Sol. to HW 5

1. (a) Set $\alpha_i \equiv \mathbb{E}[\#\text{ visits to } 1 \text{ before reaching } 2 \mid X_0 = i]$.

We want α_2.

$$\begin{cases} \alpha_2 = p_{21}(1+\alpha_1) + p_{23}\alpha_3 = \frac{1}{2}(1+\alpha_1) + \frac{1}{2}\alpha_3 \\ \left.\begin{aligned} \alpha_1 &= p_{12}\cdot 0 + p_{13}\alpha_3 + p_{14}\alpha_4 = \tfrac{1}{3}\alpha_3 + \tfrac{1}{3}\alpha_4 \\ \alpha_3 &= p_{32}\cdot 0 + p_{31}(1+\alpha_1) + p_{34}\alpha_4 = \tfrac{1}{3}(1+\alpha_1) + \tfrac{1}{3}\alpha_4 \\ \alpha_4 &= \tfrac{1}{3}(1+\alpha_1) + \tfrac{1}{3}\alpha_3 + \tfrac{1}{3}\alpha_5 \end{aligned}\right\} (*) \\ \alpha_5 = \alpha_4 \end{cases}$$

$$(*) \Rightarrow \begin{bmatrix} 3\alpha_1 = \alpha_3 + \alpha_4 \\ 3\alpha_3 = 1 + \alpha_1 + \alpha_4 \\ 2\alpha_4 = 1 + \alpha_1 + \alpha_3 \end{bmatrix} \xRightarrow{\text{add them all}} \alpha_1 + \alpha_3 = 2$$

$$\therefore \alpha_2 = \frac{1}{2} + \frac{1}{2}(\alpha_1 + \alpha_2) = \frac{3}{2}$$

(b) Set $\beta_i \equiv \mathbb{P}\{\text{reaches } 1 \text{ before reaching } 2 \mid X_0 = i\}$.

$$\beta_2 = p_{21}\cdot 1 + p_{23}\cdot\beta_3 = \frac{1}{2} + \frac{1}{2}\beta_3$$

$$\begin{pmatrix} \beta_3 = p_{32}\cdot 0 + p_{31}\cdot 1 + p_{34}\cdot\beta_4 = \frac{1}{3} + \frac{1}{3}\beta_4 \\ \beta_4 = \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot\beta_3 + \frac{1}{3}\cdot\beta_5 \\ \beta_5 = \beta_4 \end{pmatrix} \Rightarrow \beta_3 = \frac{3}{5},\quad \beta_4 = \beta_5 = \frac{4}{5}$$

$$\therefore \beta_2 = \frac{4}{5}$$

2. $S = \{0,1,2,3\}$. Y_n = remainder of S_n divided by 4.

Note that $Y_0 = 0$

(a) $\mathbb{E}[T_0] = \mathbb{E}[\text{return time to } 0 \mid Y_0 = 0]$

$$\underline{P} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1/6 & 2/6 & 2/6 & 1/6 \\ 1 & 1/6 & 1/6 & 2/6 & 2/6 \\ 2 & 2/6 & 1/6 & 1/6 & 2/6 \\ 3 & 2/6 & 2/6 & 1/6 & 1/6 \end{array}$$

$$= \frac{1}{\pi_0}$$

But $\pi = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ $\therefore \mathbb{E}[T_0] = 4$.

(b) $\mathbb{E}[T_1] = \mathbb{E}[\text{time to reach } 1 \mid Y_0 = 0]$.

Set $\alpha_i = \mathbb{E}[\text{time to reach } 1 \mid Y_0 = i]$.

We want α_0.

$$\alpha_0 = p_{00}(1+\alpha_0) + p_{01}\cdot 1 + p_{02}\cdot(1+\alpha_2) + p_{03}\cdot(1+\alpha_3)$$

$$\alpha_2 = p_{20}(1+\alpha_0) + p_{21}\cdot 1 + p_{22}\cdot(1+\alpha_2) + p_{23}\cdot(1+\alpha_3)$$

$$\alpha_3 = p_{30}(1+\alpha_0) + p_{31}\cdot 1 + p_{32}\cdot(1+\alpha_2) + p_{33}\cdot(1+\alpha_3)$$

$$\Rightarrow \left\{ \begin{aligned} 5\alpha_0 &= 2\alpha_2 + \alpha_3 + 6 \\ 5\alpha_2 &= 2\alpha_0 + 2\alpha_3 + 6 \\ 5\alpha_3 &= 2\alpha_0 + \alpha_2 + 6 \end{aligned} \right\} \Rightarrow \alpha_0 = \frac{86}{25},\ \alpha_2 = \frac{98}{25},\ \alpha_3 = \frac{84}{25}$$

$$\therefore\ \mathbb{E}[T_1] = \alpha_0 = \frac{86}{25}.$$

3. Let X_n = # ~~successive~~ consecutive heads at the n^{th} turn.

For example, if the outcome was H H T T H T H H H T ...,

then

n	1	2	3	4	5	6	7	8	9	10	...
outcome	H	H	T	T	H	T	H	H	H	T	...
X_n	1	2	0	0	1	0	1	2	3	0	...

X_n is a Markov chain with $S = \{0,1,2,3,4\}$ (since we stop when $X_n=4$ we can ~~not~~ simply choose $p(4,4)=1$.)

The transition matrix is

$$\underline{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Also, $X_0 \equiv 0$.

We want to compute

$$\mathbb{E}[\text{time to reach } 4 \mid X_0 = 0].$$

Note that we made the state 4 as an absorbing state.

$\{0,1,2,3\}$ is a transient class and $\{4\}$ is recurrent.

Set $\alpha_i \equiv \mathbb{E}[\text{time to reach } 4 \mid X_0 = i]$.

$$\left.\begin{aligned}
\alpha_0 &= \tfrac{1}{2}(1+\alpha_0) + \tfrac{1}{2}(1+\alpha_1) \\
\alpha_1 &= \tfrac{1}{2}(1+\alpha_0) + \tfrac{1}{2}(1+\alpha_2) \\
\alpha_2 &= \tfrac{1}{2}(1+\alpha_0) + \tfrac{1}{2}(1+\alpha_3) \\
\alpha_3 &= \tfrac{1}{2}(1+\alpha_0) + \tfrac{1}{2}\cdot 1
\end{aligned}\right\} \Rightarrow \alpha_0 = 30,\ \alpha_1 = 28,\ \alpha_2 = 24,\ \alpha_3 = 16.$$

$\therefore\ \alpha_0 = 30.$

4. $\underset{R_1}{\underset{\parallel}{\{0,1\}}}$, $\underset{R_2}{\underset{\parallel}{\{2,4\}}}$: recurrent classes. $\{3,5\}$: transient.

First, compute $\alpha \equiv \mathbb{P}\{\text{chain arrives in } R_1 \mid X_0 = 5\}$.

Set $\beta \equiv \mathbb{P}\{\text{chain arrives in } R_1 \mid X_0 = 3\}$.

$$\left.\begin{aligned}
\alpha &= P_{50}\cdot 1 + P_{51}\cdot 1 + P_{53}\cdot\beta + P_{54}\cdot 0 + P_{55}\cdot\alpha \\
&= \tfrac{1}{10} + \tfrac{1}{10} + \tfrac{2}{10}\beta + \tfrac{4}{10}\alpha \\
\beta &= P_{30}\cdot 1 + P_{31}\cdot 1 + P_{34}\cdot 0 + P_{35}\cdot\alpha \\
&= \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4}\alpha
\end{aligned}\right\} \Rightarrow \begin{cases} 3\alpha = \beta + 1 \\ 4\beta = 2 + \alpha \end{cases}$$

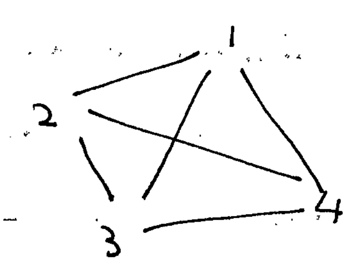
$$\Downarrow$$

$$\alpha = \frac{6}{11},\quad \beta = \frac{7}{11}$$

Second, compute the inv. distr. of $\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & & \\ 1 & & \end{array}$ $\begin{array}{c} 0 \\ 1 \end{array}\begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix} \Rightarrow \vec{\pi} = \left(\frac{3}{8}, \frac{5}{8}\right)$

$$\therefore \lim_{n\to\infty} \mathbb{P}\{X_n = 0 \mid X_0 = 5\} = \frac{3}{8}\cdot\frac{6}{11} = \frac{9}{44}.$$

5. (a) Let $p_k \equiv \mathbb{P}\{T = k \mid X_0 = 1\}$.



$p_1 = 0.$

$$p_2 = \sum_{j=2}^{N} P_{ij}\cdot\mathbb{P}\{T = 2 \mid X_0 = 1, X_1 = j\}$$

$$= \sum_{j=2}^{N} P_{ij}\cdot\frac{1}{N-1} = \frac{1}{N-1}.$$

Note that, due to the symmetry,

$$\mathbb{P}\{T=k \mid X_0=2\} = \mathbb{P}\{T=k \mid X_0=3\}.$$

Hence set $q_k = \mathbb{P}\{T=k \mid X_0=2\} = \mathbb{P}\{T=k \mid X_0=3\}$

$$= \cdots = \mathbb{P}\{T=k \mid X_0=N\}.$$

Note that $q_1 = \frac{1}{N-1}$,

for $k \geq 2$

$$q_k = \mathbb{P}\{T=k \mid X_0=2\} = p_{21} \cdot \mathbb{P}\{T=k \mid X_0=2, X_1=1\} + \sum_{j=3}^{N} p_{2j} \cdot \mathbb{P}\{T=k \mid X_0=2, X_1=j\}$$

$$= p_{21} \cdot 0 + \sum_{j=3}^{N} \frac{1}{N-1} \cdot q_{k-1}$$

$$= \frac{N-2}{N-1} q_{k-1}$$

$$\Rightarrow \quad q_k = \left(\frac{N-2}{N-1}\right)^{k-1} \cdot \frac{1}{N-1}, \quad k=1,2,3,\cdots$$

Now $p_k = \sum_{j=2}^{N} p_{1j} \cdot \mathbb{P}\{T=k \mid X_0=1, X_1=j\} = \sum_{j=2}^{N} \frac{1}{N-1} \cdot q_{k-1} = \frac{1}{N-1} q_{k-1}$

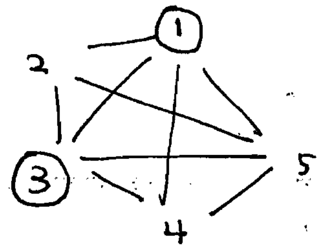
$$\Rightarrow \quad p_k = \left(\frac{N-2}{N-1}\right)^{k-2} \cdot \frac{1}{N-1}, \quad k=2,3,4,\cdots$$

$$\therefore \left.\begin{aligned} \mathbb{P}\{T=k \mid X_0=1\} &= \left(\frac{N-2}{N-1}\right)^{k-2} \cdot \frac{1}{N-1}, \quad k=2,3,4,\cdots \\ \mathbb{P}\{T=k \mid X_0=i\} &= \left(\frac{N-2}{N-1}\right)^{k-1} \cdot \frac{1}{N-1}, \quad k=1,2,3,\cdots \\ & \qquad i=2,3,\cdots,N \end{aligned}\right\} \text{geometric r.v.}$$

(b) # steps until every vertices has been visited at least once

$$= T_1 + T_2 + T_3 + \cdots + T_N$$

where T_k = # steps until a new vertex is visited starting from the moment that $k-1$ vertices has been visited at least for the first time.

By definition, $T_1 = 0$. 

Clearly, $T_2 = 1$.

For T_k, note that we have now $k-1$ vertices once visited.

The prob. that at the next step, it visits a new vertex is

$$\frac{N-k+1}{N-1}.$$

If the chain visits an "old" vertex, then it still has the same

$\frac{N-k+1}{N-1}$ probability to visit a new vertex at the next step

since the graph is complete.

Hence T_k is a geometric r.v. with $p = \frac{N-k+1}{N-1}$

$$\Rightarrow \quad \mathbb{E}[T_k] = \frac{N-1}{N-k+1}$$

$\therefore$ $\mathbb{E}$[# steps until all vertices are visited]

$$= \mathbb{E}[T_1] + \mathbb{E}[T_2] + \cdots + \mathbb{E}[T_N]$$

$$= 0 + 1 + \frac{N-1}{N-2} + \frac{N-1}{N-3} + \cdots + \frac{N-1}{2} + \frac{N-1}{1}$$

$$= (N-1) \cdot \left\{ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N-1} \right\}.$$

6. We use Thm 2.2. Take the state $k = 0$.

Then the equation to solve is

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix} = \begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \end{matrix} & \begin{bmatrix} B & A & & \\ C & B & A & 0 \\ & C & B & A \\ 0 & & \ddots & \ddots \end{bmatrix} \end{array} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix}$$

where

$A = (1-q)p$

$B = qp + (1-q)(1-p)$

$C = q(1-p)$.

$$\Rightarrow \begin{cases} y_1 = By_1 + Ay_2 \\ y_i = Cy_{i-1} + By_i + Ay_{i+1}, & i = 2,3,4,\cdots \end{cases}$$

To solve the difference eq., set $y_i = \alpha^i$ and using the actual values of A, B, C,

$$(1-q)p\cdot\alpha^2 + (-p-q+2pq)\alpha + q(1-p) = 0$$

$$\Rightarrow \begin{cases} \alpha = 1, \ \dfrac{q(1-p)}{(1-q)p} & \text{if } p \neq q \ \left(\text{then } \dfrac{q(1-p)}{(1-q)p} \neq 1\right) \\ \alpha = 1 \ \text{(double root)} & \text{if } p = q \end{cases}$$

(a) Assume that $p \neq q$.

$$\Rightarrow y_i = c_1 + c_2 r^i, \quad i = 1, 2, \cdots \quad \text{where } r \equiv \frac{q(1-p)}{(1-q)p}.$$

To satisfy $y_1 = By_1 + Ay_2$, i.e. $(p+q-2pq)y_1 = (1-q)p\,y_2$

$$\Rightarrow c_1 = -c_2$$

$$\Rightarrow y_i = c_1 \cdot (1 - r^i), \quad i = 1, 2, 3, \cdots$$

If $r > 1$, then there is no non-trivial sol. s.t $0 \leq y_i \leq 1$

i.e. $q > p$ $\Rightarrow$ recurrent

If $r < 1$, then $y_i = 1 - r^i$ is a solution

i.e. $q < p$ $\Rightarrow$ transient

(b) Assume that $p = q$: $y_i = c_1 + c_2 i$, $i = 1, 2, \cdots$

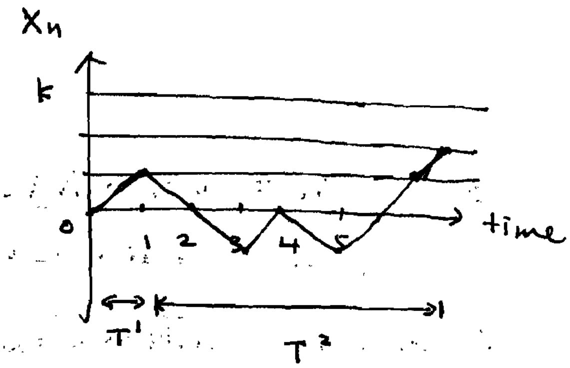
To satisfy $y_1 = By_1 + Ay_2$ (i.e. $y_1 = \frac{1}{2}y_2$), $c_1 = 0 \Rightarrow y_i = c_2 i$.

$\therefore$ There is no non-trivial sol. s.t $0 \leq y_i \leq 1$

$\therefore$ $p > q$: transient

$p \leq q$: recurrent

7. (a) $T_k = T^1 + T^2 + \cdots + T^k$



where T^i is the time it takes to reach state i, starting at the moment that the chain arrives at $i-1$.

But due to the Markov property, T^i has the same distribution as T_1 (the time it takes to go one more state up)

$$\Rightarrow \quad e(k) = \mathbb{E}[T^1] + \mathbb{E}[T^2] + \cdots + \mathbb{E}[T^k]$$

$$= \mathbb{E}[T_1] + \mathbb{E}[T_1] + \cdots + \mathbb{E}[T_1] = k \cdot e(1)$$

(b) $e(1) = \mathbb{E}[T_1] = p_{01} \cdot 1 + p_{0,-1} \, \mathbb{E}[T_1 \mid X_0 = 0, X_1 = -1]$

$$= p + (1-p)(1 + e(2))$$

Using (b), $e(2) = 2 \cdot e(1)$

$$\Rightarrow \quad e(1) = p + (1-p) \cdot (1 + 2 \cdot e(1)) \xRightarrow{\text{Solve for } e(1)} e(1) = \frac{1}{2p-1}$$

Note that $\lim_{p \downarrow \frac{1}{2}} \frac{1}{2p-1} = \infty$.

8. This is easy.