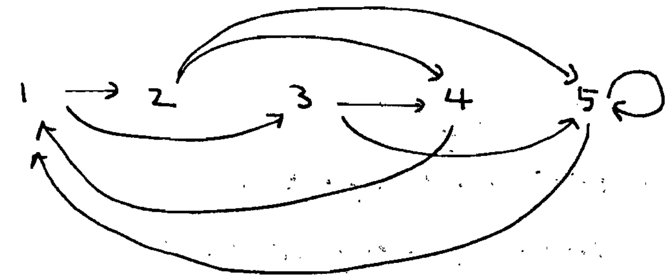
Sol. to HW 3.

1. 

(a) The chain is irreducible

(b) $d(5) = \text{g.c.d.}\{1, \cdots\} = 1$. The chain is aperiodic

(c) By (a), (b), we can use Thm 1.14.

· Let's find the inv. distn. π

$$(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)\begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/5 & 4/5 \\ 0 & 0 & 0 & 2/5 & 3/5 \\ 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \pi_1 = \pi_4 + \frac{1}{2}\pi_5 \\ \pi_2 = \frac{1}{2}\pi_1 & \longrightarrow \pi_2 = \frac{1}{2}\pi_1 \\ \pi_3 = \frac{1}{2}\pi_1 & \longrightarrow \pi_3 = \frac{1}{2}\pi_1 \\ \pi_4 = \frac{1}{5}\pi_2 + \frac{2}{5}\pi_3 & \longrightarrow \pi_4 = \frac{3}{10}\pi_1 \\ \pi_5 = \frac{4}{5}\pi_2 + \frac{3}{5}\pi_3 + \frac{1}{2}\pi_5 & \longrightarrow \pi_5 = \frac{8}{5}\pi_2 + \frac{6}{5}\pi_3 = \frac{7}{5}\pi_1 \end{cases}$$

$\Rightarrow$ π is proportional to $(1, \frac{1}{2}, \frac{1}{2}, \frac{3}{10}, \frac{7}{5})$ $\qquad 1 + \frac{1}{2} + \frac{1}{2} + \frac{3}{10} + \frac{7}{5} = \frac{37}{10}$

$\Rightarrow \pi = \left(\frac{10}{37}, \frac{5}{37}, \frac{5}{37}, \frac{3}{37}, \frac{14}{37}\right)$.

By Thm 1.14, $p_{1000}(2,1) \approx \pi(1) = \frac{10}{37}$, $p_{1000}(2,2) \approx \pi(2) = \frac{5}{37}$.

2. (a) Let X_n = the remainder of S_n after ~~dividing~~ division by 8.

State space $S = \{0, 1, 2, \cdots, 6, 7\}$.

$X_0 = 0$.

$p(0,1) = p(0,2) = \cdots = p(0,6) = \frac{1}{6}$ (i.e. the prob. that the die shows up j next time)

$p(0,0) = p(0,7) = 0$.

Similarly,

$$\underline{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix} \end{array}$$

Here $\cdot$ denotes $\frac{1}{6}$.

This chain is irreducible and aperiodic

$$d(0) = \underset{=1}{\text{g.c.d.}\{2,3,\cdots\}} \Leftarrow \begin{cases} p_2(0,0) \geq p(0,6)\,p(6,0) > 0 \\ p_3(0,0) \geq p(0,4)\,p(4,6)\,p(6,0) > 0 \end{cases}$$

Here we can use Thm 1.14

Let's ~~the~~ find the inv. distr.

We can guess that $\pi = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \cdots, \frac{1}{8})$.

It is easy to check that this vector indeed satisfies $\pi = \pi \underline{\underline{P}}$.

By the uniqueness of the inv. distr. in Thm 1.14,

$(\frac{1}{8}, \frac{1}{8}, \cdots, \frac{1}{8})$ is indeed the inv. distr.

$\therefore \lim_{n\to\infty} \mathbb{P}\{S_n \text{ is divisible by } 8\} = \lim_{n\to\infty} \mathbb{P}\{X_n = 0\} = \frac{1}{8}$.

(b) Let Z_n = the remainder of S_n after division by 5.

State space $= \{0, 1, 2, 3, 4\}$.

Note that $p(0,0) = \mathbb{P}\{Y_1 = 5\} = \frac{1}{6}$,

$p(0,1) = \mathbb{P}\{Y_1 = 1 \text{ or } 6\} = \frac{2}{6}$.

$p(0,j) = \frac{1}{6}$, $j = 2, 3, 4$.

$$\underline{\underline{P}} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \cdot & \times & \cdot & \cdot & \cdot \\ \cdot & \cdot & \times & \cdot & \cdot \\ \cdot & \cdot & \cdot & \times & \cdot \\ \cdot & \cdot & \cdot & \cdot & \times \\ \times & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \end{array}$$

where $\cdot$ denotes $\frac{1}{6}$ and $\times$ denotes $\frac{2}{6}$.

Clearly, the chain is irreducible and aperiodic $\therefore$ Thm 1.14 applies.

The inv. distr. is $\pi = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ (← check this)

$\therefore \lim_{n\to\infty} \mathbb{P}\{S_n \text{ is divisible by } 5\}$

$= \lim_{n\to\infty} \mathbb{P}\{Z_n = 0\} = \frac{1}{5}$.

3. Set $Z_n = (X_n, Y_n)$. The state space is $\{(0,0),(0,1),(1,0),(1,1)\}$

Then $p((i,i'),(j,j')) = \mathbb{P}\{Z_1 = (j,j') \mid Z_0 = (i,i')\}$

$$= \mathbb{P}\{X_1 = j \mid X_0 = i\}\,\mathbb{P}\{Y_1 = j' \mid Y_0 = i'\} = p(i,j)\,p(i',j'), \quad i,i',j,j' = 0,1.$$

Thus the transition matrix $\underline{\hat{P}}$ for Z_n is

$$\cancel{\hat{P} = \begin{array}{c|cccc} & (0,0) & (0,1) & (1,0) & (1,1) \\ \hline (0,0) & \frac{4}{9} & 2/9 & 2/9 & 1/9 \\ (0,1) & & & & \\ (1,0) & & & & \\ (1,1) & & & & \end{array}}$$

$$\underline{\hat{P}} = \begin{array}{c} \\ (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \end{array} \begin{array}{c} \begin{array}{cccc} (0,0) & (0,1) & (1,0) & (1,1) \end{array} \\ \begin{bmatrix} 9/16 & 3/16 & 3/16 & 1/16 \\ 3/8 & 3/8 & 1/8 & 1/8 \\ 3/8 & 1/8 & 3/8 & 1/8 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} \end{array}.$$

The chain Z_n is irreducible and aperiodic.

~~The (unique) invariant distribution is $\pi = (\frac{4}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9})$. (compute this!)~~

To compute the inv. distr. $\bar{\pi}$ of $\underline{\hat{P}}$,

$$\bar{\pi} = \bar{\pi}\underline{\hat{P}} \Rightarrow \bar{\pi}(I - \underline{\hat{P}}) = 0 \Rightarrow (I - \underline{\hat{P}}^t)\bar{\pi}^t = 0$$
$$\Rightarrow 16\cdot(I - \underline{\hat{P}}^t)\bar{\pi}^t = 0$$

$$16\cdot(I - \underline{\hat{P}}^t) = \begin{bmatrix} 7 & -6 & -6 & -4 \\ -3 & 10 & -2 & -4 \\ -3 & -2 & 10 & -4 \\ -1 & -2 & -2 & 12 \end{bmatrix}$$ Do Gauss elimination

$$\xrightarrow[\text{to } r_4]{\text{add } r_1+r_2+r_3} \begin{bmatrix} 7 & -6 & -6 & -4 \\ -3 & 10 & -2 & -4 \\ -3 & -2 & 10 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[r_3.]{\text{subtract } r_2 \text{ from}} \begin{bmatrix} 7 & -6 & -6 & -4 \\ -3 & 10 & -2 & -4 \\ 0 & -12 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

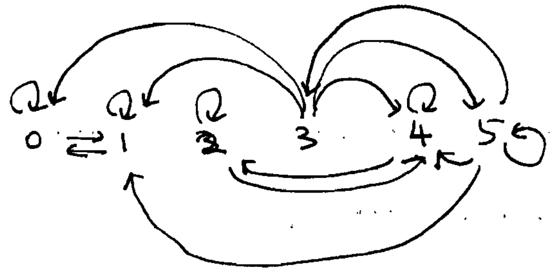
Thus $\bar{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$ satisfies

$$\left\{\begin{array}{l} 7\pi_1 - 6\pi_2 - 6\pi_3 - 4\pi_4 = 0 \\ -3\pi_1 + 10\pi_2 - 2\pi_3 - 4\pi_4 = 0 \\ -12\pi_2 + 12\pi_3 = 0 \end{array}\right. \longrightarrow \pi_2 = \pi_3 \longrightarrow \left\{\begin{array}{l} 7\pi_1 - 12\pi_2 - 4\pi_4 = 0 \\ -3\pi_1 + 8\pi_2 - 4\pi_4 = 0 \end{array}\right. \longrightarrow 4\pi_1 - 8\pi_2 = 0$$

$$\Rightarrow \bar{\pi} = c\cdot(2,1,1,\tfrac{1}{2}) \Rightarrow \bar{\pi} = \left(\tfrac{4}{9}, \tfrac{2}{9}, \tfrac{2}{9}, \tfrac{1}{9}\right).$$

We can use Thm 1.14 (or Thm 1.3)

$$\therefore \lim_{n\to\infty} \mathbb{P}\{X_n = Y_n\} = \lim_{n\to\infty} \mathbb{P}\{Z_n = (0,0) \text{ or } (1,1)\} = \frac{4}{9} + \frac{1}{9} = \frac{5}{9}$$

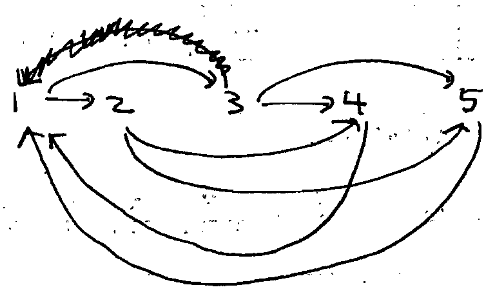
4. 

$\{0,1\}, \{2,4\}, \{3,5\}$ are communication classes.

$d(0) = d(1) = 1$ as $p(0,0) > 0$

$d(2) = d(4) = 1$ as $p(2,2) > 0$

$d(3) = d(5) = 1$ as $p(5,5) > 0$.

5. 

(a) The chain is irreducible

(b) The period is 3.

$d(1) = \text{g.c.d.}\{3, 6, 9, \cdots\} = 3$.

(c) $p_{25}(2,2) = \mathbb{P}\{X_{25} = 2 \mid X_0 = 2\}$.

Since $d(2) = 3$, $X_n \neq 2$ for n not a multiple of 3.

$\Rightarrow \quad p_{25}(2,2) = 0$.

From the chain, it is clear that if $X_0 = 2$, then X_1 is either 4 or 5, $X_2 = 1$.

Then $X_3 = 2$ or 3, $X_4 = 4$ or 5, $X_5 = 1$, $X_6 = 2$ or 3, $\cdots$

i.e. if $X_0 = 2$, $X_{3n+1} = 4$ or 5

$X_{3n+2} = 1$

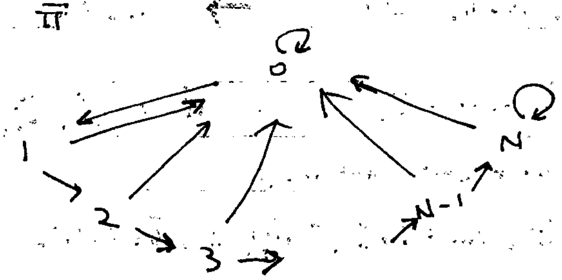
$X_{3n} = 2$.

$\Rightarrow \quad p_{25}(2,1) = 0$.

6. (a) This follows from the description of the model.

(b) The chain is irreducible and aperiodic: Then 1.14 applies.

We need to find π



$$\overline{\pi} = \overline{\pi}\,\underline{\underline{P}}$$

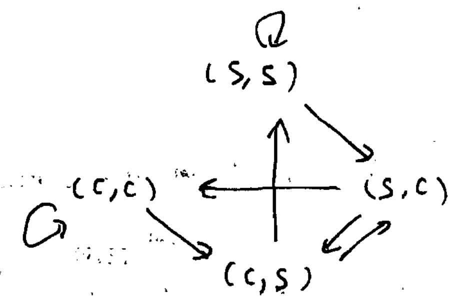
$$\Rightarrow \quad \begin{aligned} \pi_0 &= p\cdot(\pi_0 + \pi_2 + \cdots + \pi_N) \\ \pi_1 &= (1-p)\pi_0 \\ \pi_2 &= (1-p)\pi_1 \\ &\vdots \\ \pi_{N-1} &= (1-p)\pi_{N-2} \\ \pi_N &= (1-p)\pi_{N-1} + (1-p)\pi_N \end{aligned} \quad \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \Rightarrow \quad \begin{aligned} \pi_i &= (1-p)^i\pi_0, \quad 1 \le i \le N-1 \\ \pi_N &= \frac{1-p}{p}\pi_{N-1} = \frac{(1-p)^N}{p}\pi_0 \end{aligned}$$

$$\Rightarrow \quad \overline{\pi} = \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & \cdots & N-1 & N \end{array} \\ \left(p,\ (1-p)p,\ (1-p)^2p,\ (1-p)^3p,\ \cdots,\ (1-p)^{N-1}p,\ (1-p)^N\right) \end{array}$$

(check that $\sum_{i=0}^{N}\pi_i = 1$)

$$\therefore \lim_{n\to\infty} \mathbb{P}\{X_n = i\} = \begin{cases} (1-p)^i p, & i = 0,1,\cdots,N-1 \\ (1-p)^N & i = N. \end{cases}$$

7. $S = \{(S,S), (S,C), (C,S), (C,C)\}$



$$\underline{\underline{P}} = \begin{array}{c|cccc} & (S,S) & (S,C) & (C,S) & (C,C) \\ \hline (S,S) & 0.8 & 0.2 & 0 & 0 \\ (S,C) & 0 & 0 & 0.4 & 0.6 \\ (C,S) & 0.6 & 0.4 & 0 & 0 \\ (C,C) & 0 & 0 & 0.1 & 0.9 \end{array}$$

The chain is irreducible and aperiodic.

The inv. distr. is $\overline{\pi} = \left(\frac{3}{11}, \frac{1}{11}, \frac{1}{11}, \frac{6}{11}\right)$

$$\lim_{n\to\infty} \mathbb{P}\{\text{day } n \text{ is sunny}\} = \lim_{n\to\infty}\mathbb{P}\{X_n = (S,S), X_n = (C,S)\}$$

$$= \frac{3}{11} + \frac{1}{11} = \frac{4}{11}$$

8. (a) There are integers a, b s.t. $am + bn = 1$.

One of a, b is positive, and the other is negative.

Suppose that $a > 0$, $b < 0$. (The other case is similar).

Set $a = A$, $b = -B$, $A, B > 0$: $Am - Bn = 1$.

Take $N = Bn^2$.

For $0 \le k \le n-1$,

$N + k = Bn^2 + k(Am - Bn) = kAm + (n-k)Bn \in I$

$I \equiv \{xm + yn \mid x, y > 0, \text{integers}\}$.

$\Rightarrow N, N+1, \cdots, N+n-1 \in I$.

Also $N + \ell n + k = N + k + \ell n = kAm + ((n-k)B + \ell)n \in I$

for $\ell \ge 0$, $0 \le k \le N-1$.

$\therefore \{N, N+1, N+2, \cdots\} \subset I$.

(b) We claim that there are $m, n \in J$, $m \ne n$, s.t. $g.c.d.\{m, n\} = d$.

[Suppose not: Then for all $M, N \in J$, $g.c.d.\{M, N\} = k_{M,N}\, d$ for some $k_{M,N} \ge 2$

(Recall that $d \mid M$ for all $M \in J$.)

Set $k = \min\{k_{M,N} \mid M, N \in J\}$. Then $k \ge 2$ since $k_{M,N} \ge 2$ for all $M, N \in J$.

Then for any $M, N \in J$, $k \le k_{M,N}$, hence $kd \mid M$ as $k_{M,N}\, d \mid M$

$\Rightarrow g.c.d.\, J \ge kd$ contraction to $g.c.d.\, J = d$.

Let $m \ne n$, $m, n \in J$ be s.t. $g.c.d.\{m, n\} = d$.

Set $m' = \frac{m}{d}$, $n' = \frac{n}{d}$ $\Rightarrow$ $g.c.d.\{m', n'\} = 1$.

By (a), there is N s.t.

$\{N, N+1, N+2, \cdots\} \subset \{xm' + yn' \mid x, y > 0, \text{integers}\}$

$\Rightarrow \{Nd, (N+1)d, (N+2)d, \cdots\} \subset \{xm + yn \mid x, y > 0, \text{integers}\} \subset J$ #

J is closed under addition.