Sol. to HW 8

1. If the current population is i $(i \neq 0)$, the probability that i jumps to $i+1$ is $\dfrac{i\lambda}{i\lambda + i\mu} = \dfrac{\lambda}{\lambda+\mu}$

and i jumps to $i-1$ is $\dfrac{i\mu}{i\lambda+i\mu} = \dfrac{\mu}{\lambda+\mu}$, when it jumps.

Note that these probabilities are ~~independent of i~~ the same for all i.

Let $Z \equiv$ the population at the time of first death.

$$\mathbb{P}\{Z = k\} = \mathbb{P}\{k \text{ births before the first death}\}$$
$$= \left(\frac{\lambda}{\lambda+\mu}\right)^k \cdot \frac{\mu}{\lambda+\mu}, \quad k = 0,1,2,\cdots$$

2. The invariant distribution is $\pi_i = (1-\rho)\rho^i$, $i = 0,1,2,\cdots$

where $\rho = \dfrac{\lambda}{\mu}$. $\Rightarrow \lim_{t\to\infty} \mathbb{P}\{X_t = i\} = (1-\rho)\rho^i$, $i = 0,1,2,\cdots$

Hence $\mathbb{P}\{\text{immediately served}\} \approx \lim_{t\to\infty} \mathbb{P}\{X_t = 0\} = 1-\rho$.

If there are k customers in the post office when you entered, then ~~I~~ you spend, on average, $\dfrac{(k+1)}{\mu}$ time in the post office.

(Here $\dfrac{1}{\mu}$ is the expected service time for each ~~per~~ customer, and there are $k+1$ customers including yourself.)

Hence $\mathbb{E}[\text{time spent in the post office}]$

$$= \sum_{k=0}^{\infty} \mathbb{E}[\text{time spent} \mid \text{already } k \text{ customers}] \cdot \mathbb{P}\{\text{already } k \text{ customers}\}$$

$$\approx \sum_{k=0}^{\infty} \frac{k+1}{\mu} \cdot \mathbb{P}\{X_t = k\} = \frac{1}{\mu} \cdot \sum_{k=0}^{\infty} (k+1) \cdot (1-\rho)^k \rho$$

$$= \frac{1}{\mu} \cdot \sum_{k=1}^{\infty} k \cdot (1-\rho)^{k-1} \rho = \frac{1}{\mu\rho}$$

$\mathbb{E}$ of ~~a~~ geometric r.v.

3. The forward equations for $p_i(t) \equiv \mathbb{P}\{X_t = i\}$ are

$$(p_0', p_1', p_2', p_3', \cdots) = (p_0, p_1, p_2, p_3, \cdots) \begin{bmatrix} -\nu & \nu & & \\ \mu & -(\lambda+\nu+\mu) & \lambda+\nu & 0 \\ & 2\mu & -(2\lambda+\nu+2\mu) & 2\lambda+\nu \\ 0 & & 3\mu & \ddots \end{bmatrix}$$

$$\Rightarrow \begin{cases} p_0'(t) = -\nu p_0(t) + \mu \cdot p_1(t) \\ p_i'(t) = ((i-1)\lambda+\nu) p_{i-1} - (i\lambda + \nu + i\mu) p_i + (i+1)\mu\, p_{i+1}, \quad i = 1, 2, 3, \cdots \end{cases}$$

$$M(t) = \mathbb{E}[X_t] = \sum_{i=0}^{\infty} i \cdot p_i(t) = \sum_{i=1}^{\infty} i \cdot p_i(t)$$

$$M'(t) = \sum_{i=1}^{\infty} i \cdot p_i'(t) = \sum_{i=1}^{\infty} \left\{ i\,((i-1)\lambda+\nu) p_{i-1} - i\,(i(\lambda+\mu)+\nu) p_i + i(i+1)\mu\, p_{i+1} \right\}$$

$$= \sum_{j=0}^{\infty} (j+1)(j\lambda+\nu) p_j - \sum_{j=1}^{\infty} j\,(j(\lambda+\mu)+\nu) p_j + \sum_{j=2}^{\infty} (j-1) j \cdot \mu\, p_j$$

$$= \nu p_0 + (\lambda - \mu + \nu) p_1 + \sum_{j=2}^{\infty} (j(\lambda-\mu)+\nu) p_j$$

$$= \left(\sum_{j=1}^{\infty} j \cdot p_j \right) \cdot (\lambda - \mu) + \left(\sum_{j=0}^{\infty} p_j \right) \cdot \nu$$

$$= (\lambda-\mu) \cdot M(t) + \nu. \qquad \left(\text{as } \sum_{j=0}^{\infty} p_j = \sum_{j=0}^{\infty} \mathbb{P}\{X_t = j\} = 1 \right)$$

It is direct to check that

$$M(t) = \begin{cases} \frac{\nu}{\lambda-\mu}\left(e^{(\lambda-\mu)t} - 1\right) + J e^{(\lambda-\mu)t}, & \lambda \neq \mu \\ \nu t + J, & \lambda = \mu \end{cases}$$

solves the equation $M' = \nu + (\lambda-\mu) M$,

~~and the~~ and the initial condition $M(0) = \mathbb{E}[X_0] = J$.

4. (a) $G(s) = \sum_{k=0}^{\infty} s^k \cdot \mathbb{P}\{X=k\}$

- $G'(s) = \sum_{k=1}^{\infty} k \cdot s^{k-1} \cdot \mathbb{P}\{X=k\} \Rightarrow G'(1) = \mathbb{E}[X]$.

  $\uparrow$ $k=0$ term is a constant in s. Hence $\frac{d}{ds}$ is zero

- As $s \downarrow 0$, $s^k \to 0$ for $k = 1, 2, 3, \dots$

  $\Rightarrow \lim_{s \downarrow 0} G(s) = \mathbb{P}\{X=0\}$.

- $\lim_{s \downarrow 0} G'(s) = \mathbb{P}\{X=1\}$ $\leftarrow$ only the term for $k=1$ does not vanish.

- $G''(s) = \sum_{k=2}^{\infty} k(k-1) \cdot s^{k-1} \cdot \mathbb{P}\{X=k\} \Rightarrow G''(1) = \mathbb{E}[X(X-1)]$

  $\Rightarrow \mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = G''(1) + G'(1) - (G'(1))^2$

(b) • Poisson : $G(s) = \sum_{k=0}^{\infty} s^k \cdot \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{(s\lambda)^k}{k!} \underset{\substack{\uparrow \\ \text{exponential series}}}{=} e^{-\lambda} \cdot e^{s\lambda} = e^{\lambda(s-1)}$.

• Geometric : $G(s) = \sum_{k=1}^{\infty} s^k \cdot (1-p)^{k-1} p = sp \cdot \sum_{k=1}^{\infty} (s(1-p))^{k-1} \underset{\substack{\uparrow \\ \text{geometric series}}}{=} \frac{sp}{1 - s(1-p)}$

(for $|s| < \frac{1}{1-p}$)

• binomial : $G(s) = \sum_{k=0}^{n} s^k \cdot \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \cdot (sp)^k (1-p)^{n-k}$

$= (sp + (1-p))^n$

$\uparrow$ binomimial identity (theorem)

5. (a) It is straightforward to check that A is the generator. The differential equations follow from $\bar{p}' = \bar{p} A$ (forward eq.)

where $\bar{p} = (p_0(t), p_1(t), p_2(t), \dots)$

(b) $\frac{\partial G}{\partial s} = \sum_{j=1}^{\infty} j s^{j-1} p_j(t)$

$\frac{\partial G}{\partial t} = \sum_{j=0}^{\infty} s^j \cdot p_j'(t) = p_0'(t) + \sum_{j=1}^{\infty} s^j \cdot p_j'(t)$

$\overset{(a)}{=} \delta \cdot \sum_{j=1}^{\infty} p_j(t) + \sum_{j=1}^{\infty} s^j \cdot [(j-1)\lambda p_{j-1} - (\delta + j\lambda) p_j]$

5. (a) This is straightforward.

(b) $\bar{p}' = \bar{p} A$ implies that $\begin{cases} p_0'(t) = \delta\,(p_1(t) + p_2(t) + p_3(t) + \cdots) \\ p_i'(t) = (i-1)\lambda\, p_{i-1}(t) - (\delta + i\lambda)\, p_i(t), \quad i = 1,2,3,\ldots \end{cases}$

where $p_i(t) = \mathbb{P}\{X_t = i\}$.

$$M'(t) = \sum_{i=1}^{\infty} i \cdot p_i'(t) = \sum_{i=1}^{\infty} i\,\{(i-1)\lambda\, p_{i-1}(t) - (\delta + i\lambda)\, p_i(t)\}$$

$$= \sum_{j=0}^{\infty} (j+1)\, j\, \lambda\, p_j(t) - \delta \cdot \sum_{j=1}^{\infty} j\, p_j(t) - \sum_{j=1}^{\infty} j^2 \cdot \lambda \cdot p_j(t)$$

this term is 0 when $j=0$. Hence we can replace it by $\sum_{j=1}^{\infty}$

$$= \sum_{j=1}^{\infty} \left((j+1)j - j^2\right) \lambda\, p_j(t) - \delta \cdot \sum_{j=1}^{\infty} j \cdot p_j(t)$$

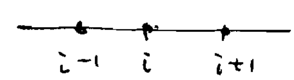
$$= \sum_{j=1}^{\infty} \lambda \cdot j\, p_j(t) - \delta \cdot \sum_{j=1}^{\infty} j \cdot p_j(t) = (\lambda - \delta)\, M(t).$$

(c) $M(t)$ is the solution to $\begin{cases} M' = (\lambda - \delta) M \\ M(0) = J \end{cases}$

$\Rightarrow \quad M(t) = J\, e^{(\lambda-\delta)t}$.

Hence $$\lim_{t\to\infty} \mathbb{E}[X_t] = \begin{cases} \infty & \text{if } \lambda > \delta \\ J & \text{if } \lambda = \delta \\ 0 & \text{if } \lambda < \delta \end{cases}.$$

6. (a) The waiting time of departure from state i is an exponential r.v. with parameter $\lambda_i + \mu_i$.



$\Rightarrow$ It takes $\frac{1}{\lambda_i + \mu_i}$, on average, to jump out of i.

When it jumps, it goes to $i-1$ with prob. $\frac{\mu_i}{\lambda_i + \mu_i}$.

and it goes to $i+1$ with prob. $\frac{\lambda_i}{\lambda_i + \mu_i}$

Conditioned on the first case, E_i is $\frac{1}{\lambda_i + \mu_i}$.

Conditioned on the second case, E_i is $\frac{1}{\lambda_i + \mu_i} + E_{i+1} + E_i$

$i+2 \to i+1 \qquad i+1 \to i$.

Hence $$E_i = \frac{\mu_i}{\lambda_i + \mu_i} \cdot \frac{1}{\lambda_i + \mu_i} + \frac{\lambda_i}{\lambda_i + \mu_i}\left(\frac{1}{\lambda_i + \mu_i} + E_{i+1} + E_i\right).$$

Solving for E_i, we find $E_i = \frac{\lambda_i}{\mu_i} E_{i+1} + \frac{1}{\mu_i}$.

(b) $\lambda_i = \lambda$, $\mu_i = \mu$. ~~Hence $E_i = \frac{\lambda_i}{\mu_i} E_{i+1} + \frac{1}{\mu_i}$~~

Hence $E_i = \frac{\lambda}{\mu} E_{i+1} + \frac{1}{\mu}$, $i = 1, 2, 3, \ldots$

$$\Rightarrow E_1 = \frac{\lambda}{\mu} E_2 + \frac{1}{\mu} = \frac{\lambda}{\mu}\left(\frac{\lambda}{\mu} E_3 + \frac{1}{\mu}\right) + \frac{1}{\mu} = \left(\frac{\lambda}{\mu}\right)^2 E_3 + \left(\frac{\lambda}{\mu} + 1\right)\frac{1}{\mu}$$

$$= \left(\frac{\lambda}{\mu}\right)^2 \left(\frac{\lambda}{\mu} E_4 + \frac{1}{\mu}\right) + \left(\frac{\lambda}{\mu} + 1\right)\frac{1}{\mu} = \left(\frac{\lambda}{\mu}\right)^3 E_4 + \left(\left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right) + 1\right)\frac{1}{\mu}$$

$$= \cdots$$

$$= \rho^i E_{i+1} + \left(\rho^{i-1} + \rho^{i-2} + \cdots + \rho + 1\right)\frac{1}{\mu}, \quad \rho \equiv \frac{\lambda}{\mu}$$

$$= \rho^i E_{i+1} + \frac{1-\rho^i}{1-\rho} \cdot \frac{1}{\mu}. \quad \text{for all } i = 1, 2, 3, \cdots$$

Take the limit $i \to \infty \Rightarrow E_1 = \frac{1}{1-\rho} \cdot \frac{1}{\mu} = \frac{1}{\mu - \lambda}$.

Here we assumed that $\lim_{i \to \infty} E_{i+1}$ is bounded (so that $\rho^i E_{i+1} \to 0$).

Think about why this is reasonable.

(Alternatively, note that $E_i = E_{i+1}$ since $\lambda_i = \lambda$, $\mu_i = \mu$ for all i.

Hence $E_i = \frac{\lambda}{\mu} E_{i+1} + \frac{1}{\mu}$ implies that $E_i = \frac{\lambda}{\mu} E_i + \frac{1}{\mu} \Rightarrow E_i = \frac{1}{\mu - \lambda}$ for all i)

(c) $E_i = \frac{\lambda}{\mu} E_{i+1} + \frac{1}{i\mu}$. $i = 1, 2, 3, \cdots$

$$E_1 = \frac{\lambda}{\mu} E_2 + \frac{1}{\mu} = \frac{\lambda}{\mu}\left(\frac{\lambda}{\mu} E_3 + \frac{1}{2\mu}\right) + \frac{1}{\mu} = \cdots$$

$$= \left(\frac{\lambda}{\mu}\right)^i E_{i+1} + \left(\frac{1}{i}\left(\frac{\lambda}{\mu}\right)^{i-1} + \frac{1}{i-1}\cdot\left(\frac{\lambda}{\mu}\right)^{i-2} + \cdots + \frac{\lambda}{2\mu} + 1\right)\cdot\frac{1}{\mu}$$

$$= \rho^i \cdot E_{i+1} + \left(\frac{1}{i}\rho^{i-1} + \frac{1}{i-1}\cdot\rho^{i-2} + \cdots + \frac{1}{2}\rho + 1\right)\cdot\frac{1}{\mu}$$

$$= \rho^i E_{i+1} + \left(\frac{\rho^i}{i} + \frac{\rho^{i-1}}{i-1} + \cdots + \frac{\rho^2}{2} + \rho\right)\cdot\frac{1}{\mu\rho}$$

Take $i \to \infty$.

$$\Rightarrow E_1 = \left(\rho + \frac{\rho^2}{2} + \frac{\rho^3}{3} + \cdots\right)\cdot\frac{1}{\lambda} = -\frac{1}{\lambda}\cdot\log(1-\rho) = \frac{1}{\lambda}\cdot\log\left(\frac{\mu}{\mu-\lambda}\right).$$