}.$$

- (e) Assume that initially the population X_0 was non-zero. Compute $\mathbb{P}\{\tilde{X}_t = 0\}$ and $\mathbb{P}\{\tilde{X}_t = 1\}$ for each time $t > 0$. Compare your answer with what you obtained in (d).

(Hint: You may either solve the forward equation, or compute $e^{t\tilde{A}}$.)

Problem 2. Consider the simple population model X_t , which is the birth-and-death process with $\lambda = i\lambda$ and $\mu_i = i\mu$. Let $X_0 = J$.

- (a) Show that $\mathbb{E}[X_t] = J e^{(\lambda - \mu)t}$.
- (b) Show that the generating function $G(s, t) = \sum_{j=0}^{\infty} s^j p_j(t) = \mathbb{E}[s^{X_t}]$ is given by

$$G(s, t) = \left(\frac{\mu(1-s) - (\mu - \lambda s)e^{-(\lambda - \mu)t}}{\lambda(1-s) - (\mu - \lambda s)e^{-(\lambda - \mu)t}} \right)^J, \quad \mu \neq \lambda$$

and

$$G(s, t) = \left(\frac{\lambda(1-s)t + s}{\lambda(1-s)t + 1} \right)^J, \quad \mu = \lambda.$$

- (c) Compute the extinction probability

$$q \equiv \lim_{t \rightarrow \infty} \mathbb{P}\{X_t = 0\}.$$

Problem 3. Let X_n be a branching process with $X_0 = 1$. Let Y be the random variable representing the number of offsprings from an individual and set $a_k = \mathbb{P}\{Y = k\}$, $k = 0, 1, 2, \dots$. Let μ and σ^2 be the expected value and the variance of Y , respectively, and set $\phi(s) = \mathbb{E}[s^Y]$. Let

$$\phi_n(s) = \mathbb{E}[s^{X_n}]$$

be the generating function of X_n as usual. Recall that $\phi_{n+1}(s) = \phi(\phi_n(s))$.

- Using that $\text{Var}(Y) = \phi''(1) + \phi'(1) - (\phi'(1))^2$ (why? see Problem 4, HW8), show that

$$\text{Var}(X_{n+1}) = \mu \text{Var}(X_n) + \sigma^2 \mu^{2n}, \quad n = 1, 2, 3, \dots$$

(Be careful when you use the chain rule.)

2. Show that

$$\text{Var}(X_n) = \begin{cases} n\sigma^2, & \mu = 1, \\ \frac{\sigma^2 \mu^{n-1} (\mu^n - 1)}{\mu - 1}, & \mu \neq 1. \end{cases}$$

(Hint: $1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$.)

3. Compute

$$\lim_{n \rightarrow \infty} \text{Var}(X_n).$$

Problem 4. Consider a branching process with the probability mass function of Y is given by

$$a_k \equiv \mathbb{P}\{Y = k\} = (1-p)^k p, \quad k = 0, 1, 2, 3, \dots$$

Suppose that $X_0 = 0$. Let

$$T = \min\{n \geq 1 : X_n = 0\}$$

be the extinction time. Compute $\mathbb{P}\{T = n\}$ for $n = 1, 2, 3$.

Problem 5. Consider the following variation of the branching process. Let Y be the random variable representing the number of offsprings from an individual with expected value μ . But after an individual produces Y number of offsprings, the individual dies only with probability r (where $0 < r < 1$). In the original branching process, $r = 1$. What is the conditions on Y and r so that the extinction occurs with probability 1?