HW2. Solutions

1. (a) If $X_n = i$,

white i	white $N-i$
black $N-i$	black i
urn A	urn B

For $1 \le i \le N-1$

$$p(i, i-1) = \mathbb{P}\{\text{pick a white ball in A, pick a black ball in B}\}$$
$$= \frac{i}{N} \cdot \frac{i}{N}$$

$$p(i,i+1) = \mathbb{P}\{\text{pick a black ball in A, pick a white ball in B}\} = \left(\frac{N-i}{N}\right)^2$$

$$p(i,i) = 2\,\frac{i(N-i)}{N^2}$$

For $i=0$, $p(0,1)=1$ since we ~~always~~ pick a black ball in A, and we pick a white ball in B for sure.

For $i=N$, $p(N,N-1)=1$.

Hence $S = \{0,1,2,\cdots,N\}$, and

$$\left.\begin{aligned} p(i,i-1) &= \left(\frac{i}{N}\right)^2 \\ p(i,i) &= \frac{2i(N-i)}{N^2} \\ p(i,i+1) &= \left(\frac{N-i}{N}\right)^2 \end{aligned}\right\} \text{ for } 1 \le i \le N-1$$

and $p(0,1) = p(N,N-1) = 1$.

(b) If $X_n = i$,

i	$N-i$
A	B

For $1 \le i \le N-1$,

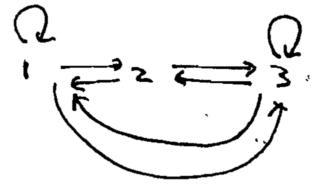
$$p(i,i-1) = \mathbb{P}\{\text{a ball in A is selected ~~and~~, the urn B is chosen}\}$$
$$= \frac{i}{N}\cdot(1-p)$$

$$p(i,i+1) = \mathbb{P}\{\text{a ball in B ~~is select~~ is selected and the urn A is chosen}\}$$
$$= \frac{N-i}{N}\cdot p$$

$$p(i,i) = \frac{i}{N}\cdot p + \frac{N-i}{N}\cdot(1-p)$$

For $i=0$, $p(0,0) = \mathbb{P}\{\text{the urn B is chosen}\}$ since a ball is chosen from B for sure.
$$= 1-p$$
$$p(0,1) = p$$

For $i=N$, $p(N,N-1) = 1-p$, $p(N,N) = p$.

#2.  It is easy to check that $p_2(i,j) > 0$ for any $i, j = 1, 2, 3$

Hence Thm 1.4 applies (with $n = 2$).

We ~~first~~ compute the invariant distribution $\bar{\pi}$: $\bar{\pi}\underline{\underline{P}} = \bar{\pi}$

$\Rightarrow \bar{\pi}(\underline{\underline{P}} - I) = 0 \Rightarrow (\underline{\underline{P}}^t - I)\bar{\pi}^t = 0$.

$$\underline{\underline{P}}^t - I = \frac{1}{10}\begin{bmatrix} -6 & 6 & 2 \\ 2 & -10 & 5 \\ 4 & 4 & -7 \end{bmatrix}$$

Do the Gauss elimination to the matrix, multiplied by 10:

$$\begin{bmatrix} -6 & 6 & 2 \\ 2 & -10 & 5 \\ 4 & 4 & -7 \end{bmatrix} \xrightarrow{\text{add row } 1 \times \frac{1}{3} \text{ to row } 2} \begin{bmatrix} -6 & 6 & 2 \\ 0 & -8 & \frac{17}{3} \\ 4 & 4 & -7 \end{bmatrix} \xrightarrow{\text{add } \frac{2}{3} \times \text{row } 1 + \text{row } 2 \text{ to row } 3} \begin{bmatrix} -6 & 6 & 2 \\ 0 & -8 & \frac{17}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

Hence
$$-6\pi_1 + 6\pi_2 + 2\pi_3 = 0$$
$$-8\pi_2 + \frac{17}{3}\pi_3 = 0$$
$$\Rightarrow \bar{\pi} = \left(\frac{25}{24}\pi_3, \frac{17}{24}\pi_3, \pi_3\right)$$

The condition ~~To make~~ $\pi_1 + \pi_2 + \pi_3 = 0$ is satisfied if we take $\pi_3 = \frac{4}{11}$

$$\therefore \bar{\pi} = \left(\frac{25}{66}, \frac{17}{66}, \frac{4}{11}\right) \Rightarrow \lim_{n\to\infty} \phi_n = \left(\frac{25}{66}, \frac{17}{66}, \frac{4}{11}\right)$$

Hence $\lim_{n\to\infty} \mathbb{P}\{X_n = 1\} = \frac{25}{66}$.

#3. $\bar{\pi}\underline{\underline{P}} = \bar{\pi}$: ~~implies~~

$$(\pi_0, \pi_1, \pi_2, \cdots, \pi_N)\begin{bmatrix} 0 & 1 & & & & & \\ 1-p & 0 & p & & & & \\ & 1-p & 0 & p & & & \\ & & 1-p & 0 & p & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1-p & 0 & p \\ & & & & & 1 & 0 \end{bmatrix} = (\pi_0, \pi_1, \pi_2, \cdots, \pi_N)$$

$$\Rightarrow \begin{cases} (1-p)\pi_1 = \pi_0 & \text{— (1)} \\ \pi_0 + (1-p)\pi_2 = \pi_1 & \text{— (2)} \\ p\pi_{n-1} + (1-p)\pi_{n+1} = \pi_n \quad \text{for } 2 \le n \le N-2 & \text{— (3)} \\ p\pi_{N-2} + \pi_N = \pi_{N-1} & \text{— (4)} \\ p\pi_{N-1} = \pi_N & \text{— (5)} \end{cases}$$

Now let's solve the difference equation

$$p\pi_{n-1} + (1-p)\pi_{n+1} = \pi_n, \qquad 2 \le n \le N-2. \quad \cdots (*)$$

Let's try the solution of the form

$$\pi_n = \alpha^n, \qquad 1 \le n \le N-1$$ ← (Note that here $1 \le n \le N-1$ since for the case $n=2$ in $(*)$, we have π_1 term)

$$\Rightarrow \quad p + (1-p)\alpha^2 = \alpha \quad \Rightarrow \quad (1-p)\alpha^2 - \alpha + p = 0$$

$$\Rightarrow \quad (\alpha-1)\big((1-p)\alpha - p\big) = 0 \quad \Rightarrow \quad \alpha = 1, \ \frac{p}{1-p}.$$

Hence $$\pi_n = A\cdot 1^n + B\cdot\left(\frac{p}{1-p}\right)^n, \qquad 1 \le n \le N-1,$$

for some constants A, B.

Now Equations (1) and (2) imply that

$$(1) \Rightarrow \pi_0 = (1-p)\cdot\left(A + B\cdot\frac{p}{1-p}\right) = (1-p)A + pB$$

$$(2) \Rightarrow \pi_0 = \pi_1 - (1-p)\pi_2 = A + B\cdot\frac{p}{1-p} - (1-p)\cdot\left(A + B\cdot\frac{p^2}{(1-p)^2}\right)$$

$$= pA + pB.$$

Hence $$\left.\begin{aligned}\pi_0 &= (1-p)A + pB\\ &= pA + pB\end{aligned}\right\} \Rightarrow A = 0, \quad \pi_0 = pB.$$

Also, $$(4) \Rightarrow \pi_N = \pi_{N-1} - p\pi_{N-2} = \left(\cancel{A} + B\left(\frac{p}{1-p}\right)^{N-1}\right) - p\cdot\left(\cancel{A} + B\left(\frac{p}{1-p}\right)^{N-2}\right)$$

$$= \frac{p^N}{(1-p)^{N-1}}\cdot B$$

$$(5) \Rightarrow \pi_N = p\pi_{N-1} = \frac{p^N}{(1-p)^{N-1}}\cdot B.$$

Hence
$$\begin{cases}\pi_0 = pB\\ \pi_n = \left(\frac{p}{1-p}\right)^n\cdot B, & 1 \le n \le N-1\\ \pi_N = \frac{p^N}{(1-p)^{N-1}}\cdot B\end{cases}$$

Now the sum $\pi_0 + \pi_1 + \cdots + \pi_N = 1$

$$\Rightarrow \left\{p + \frac{p}{1-p} + \left(\frac{p}{1-p}\right)^2 + \left(\frac{p}{1-p}\right)^3 + \cdots + \left(\frac{p}{1-p}\right)^{N-1} + \frac{p^N}{(1-p)^{N-1}}\right\} B = 1.$$

Recall that $1 + x + x^2 + \cdots + x^m = \dfrac{1 - x^{m+1}}{1 - x}$ for $x \neq 1$.

Hence the sum in the middle

$$\frac{p}{1-p} + \left(\frac{p}{1-p}\right)^2 + \cdots + \left(\frac{p}{1-p}\right)^{N-1}$$

$$= \frac{p}{1-p} \left\{ 1 + \frac{p}{1-p} + \left(\frac{p}{1-p}\right)^2 + \cdots + \left(\frac{p}{1-p}\right)^{N-2} \right\}$$

$$= \frac{p}{1-p} \cdot \frac{1 - \left(\frac{p}{1-p}\right)^{N-1}}{1 - \frac{p}{1-p}} = \frac{p}{1-2p} \cdot \left\{ 1 - \left(\frac{p}{1-p}\right)^{N-1} \right\}$$

$\Rightarrow$ the whole sum

$$= p + \left(\frac{p}{1-2p} - \frac{p^N}{(1-2p)(1-p)^{N-1}} \right) + \frac{p^N}{(1-p)^{N-1}}$$

$$= \frac{2p(1-p)}{1-2p} \cdot \left\{ 1 - \left(\frac{p}{1-p}\right)^N \right\}$$

$$\therefore \quad B = \frac{1}{2p(1-p) \cdot \left(1 - \left(\frac{p}{1-p}\right)^N \right)}$$

Hence

$$\begin{cases} \pi_0 = pB \\ \pi_n = \left(\frac{p}{1-p}\right)^n B & 1 \le n \le N-1 \\ \pi_N = \dfrac{p^N}{(1-p)^{N-1}} \cdot B \end{cases}$$

with B give by

#4. Let X_n be the # of ~~newspa~~ papers in the bin at 8:01 pm.

Then $S = \{0, 1, 2, 3, 4\}$

Note that $p(4,0) = 1$ since ~~d~~ if there were 4 papers in the bin, the next day it will have 5 papers, and hence will be emptied ~~for~~ sure.

For $i = 0,1,2,3$, $p(i,0) = \frac{1}{3}$, $p(i,i+1) = \frac{2}{3}$

Hence

$$\underline{\underline{P}} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

#5. $$\underline{\underline{P}}^3 = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \frac{13}{27} & \frac{14}{27} \\ \frac{14}{27} & \frac{13}{27} \end{pmatrix} \end{array}$$

With $\Phi_0 = (1, 0)$,

$$\Phi_3 = \Phi_0 \underline{\underline{P}}^3 = (1,0) \begin{pmatrix} \frac{13}{27} & \frac{14}{27} \\ \frac{14}{27} & \frac{13}{27} \end{pmatrix} = \left(\frac{13}{27}, \frac{14}{27}\right).$$

Hence $\mathbb{P}\{X_3 = 1\} = \phi_3(1) = \frac{14}{27}$

#6. $\bar{\pi} = \left(\frac{11}{39}, \frac{16}{39}, \frac{12}{39}\right)$. ~~Hence $\lim_{n\to\infty} \Phi_n = \pi$~~

From ~~B~~ the theorem 1.3 (Perron - Frobenius), ~~since all $p(i,j) > 0$~~, since all entries of $\underline{P}$ are strictly positive, $\lim_{n\to\infty} \bar{\Phi}_n = \bar{\pi}$.

$$\therefore \lim_{n\to\infty} \mathbb{P}\{X_n = 1\} = \pi(1) = \frac{11}{39}.$$

#7. $$\bar{\pi}(\underline{P} - I) = 0 \Rightarrow (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) \begin{bmatrix} -1 & 1/3 & 2/3 & 0 & 0 \\ 0 & -1 & 0 & 1/4 & 3/4 \\ 0 & 0 & -1 & 1/2 & 1/2 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix} = (0,0,0,0,0)$$

$$\Rightarrow \begin{cases} -\pi_1 + \pi_4 + \pi_5 = 0 \\ \left.\begin{aligned} \tfrac{1}{3}\pi_1 - \pi_2 &= 0 \\ \tfrac{2}{3}\pi_1 - \pi_3 &= 0 \end{aligned}\right\} \Rightarrow (\pi_1, \pi_2, \pi_3) = \left(a, \tfrac{1}{3}a, \tfrac{2}{3}a\right) \text{ for some } a. \\ \tfrac{1}{4}\pi_2 + \tfrac{1}{2}\pi_3 - \pi_4 = 0 \longrightarrow \pi_4 = \tfrac{5}{12}a \\ \tfrac{3}{4}\pi_2 + \tfrac{1}{2}\pi_3 - \pi_5 = 0 \longrightarrow \pi_5 = \tfrac{7}{12}a \end{cases}$$

We can check that $(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = \left(a, \frac{1}{3}a, \frac{2}{3}a, \frac{5}{12}a, \frac{7}{12}a\right)$ satisfies $-\pi_1 + \pi_4 + \pi_5 = 0$.

a is determined by the condition $\pi_1 + \pi_2 + \cdots + \pi_5 = 1$

$\Rightarrow a = 3$. $\quad \therefore \bar{\pi} = \left(\frac{1}{3}, \frac{1}{9}, \frac{2}{9}, \frac{5}{36}, \frac{7}{36}\right)$

#8. (a)

$$\underline{P} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(b) We guess that

$\bar{\pi}$ is proportional to $(3, 2, 3, 3, 1)$.

~~To be a probability vector, we need to divide~~

Here each entry equals the number of edges connected to each vertex.

It is easy to check that

$$(3, 2, 3, 3, 1) \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = (3, 2, 3, 3, 1)$$

To make it a probability vector, divide by $3+2+3+3+1 = 12$.

$\therefore$ By uniqueness, $\bar{\pi} = \left(\frac{3}{12}, \frac{2}{12}, \frac{3}{12}, \frac{3}{12}, \frac{1}{12}\right)$.

#9. (a) $\bar{w}\underline{P}^n = \bar{w}\underline{P} \cdot \underline{P}^{n-1} = \bar{w} \cdot \underline{P}^{n-1} = \cdots = \bar{w}$

By the uniqueness of the left-eigenvector for the eigenvalue 1 of $\underline{P}^n$ (combine Thm 1.4 and Thm 1.2), $\bar{w} = c\hat{\pi}$ for some constant $c \neq 0$.

$\Rightarrow \bar{\pi} = \hat{\pi}$ is the unique invariant distribution of $\underline{P}$.

(b) $\bar{v}\underline{P}^n = \bar{v}\underline{P} \cdot \underline{P}^{n-1} = \lambda \bar{v}\underline{P}^{n-1} = \cdots = \lambda^n \bar{v}$.

If $\lambda^n = 1$, then $\bar{v} = c'\hat{\pi} = c'\bar{\pi}$ $\Rightarrow \lambda = 1$.

If $\lambda^n \neq 1$, then as λ^n is an eigenvalue of $\underline{P}^n$, by (b) of Thm 1.2, $|\lambda^n| < 1$. Hence $|\lambda| < 1$.

Thus we showed that $\underline{P}$ satisfies the conditions (a) and (b) of Thm 1.2.