Sol. to HW 6

1. (a) We compute the inv. distri. $\bar{\pi}$ to determine if the chain is positive recurrent.

$$(\pi_0, \pi_1, \pi_2, \cdots) = (\pi_0, \pi_1, \pi_2, \cdots) \begin{bmatrix} 1-p & p & & & \\ C & B & A & & 0 \\ & C & B & A & \\ & & C & B & A \\ & 0 & & & \ddots \end{bmatrix}$$

$A = (1-q)p$

$B = qp + (1-q)(1-p)$

$C = q(1-p)$

$$\Rightarrow \begin{cases} \pi_0 = (1-p)\pi_0 + C\pi_1 & \rightarrow \pi_0 = \frac{q(1-p)}{p}\pi_1 \rightarrow \pi_1 = \frac{q(1-p)}{(1-q)p}\pi_2 \\ \pi_1 = p\pi_0 + B\pi_1 + C\pi_2 \\ \pi_i = A\pi_{i-1} + B\pi_i + C\pi_{i+1}, & i \geq 2 \end{cases}$$

$$\Rightarrow \pi_i = \begin{cases} C_1 + C_2 \cdot \left(\frac{(1-q)p}{q(1-p)}\right)^i, & i = 1,2,3,\cdots \quad \text{if } p \neq q \\ C_1 + C_2 \cdot i & \text{if } p = q \end{cases}$$

The condition $\pi_1 = \frac{q(1-p)}{(1-q)p}\pi_2$ is satisfied only if $C_1 = 0$ when $p \neq q$ and $C_2 = 0$ when $p = q$

$$\Rightarrow \pi_i = \begin{cases} C_2 \left(\frac{(1-q)p}{q(1-p)}\right)^i, & i = 1,2,3,\cdots \quad \text{if } p \neq q \\ C_1, & \text{if } p = q. \end{cases}$$

$\Sigma \pi_i = 1$ can be satisfied only if $\frac{(1-q)p}{q(1-p)} < 1$ i.e. $p < q$

$\therefore$ $p < q \Rightarrow$ positive recurrent

$p \geq q \Rightarrow$ transient (HW5, Problem 6)

$p = q \Rightarrow$ null recurrent

When $p < q$,

$$\begin{cases} \pi_i = \frac{q-p}{q(1-q)} \cdot \left(\frac{(1-q)p}{q(1-p)}\right)^i, & i = 1,2,3,\cdots \\ \pi_0 = \frac{q-p}{q} \end{cases}$$

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \sum_{i=0}^{\infty} i \cdot \pi_i = \frac{\frac{q-p}{q(1-q)} \cdot a}{1-a} \cdot \sum_{i=1}^{\infty} i \cdot a^{i-1}(1-a), \quad a \equiv \frac{(1-q)p}{q(1-p)}$$

expectation of Geom(a)

$$= \frac{\frac{q-p}{q(1-q)} \cdot a}{1-a} \cdot \frac{1}{a} = \frac{1-p}{1-q}.$$

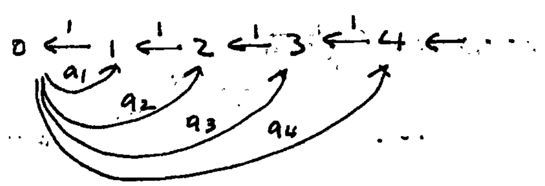
1(b) In HW5, Problem 6, with $k=0$,

$$y_i = c_1\left\{1-\left(\frac{q(1-p)}{(1-q)p}\right)^i\right\}, \quad i=1,2,3,\cdots, \text{ when } p>q.$$

The maximal sol. satisfying $0\le y_i \le 1$ for all i is achieved when $c_1 = 1$.

$$\therefore\ P\{X_n \neq 0, \text{ for all } n\ge 1 \mid X_0 = i\} = y_i = 1-\left(\frac{q(1-p)}{(1-q)p}\right)^i, \quad i=1,2,3,\cdots$$

2.



$$\bar{\pi} = \bar{\pi}\begin{bmatrix} 0 & a_1 & a_2 & a_3 & \cdots \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & & & & \ddots \end{bmatrix}$$

$$\Rightarrow \begin{cases} \pi_0 = \pi_1 \\ \pi_1 = a_1\pi_0 + \pi_2 \\ \pi_2 = a_2\pi_0 + \pi_3 \\ \vdots \end{cases} \Rightarrow \begin{cases} \pi_1 = \pi_0 = (a_1+a_2+a_3+a_4+\cdots)\pi_0 \\ \pi_2 = \pi_1 - a_1\pi_0 = (1-a_1)\pi_0 = (a_2+a_3+a_4+\cdots)\pi_0 \\ \pi_3 = \pi_2 - a_2\pi_0 = (1-a_1-a_2)\pi_0 = (a_3+a_4+\cdots)\pi_0 \\ \pi_4 = (1-a_1-a_2-a_3)\pi_0 = (a_4+\cdots)\pi_0 \\ \vdots \end{cases}$$

$$\sum_{i=0}^{\infty}\pi_i = \pi_0\cdot\left\{1 + \overbrace{(a_1+a_2+a_3+\cdots)}^{\pi_1} + \overbrace{(a_2+a_3+\cdots)}^{\pi_2} + (a_3+\cdots) + \cdots\right\}$$

$$= \pi_0\cdot(1 + 1\cdot a_1 + 2\cdot a_2 + 3\cdot a_3 + 4\cdot a_4 + \cdots)$$

$$= \pi_0\cdot\left(1+\sum_{m=1}^{\infty} m\cdot a_m\right) = \pi_0(1+\mu), \quad \mu \equiv \sum_{m=1}^{\infty} m\cdot a_m$$

(expectation)

$\therefore$ positive recurrent if $\mu<\infty$

null recurrent if $\mu=\infty$

When $\mu<\infty$, $\begin{cases} \pi_i = \frac{1}{\mu+1}\cdot(a_i + a_{i+1} + a_{i+2} + \cdots), & i=1,2,3,\cdots \\ \pi_0 = \frac{1}{\mu+1} \end{cases}$

This means that the bulb lasts at least i units of time.

3. (a) $\mathbb{P}\{X_{n+1}=i+1 \mid X_n=i\} = \mathbb{P}\{T \geq i+1 \mid T \geq i\} = \dfrac{\mathbb{P}\{T\geq i+1\}}{\mathbb{P}\{T\geq i\}}$

$$= \frac{(1-p)^i p + (1-p)^{i+1} p + (1-p)^{i+2} p + \cdots}{(1-p)^{i-1} p + (1-p)^i p + (1-p)^{i+1} p + \cdots} = \frac{(1-p)^i}{(1-p)^{i-1}} = 1-p.$$

$$\mathbb{P}\{X_{n+1}=0 \mid X_n=i\} = \mathbb{P}\{T=i \mid T\geq i\} = \frac{\mathbb{P}\{T=i\}}{\mathbb{P}\{T\geq i\}} = p.$$

Hence
$$\underline{\underline{P}} = \begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & \end{array} \\ \hline \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \end{array} & \begin{bmatrix} p & 1-p & & & \mathcal{O} \\ p & 0 & 1-p & & \\ p & 0 & 0 & 1-p & \\ p & 0 & 0 & 0 & 1-p \\ \vdots & & \mathcal{O} & & \ddots \end{bmatrix} \end{array}$$

(b) This chain is clearly irreducible and aperiodic.

Let's solve the "π-eq's":
$$(\pi_0, \pi_1, \pi_2, \cdots) = (\pi_0, \pi_1, \pi_2, \cdots) \begin{bmatrix} p & 1-p & & \\ p & 0 & 1-p & \\ p & 0 & 0 & 1-p \\ \vdots & & & \ddots \end{bmatrix}$$

$$\Rightarrow \begin{cases} \pi_0 = p(\pi_0+\pi_1+\pi_2+\cdots) \\ \left.\begin{array}{l} \pi_1 = (1-p)\pi_0 \\ \pi_2 = (1-p)\pi_1 \\ \pi_3 = (1-p)\pi_2 \end{array}\right\} \Rightarrow \pi_i = (1-p)^i \pi_0 \\ \quad \vdots \end{cases}$$

As $\sum_{i=0}^{\infty} (1-p)^i = \frac{1}{p}$ converges for all $p\ (\neq 0)$, the chain is positive recurrent for all $p \neq 0$, and $\pi_i = (1-p)^i p$, $i=0,1,2,\cdots$

$$\therefore \mathbb{P}\{X_{1000}=5\} \approx \pi_5 = (1-p)^5 p.$$

(c) $\lim\limits_{n\to\infty} \mathbb{E}[X_n] = \lim\limits_{n\to\infty} \sum_{i=0}^{\infty} i \cdot \mathbb{P}\{X_n=i\} = \sum_{i=0}^{\infty} i\cdot\pi_i = \mathbb{E}[\xi]$

where ξ is a discrete r.v. with mass function π_i, $i=0,1,2,\cdots$

Note that $\eta \equiv \xi+1$ is a geometric r.v. with parameter p.

$$\Rightarrow \mathbb{E}[\xi] = \mathbb{E}[\eta-1] = \frac{1}{p}-1$$

$$\therefore \lim_{n\to\infty} \mathbb{E}[X_n] = \frac{1-p}{p}$$

4. (a)
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix} = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ \ \end{matrix} & \begin{bmatrix} a_1 & a_2 & a_3 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ 0 & a_0 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \end{array} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix}$$

$$\begin{cases} y_1 = a_1 y_1 + a_2 y_2 + a_3 y_3 + \cdots & \text{— (1)} \\ y_i = \underset{\substack{\uparrow \\ \text{right-hand-side}}}{a_0 y_{i-1}} + a_1 y_i + a_2 y_{i+1} + \cdots, \quad i = 2,3,4,\ldots & \text{— (2)} \end{cases}$$

- $\Rightarrow$ RHS of (2) $= a_0(C_1 + C_2\mu^{-i+1}) + a_1(C_1 + C_2\mu^{-i}) + a_2(C_1 + C_2\mu^{-i-1}) + \cdots$

$$= (a_0 + a_1 + a_2 + \cdots)\, C_1 + \left(a_0 + \frac{a_1}{\mu} + \frac{a_2}{\mu^2} + \cdots\right) \cdot \frac{C_2}{\mu^{i-1}}$$

But $a_0 + a_1 + a_2 + \cdots = p + (1-p)p + (1-p)^2 p + \cdots = 1$

$$a_0 + \frac{a_1}{\mu} + \frac{a_2}{\mu^2} + \cdots = p + \frac{1-p}{\mu}\cdot p + \left(\frac{1-p}{\mu}\right)^2 p + \cdots = \frac{1}{1 - \frac{1-p}{\mu}} \cdot p = \frac{p}{1-p} = \frac{1}{\mu}$$

(using $1 + r + r^2 + \cdots = \frac{1}{1-r}$)

$\Rightarrow$ RHS of (2) $= C_1 + \frac{1}{\mu}\cdot\frac{C_2}{\mu^{i-1}} =$ LHS of (2) $\therefore y_i = C_1 + C_2\mu^{-i}$ is a solution

- RHS of (1) $= a_1(C_1 + C_2\mu^{-1}) + a_2(C_1 + C_2\mu^{-2}) + a_3(C_1 + C_2\mu^{-3}) + \cdots$

$= (a_1 + a_2 + a_3 + \cdots)\, C_1 + \left(\frac{a_1}{\mu} + \frac{a_2}{\mu^2} + \frac{a_3}{\mu^3} + \cdots\right) \cdot C_2$

$= (1 - a_0)\, C_1 + \left(\frac{1}{\mu} - a_0\right) C_2$

$= (1-p)\, C_1 + \frac{p^2}{1-p} \cdot C_2$

LHS of (1) $= C_1 + C_2\mu^{-1} = C_1 + \frac{p}{1-p} C_2$

To satisfy (1), we need $C_1 + C_2 = 0$.

$\therefore y_i = C_1(1 - \mu^{-i}), \quad i = 1,2,3,\ldots$

If $\mu > 1$, then $y_i = 1 - \mu^{-i}$ satisfies $0 \le y_i \le 1$, for all i

If $\mu < 1$, then for any C_1, there is i such that $|y_i| > 1$.

i.e. no solution y.

When $\mu = 1$, ~~the~~ it can be shown that the solutions to (2) is

$y_i = C_1 + C_2 i$. To satisfy (1), we obtain $C_1 = 0 \Rightarrow y_i = C_2 i$ $\therefore$ No solutn

Thus the chain is transient $\Leftrightarrow$ $\mu > 1$.

(b) $$(\pi_0, \pi_1, \pi_2, \cdots) = (\pi_0, \pi_1, \pi_2, \cdots)\begin{bmatrix} a_0 & a_1 & a_2 & \text{—} \\ a_0 & a_1 & a_2 & \text{—} \\ 0 & a_0 & a_1 & \text{—} \\ | & | & | & \end{bmatrix}$$

$$\Rightarrow \pi_i = a_i \pi_0 + a_i \pi_1 + a_{i-1}\pi_2 + \cdots + a_1 \pi_i + a_0 \pi_{i+1}, \quad i = 0,1,2,3,\cdots$$

Setting $\pi_i = C\mu^i$,

$$\text{RHS} = C\cdot(a_i + a_i\mu + a_{i-1}\mu^2 + \cdots + a_1\mu^i + a_0\mu^{i+1})$$

$$= C\cdot p\cdot\left((1-p)^i + (1-p)^i\mu + (1-p)^{i-1}\mu^2 + \cdots + (1-p)\cdot\mu^i + \mu^{i+1}\right)$$

$$= C\cdot p\cdot\mu^{i+1}\cdot\left(\frac{(1-p)^i}{\mu^{i+1}} + \left(\frac{1-p}{\mu}\right)^i + \left(\frac{1-p}{\mu}\right)^{i-1} + \cdots + \frac{1-p}{\mu} + 1\right)$$

$$= C\cdot p\cdot\mu^{i+1}\left[\frac{1}{\mu}\cdot\left(\frac{1-p}{\mu}\right)^i + \frac{\left(\frac{1-p}{\mu}\right)^{i+1} - 1}{\frac{1-p}{\mu} - 1}\right] \quad \text{using } 1+r+\cdots+r^N = \frac{1-r^{N+1}}{1-r}$$

$$= C\cdot p\cdot\mu^{i+1}\cdot\left(\frac{p^i}{\mu} + \frac{p^{i+1}-1}{p-1}\right)$$

$$= C\mu^i\cdot\left(p^{i+1} + \frac{(p^{i+1}-1)}{p-1}p\mu\right) = C\mu^i\cdot\left(p^{i+1} - (p^{i+1}-1)\right)$$

$$= C\mu^i = \text{LHS}.$$

$\therefore$ $\pi_i = C\mu^i$ is a solution

If $\mu < 1$, $\pi_i = (1-\mu)\cdot\mu^i$, $i = 0,1,2,\cdots$ is the solution

If $\mu \geq 1$, no solution since $\sum_{i=0}^{\infty}\mu^i = \infty$.

$\therefore$ positive recurrent $\Leftrightarrow$ $\mu < 1$.

In this case, $$\lim_{n\to\infty}\mathbb{E}[X_n] = \lim_{n\to\infty}\sum_{i=0}^{\infty} i\cdot\mathbb{P}\{X_n = i\}$$

$$= \lim_{n\to\infty}\sum_{i=0}^{\infty} i\cdot\mu^i(1-\mu) = \frac{1}{1-\mu} - 1 = \frac{1-p}{2p-1}.$$

5. $(\pi_0, \pi_1, \pi_2, \cdots) = (\pi_0, \pi_1, \pi_2, \cdots) \begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$

Let's guess that

$\bar{\pi} = (a_0, a_1, a_2, \cdots)$

Then $\quad \text{RHS} = (a_0, a_1, a_2, \cdots) \begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$

$$= \left((a_0 + a_1 + a_2 + \cdots) a_0,\ (a_0 + a_1 + a_2 + \cdots) a_1,\ (a_0 + a_1 + a_2 + \cdots) a_2, \cdots\right)$$

$$= (a_0, a_1, a_2, \cdots) = \text{LHS}$$

as $a_0 + a_1 + a_2 + \cdots = 1$

Also, $\sum \pi(i) = 1$.

Thus $\bar{\pi} = (a_0, a_1, a_2, \cdots)$ is the inv. distribution.

The chain is positive recurrent for all a_i's s.t $a_i > 0$, $\sum_0^\infty a_i = 1$.