Sol. to HW 10

1. • $\mathbb{E}[X_3 \mid X_1] = \mathbb{E}[X_3 - X_1 \mid X_1] + \mathbb{E}[X_1 \mid X_1]$.

But X_1 and $X_3 - X_1$ are independent. So,

$\mathbb{E}[X_3 \mid X_1] = \mathbb{E}[X_3 - X_1] + X_1$.

Now $X_3 - X_1 \sim \mathrm{Pois}(2\lambda)$. Hence $\mathbb{E}[X_3 \mid X_1] = 2\lambda + X_1$

• Given X_3, $X_1 \sim \mathrm{Bin}(X_3, \frac{1}{3})$. (See HW 7, problem 1 (c)).

Hence $\mathbb{E}[X_1 \mid X_3] = \frac{1}{3} X_3$.

2. • $\mathbb{E}[Y_{n+1} \mid X_0, X_1, \cdots, X_n] \underset{\uparrow}{=} \frac{1}{\cos^{n+1}\theta} \cdot \mathbb{E}[\cos(\theta(X_{n+1} - \frac{1}{2}N)) \mid X_n]$

Markov property

As $X_{n+1} = X_n + 1$ or $X_{n+1} = X_n - 1$ with prob. $\frac{1}{2}$ each,

$= \frac{1}{\cos^{n+1}\theta} \cdot \left\{ \cos(\theta(X_n + 1 - \frac{1}{2}N)) \cdot \frac{1}{2} + \cos(\theta(X_n - 1 - \frac{1}{2}N)) \cdot \frac{1}{2} \right\}$

$= \frac{1}{\cos^{n+1}\theta} \cdot 2 \cdot \cos(\theta(X_n - \frac{1}{2}N)) \cdot \cos\theta \cdot \frac{1}{2}$ using $\cos(A) + \cos(B) = 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2})$

$= Y_n$.

• For each n, $|Y_n| \le \frac{1}{\cos^n \theta} \Rightarrow \mathbb{E}[|Y_n|] < \infty$.

as $\cos\theta \ge \cos(\frac{\pi}{N}) > 0$.

Hence $\{Y_n\}$ is a martingale wrt $\{X_n\}$.

~~$T = \min\{n : X_n = 0 \text{ or } X_n = N\}$ is a stopping time and $P\{T < \infty\}$ since the chain is finite~~

3. • $\mathbb{E}[Z_{n+1} \mid X_0, \cdots, X_n] = \mathbb{E}[X_{n+1} - (n+1)(2p-1) \mid X_n]$

$= X_n + 1 \cdot p + (-1) \cdot (1-p) - (n+1)(2p-1) = X_n - n \cdot (2p-1) = Z_n$.

• $\mathbb{E}[|Z_n|] \le \mathbb{E}[|X_n|] + n \cdot (2p-1) \le N + n \cdot (2p-1) < \infty$.

$\therefore$ $\{Z_n\}$ is a martingale wrt $\{X_n\}$.

$T \equiv \min\{n : X_n = 0 \text{ or } X_n = N\}$ is a stopping time s.t $\mathbb{P}\{T < \infty\} = 1$.

Now $|Z_{T\wedge n}| = |X_{T\wedge n} - (T\wedge n)\cdot(2p-1)|$

$$\leq N + (T\wedge n)\cdot(2p-1) \leq N + T\cdot(2p-1) \equiv W.$$

and $\mathbb{E}[W] = N + (2p-1)\cdot\mathbb{E}[T] < \infty$.

Hence by optimal stopping thm (Cor 16),

$$\mathbb{E}[Z_T] = \mathbb{E}[Z_0] = i.$$

$$\parallel$$

$$\mathbb{E}[X_T - T\cdot(2p-1)]$$

$$\parallel$$

$$\mathbb{E}[X_T] - (2p-1)\cdot\mathbb{E}[T]$$

$$\Rightarrow \mathbb{E}[T] = \frac{\mathbb{E}[X_T] - i}{2p-1}$$

But $\mathbb{E}[X_T] = \mathbb{E}[X_T \mid X_T = 0]\cdot\mathbb{P}\{X_T = 0\} + \mathbb{E}[X_T \mid X_T = N]\cdot\mathbb{P}\{X_T = N\}$

$$= 0 + N\cdot\frac{1-\left(\frac{1-p}{p}\right)^i}{1-\left(\frac{1-p}{p}\right)^N}$$

$\therefore \mathbb{E}[T] = \dfrac{N}{2p-1}\cdot\left(\dfrac{1-\left(\frac{1-p}{p}\right)^i}{1-\left(\frac{1-p}{p}\right)^N} - \dfrac{i}{N}\right)$.

(in ~~from~~ the problem, we know that $\mathbb{P}\{$chain reaches N before $0\} = \dfrac{1-\left(\frac{1-p}{p}\right)^i}{1-\left(\frac{1-p}{p}\right)^N}$.

4.(1) $\cdot$ $\mathbb{E}[Y_{n+1} \mid X_0, X_1, \cdots, X_n] = m(t)^{-n-1}\cdot\mathbb{E}[e^{t(X_1+\cdots+X_{n+1})} \mid X_0, \cdots, X_n]$

$$= m(t)^{-n-1}\, e^{t(X_1+\cdots+X_n)}\cdot\underbrace{\mathbb{E}[e^{tX_{n+1}} \mid X_0, \cdots, X_n]}_{\parallel}$$

$$= m(t)^{-n-1}\, e^{tS_n}\cdot m(t) = Y_n. \qquad \mathbb{E}[e^{tX_{n+1}}] = \mathbb{E}[e^{tX_1}]$$

$\cdot$ $\mathbb{E}[|Y_n|] \underset{m(t)>0}{=} m(t)^{-n}\,\mathbb{E}[e^{t(X_1+\cdots+X_n)}] = m(t)^{-n}\cdot\mathbb{E}[e^{tX_1}]\cdots\mathbb{E}[e^{tX_n}]$

$$= m(t)^{-n}\cdot m(t)^n = 1 < \infty.$$

(2) $m(t)-1 = pe^{t} + (1-p)e^{-t} - 1 = e^{-t}\{pe^{2t} - e^{t} + (1-p)\}$

$$= e^{-t}(e^{t}-1)(pe^{t}-(1-p)) = pe^{-t}(e^{t}-1)(e^{t}-\tfrac{1-p}{p}).$$

But $e^{t} \geq 1$ and since $p > \frac{1}{2}$, $\frac{1-p}{p} < 1 \leq e^{t}$.

Hence $m(t)-1 \geq 0$.

(3). Clearly, T_a is a stopping time s.t $\mathbb{P}\{T_a < \infty\} = 1$.

$$|Y_{T\wedge n}| = m(t)^{-(T\wedge n)} \cdot e^{t S_{T\wedge n}}.$$

But $S_{T\wedge n} \leq b$ since for $n < T$, $-a < S_n < b$.

Also, as $m(t) \geq 1$, $m(t)^{-1} \leq 1$ and $m(t)^{-(T\wedge n)} \leq 1$.

$$\Rightarrow |Y_{T\wedge n}| \leq 1 \cdot e^{tb} \equiv W.$$

Clearly $\mathbb{E}[W] = e^{tb} < \infty$.

Thus, by OST (cor 16), $\mathbb{E}[m(t)^{-T_a} e^{t S_{T_a}}] = \mathbb{E}[Y_0] = 1$.

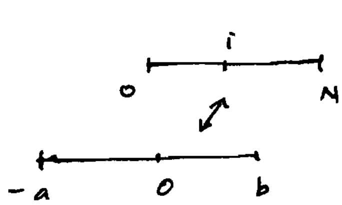
(4) As $a \to \infty$, the chain will more likely ~~exit~~ reach b before reaching $-a$.

More precisely, $\lim_{a\to\infty} \mathbb{P}\{T_a = -a\} = 0$.

This can be seen from, for example, problem 3,

$$\mathbb{P}\{T_a = b\} = \frac{1-\left(\frac{1-p}{p}\right)^{a}}{1-\left(\frac{1-p}{p}\right)^{a+b}} \longrightarrow 1 \quad \text{as} \quad a \to \infty$$

(setting $i = a$, $N = a+b$



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Also, $e^{-ta} \to 0$ as $a \to \infty$.

Moreover, ~~Hence~~ as $m(t) \geq 1$, $\mathbb{E}[\, m(t)^{-T_a} \mid S_{T_a} = -a] \leq 1$.

$$\Rightarrow \quad e^{-ta}\, \mathbb{E}[m(t)^{-T_a} \mid S_{T_a} = -a] \cdot \mathbb{P}\{S_{T_a} = -a\}$$

$$\leq e^{-ta} \cdot \mathbb{P}\{S_{T_a} = -a\} \longrightarrow 0 \quad . \quad \text{as} \quad a \to \infty .$$

Therefore, from $e^{tb}\, \mathbb{E}[m(t)^{-T_a} \mid S_{T_a} = b] \cdot \mathbb{P}\{S_{T_a} = b\}$

$$+ e^{-ta}\, \mathbb{E}[m(t)^{-T_a} \mid S_{T_a} = -a] \cdot \mathbb{P}\{S_{T_a} = -a\} = 1 ,$$

we obtain, by taking $a \to \infty$,

$$e^{tb}\, \mathbb{E}[m(t)^{-T}] = 1 .$$

Hence $\mathbb{E}[m(t)^{-T}] = e^{-tb}$.

Now set $s \equiv m(t)^{-1} = \dfrac{1}{pe^{t} + (1-p)e^{-t}}$. We solve e^{-t} in terms of s.

$$\Rightarrow \quad pe^{t} + (1-p)e^{-t} = \frac{1}{s} \quad \Rightarrow \quad (1-p)e^{-t} - \frac{1}{s} + pe^{t} = 0$$

$$\Rightarrow \quad (1-p)(e^{-t})^2 - \frac{1}{s}e^{-t} + p = 0$$

$$\Rightarrow \quad e^{-t} = \frac{\frac{1}{s} \pm \sqrt{\left(\frac{1}{s}\right)^2 - 4(1-p)p}}{2(1-p)} = \frac{1 \pm \sqrt{1 - 4(1-p)ps^2}}{2(1-p)s} .$$

But $\dfrac{1 + \sqrt{1 - 4(1-p)ps^2}}{2(1-p)s} \geq 1$ for $0 < s \leq 1$ (check this.)

Hence $$e^{-t} = \frac{1 - \sqrt{1 - 4(1-p)ps^2}}{2(1-p)s} .$$

$$\Rightarrow \quad \mathbb{E}[s^{T}] = \mathbb{E}[m(t)^{-T}] = (e^{-t})^{b} = \left(\frac{1 - \sqrt{1 - 4(1-p)ps^2}}{2(1-p)s} \right)^{b} .$$

(5) Set $G(s) = \left(\dfrac{1 - \sqrt{1 - 4p(1-p)s^2}}{2(1-p)s} \right)^b . = \mathbb{E}[s^T]$

$\mathbb{E}[T] = G'(1) = \dfrac{b}{2p-1}$ ← by a direct Calculus computation

$\mathrm{Var}(T) = G''(1) + G'(1) - (G''(1))^2 = \dfrac{4b \cdot p(1-p)}{(2p-1)^3}$

5. Set $\bar{Y}_n \equiv \max\{Y_0, Y_1, Y_2, \cdots, Y_n\}$.

By Doob's maximal inequality,

$$\mathbb{P}\{\bar{Y}_n < \tfrac{3}{4}\} = 1 - \mathbb{P}\{\bar{Y}_n \geq \tfrac{3}{4}\} \geq 1 - \frac{\mathbb{E}[Y_n]}{\frac{3}{4}}$$

But $\mathbb{E}[Y_n] \underset{\uparrow}{=} \mathbb{E}[Y_0] = \frac{1}{2}$

$\{Y_n\}$ is a martingale

$\Rightarrow \mathbb{P}\{\bar{Y}_n < \frac{3}{4}\} \geq 1 - \frac{4}{3} \cdot \frac{1}{2} = \frac{1}{3}$ ~~for all~~ n.

$\Rightarrow \mathbb{P}\{Y_k < \frac{3}{4}$ for all $k = 0, 1, 2, \cdots, n\} \geq \frac{1}{3}$.

This is true for all n. Hence taking $n \to \infty$,

$$\mathbb{P}\{Y_k < \tfrac{3}{4} \text{ for all } k = 0, 1, 2, \cdots\} \geq \tfrac{1}{3}.$$

$$\Rightarrow \mathbb{P}\{Y_k \geq \tfrac{3}{4} \text{ for some } k\} \leq 1 - \tfrac{1}{3} = \tfrac{2}{3}.$$