Sol. to HW 4

1. $\{1,2\}$, $\{5,6\}$ > recurrent classes. $\{3,4\}$: transient class.

$\{1,2\}$: $\underline{\underline{P}}_1 = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1/2 & 1/2 \\ 2 & 1/4 & 3/4 \end{array} \Rightarrow$ irreducible, aperiodic : Thm 1.14 applies

$\bar{\pi}^1 = \left(\frac{1}{3}, \frac{2}{3}\right)$

$\Rightarrow \lim_{n\to\infty} \underline{\underline{P}}_1^{\,n} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1/3 & 2/3 \\ 2 & 1/3 & 2/3 \end{array}$

$\{5,6\}$: $\underline{\underline{P}}_2 = \begin{array}{c|cc} & 5 & 6 \\ \hline 5 & 1/2 & 1/2 \\ 6 & 1/2 & 1/2 \end{array} \Rightarrow$ irreducible, aperiodic, $\bar{\pi}^2 = \left(\frac{1}{2}, \frac{1}{2}\right)$.

$$\Rightarrow \lim_{n\to\infty} \underline{\underline{P}}^n = \begin{array}{c|cc|cc|cc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1/3 & 2/3 & 0 & & 0 & \\ 2 & 1/3 & 2/3 & & & & \\ \hline 3 & * & & 0 & & * & \\ 4 & & & & & & \\ \hline 5 & 0 & & 0 & & 1/2 & 1/2 \\ 6 & & & & & 1/2 & 1/2 \end{array}$$

Initial distribution $\bar{\phi}_0 = \overset{1}{(}\tfrac{1}{2}, \overset{2}{0}, \overset{3}{0}, \overset{4}{0}, \overset{5}{0}, \overset{6}{\tfrac{1}{2}})$

$$\Rightarrow \lim_{n\to\infty} \bar{\phi}_n = \lim_{n\to\infty} \bar{\phi}_0 \underline{\underline{P}}^n = \left(\tfrac{1}{2}, 0, 0, 0, 0, \tfrac{1}{2}\right) \left[\begin{array}{cc|cc|cc} 1/3 & 2/3 & 0 & & 0 & \\ 1/3 & 2/3 & & & & \\ \hline * & & 0 & & 0 & \\ \hline 0 & & 0 & & 1/2 & 1/2 \\ & & & & 1/2 & 1/2 \end{array}\right]$$

$$= \left(\tfrac{1}{6}, \tfrac{1}{3}, 0, 0, \tfrac{1}{4}, \tfrac{1}{4}\right)$$

$$\therefore \lim_{n\to\infty} \mathbb{P}\{X_n = i\} = \begin{cases} 1/6 & , i=1 \\ 1/3 & , i=2 \\ 0 & . i=3,4 \\ 1/4 & , i=5,6 \end{cases}$$

2. The chain is irreducible and period $= 2$: Thm 1.15 applies.

$\bar{\pi} = \bar{\pi}\, \underline{\underline{P}}$

$$\Rightarrow \begin{cases} \pi_0 = (1-p)\pi_1 \\ \pi_1 = \pi_0 + (1-p)\pi_2 \\ \pi_2 = p\pi_1 + (1-p)\pi_3 \\ \pi_3 = p\pi_2 + \pi_4 \\ \pi_4 = p\pi_3 \end{cases} \Rightarrow \bar{\pi} = \frac{1}{2(1-2p+2p^2)} \left(\overset{0}{(1-p)^3}, \overset{1}{(1-p)^2}, \overset{2}{p(1-p)}, \overset{3}{p^2}, \overset{4}{p^3} \right)$$

(a) Since $X_0 = 0$, $X_{2n} = 0$ is possible.

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}\{X_{2n} = 0\} = \lim_{n\to\infty} \mathbb{P}\{X_{2n} = 0 \mid X_0 = 0\} = 2 \cdot \pi(0) = \frac{(1-p)^3}{1-2p+2p^2}.$$

(b) $\mathbb{E}[\#\text{ steps until reaching } 2 \mid X_0 = 2]$

$$= \frac{1}{\pi(2)} = \frac{2(1-2p+2p^2)}{p(1-p)} .$$

3. The chain is irreducible and period $= 2$: Thm 1.15 applies

$$\uparrow$$

$A_1 = \{1,3\}, \quad A_2 = \{2,4\}.$

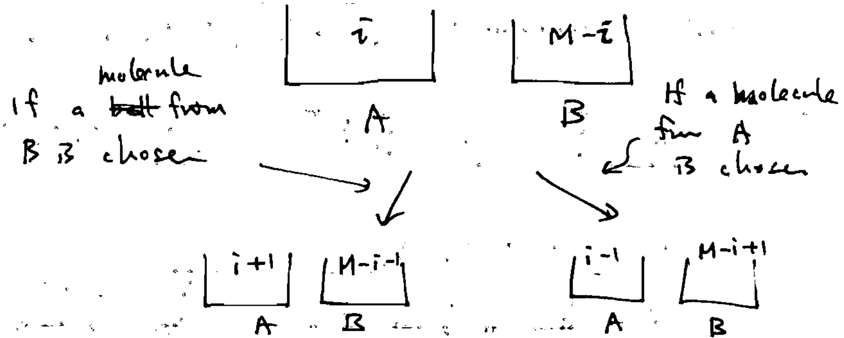
~~For~~ To find π, $(\bar{\pi} = \bar{\pi}\underline{P})$, try $(1,1,1,1)$...

The (unique) inv. distr. is $\bar{\pi} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.

$\therefore \mathbb{E}[\#\text{ steps to reach } i \mid X_0 = i] = \frac{1}{\pi(i)} = 4$, for $i = 1,2,3,4$

4. For $1 \le i \le M-1$,

$$\left.\begin{aligned} p(i, i+1) &= \frac{M-i}{M} \\ p(i, i-1) &= \frac{i}{M} . \end{aligned}\right) (*)$$



On the other hand,

$p(0,1) = 1, \quad p(M, M-1) = 1.$

(For these, the formula $(*)$ still works.)

The chain is irreducible. (and the period $= 2$). By Thm 1.15, there is unique invr. distr..

Let's check that $\bar{\pi}$ given by $\pi(i) = \binom{M}{i} 2^{-M}$, $i = 0, 1, \cdots, M$,

is the correct invr. distr.

$$\Rightarrow (\bar{\pi}\underline{P})(j) = \sum_{i=0}^{M} \pi(i) P(i,j) = \pi(j-1) P(j-1,j) + \pi(j+1) P(j+1,j)$$

$$= \binom{M}{j-1} 2^{-M} \cdot \frac{M-j+1}{M} + \binom{M}{j+1} 2^{-M} \cdot \frac{j+1}{M}$$

$$= \left\{ \frac{M-j+1}{(j-1)!\,(M-j+1)!} + \frac{(j+1)!}{(j+1)!\,(M-j-1)!} \right\} \cdot \frac{M!\,2^{-M}}{M}$$

$$= \binom{M}{j} 2^{-M} = \bar{\pi}(j) .$$

Hence ~~$\bar{\pi}$~~ $\pi(j) = \binom{M}{j} 2^{-M}$ is the (unique) invr. distr.

5. This is easy to check.

6. Let the state space be the set of all $N!$ orderings of cards. The transition probability is $p(\sigma_1, \sigma_2) = \frac{1}{N}$ if σ_2 can be obtained from σ_1 by taking out a card and placing it on top.

Otherwise, $p(\sigma_1, \sigma_2) = 0$.

The transition matrix $\underline{P} = \begin{array}{c|ccc} & 1 & 2 \cdots & N! \\ \hline 1 & & & \\ 2 & & & \\ \vdots & & & \\ N! & & & \end{array}$

has N non-zero entries (with value $\frac{1}{N}$) on each row.

Also, on each column, there are N non-zero entries since an ordering σ can be obtained from N different orderings from the previous time step.

From this, it is easy to check that $\pi = (\overset{1}{\frac{1}{N!}}, \overset{2}{\frac{1}{N!}}, \overset{\cdots}{\cdots}, \overset{N!}{\frac{1}{N!}})$ is an invariant distribution.

Now the chain is irreducible since from an arbitrary ordering, we can make it to the trivial ordering $1, 2, 3, \cdots, N$ by a succession of the shuffling procedures: pick card N first, and put it on top. Next pick card $N-1$ and put it on top, etc. Conversely, starting at the trivial ordering, we can make an arbitrary ordering by a succession of the shuffling procedures.

Also, the chain is aperiodic since from the trivial ordering, we may pick card 1 and put it on top, resulting the same trivial ordering.

Hence by Thm 1.14, $\lim_{n\to\infty} \mathbb{P}\{X_n = i\} = \pi(i) = \frac{1}{N!}$ for $i \in S$.