Sol. to HW 9

#1. (a) From $\bar{p}'(t) = \bar{p}(t)A$,

$$p_0'(t) = \delta(p_1 + p_2 + p_3 + \cdots)$$

$$p_i'(t) = (i-1)\lambda p_{i-1} - (\delta + i\lambda)p_i, \quad i = 1,2,3,\cdots$$

$$A = \begin{bmatrix} 0 & 0 & & & & \\ \delta & -(\delta+\lambda) & \lambda & & 0 & \\ \delta & 0 & -(\delta+2\lambda) & 2\lambda & & \\ \delta & 0 & 0 & -(\delta+3\lambda) & 3\lambda & \\ \vdots & & 0 & & \ddots & \ddots \end{bmatrix}$$

$$\Rightarrow \frac{\partial G}{\partial t} = \sum_{j=0}^{\infty} s^j \cdot p_j'(t) = p_0' + \sum_{j=1}^{\infty} s^j \cdot p_j'$$

$$= \delta \cdot \sum_{j=1}^{\infty} p_j + \sum_{j=1}^{\infty} s^j \cdot \left((j-1)\lambda p_{j-1} - (\delta + j\lambda)p_j\right)$$

$$= \delta(1-p_0) + \lambda \cdot (s^2 p_1 + 2s^3 p_2 + 3s^4 p_3 + \cdots)$$

$$- \delta \cdot (G - p_0) - \lambda \cdot (s p_1 + 2s^2 p_2 + 3s^3 p_3 + \cdots)$$

$$= \delta(1-G) + \lambda(s-1) \cdot (s p_1 + 2s^2 p_2 + 3s^3 p_3 + \cdots)$$

$$= \delta(1-G) + \lambda(s-1) \cdot s \cdot \frac{\partial G}{\partial s}.$$

(b) This is straightforward

(c) $\mathbb{P}\{X_t = 0\} = G(0,t) = 1 - e^{-\delta t}$

$\mathbb{E}[X_t] = \frac{\partial}{\partial s} G(1,t) = J \cdot e^{(\lambda-\delta)t}$

(d) $\mathbb{P}\{\tilde{X}_{t+\Delta t} = 0 \mid \tilde{X}_t = 1\} = \mathbb{P}\{$a disaster occurs in time interval $(t, t+\Delta t]\}$

$= \delta \Delta t + o(\Delta t)$.

Other entries are similar.

(e) Forward eq's : Set $\tilde{p}_i(t) = \mathbb{P}\{\tilde{X}_t = i\}$, $i = 0,1$

$\tilde{p}_0' = \delta \tilde{p}_1$

$\tilde{p}_1' = -\delta \tilde{p}_0$

i.e. $(\tilde{p}_0, \tilde{p}_1)' = (\tilde{p}_0, \tilde{p}_1)\begin{pmatrix} 0 & 0 \\ \delta & -\delta \end{pmatrix}$

~~Note the soln is~~

The sol. is (see Lemma 3.6 p. 23)

$$(\tilde{p}_0(t), \tilde{p}_1(t)) = (0,1)\, e^{t\begin{pmatrix} 0 & 0 \\ \delta & -\delta \end{pmatrix}}.$$

Here $\tilde{p}_0(0) = 0$, $\tilde{p}_1(0) = 1$ since at time 0, the population is non-zero.

Now $\begin{pmatrix} 0 & 0 \\ \delta & -\delta \end{pmatrix} = Q \Lambda Q^{-1}$, $\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & -\delta \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $Q^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

same as in (c)

$$\Rightarrow \quad (\tilde{p}_0(t),\ \hat{p}_1(t)) = (0,1)\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-\delta t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = (1-e^{-\delta t},\ e^{-\delta t})$$

2. $p_i(t)$ $\mathbb{P}\{X_t = i\}$ satisfies the forward eq. $\bar{p}'(t) = p(t)\begin{bmatrix} 0 & 0 & & & \\ \mu & -(\lambda+\mu) & \lambda & & 0 \\ & 2\mu & -2(\lambda+\mu) & 2\lambda & \\ & & 3\mu & \ddots & \ddots \\ 0 & & & \ddots & \end{bmatrix}$

$$p_0'(t) = \mu\, p_1(t)$$

$$p_i'(t) = (i-1)\lambda\, p_{i-1}(t) - i(\lambda+\mu)p_i(t) + (i+1)\mu\, p_{i+1}(t)$$

$, \quad i = 1, 2, 3, \cdots$

(a) Set $M(t) \equiv \mathbb{E}[X_t] = \sum_{i=1}^{\infty} i \cdot p_i(t)$.

$$M'(t) = \sum_{i=1}^{\infty} i\cdot p_i' = \sum_{i=1}^{\infty} i\cdot\left[(i-1)\lambda p_{i-1} - i(\lambda+\mu)p_i + (i+1)\mu\, p_{i+1}\right]$$

$$= \lambda\cdot\sum_{j=0}^{\infty}(j+1)j\,p_j - \lambda\cdot\sum_{j=1}^{\infty} j^2 p_j - \mu\cdot\sum_{j=1}^{\infty} j^2 p_j + \mu\cdot\sum_{j=2}^{\infty}(j-1)j\,p_j$$

$$= \lambda\cdot\sum_{j=1}^{\infty} j\cdot p_j - \mu\cdot\sum_{j=1}^{\infty} j\cdot p_j = (\lambda-\mu)M(t)$$

Solving $\begin{cases} M' = (\lambda-\mu)M \\ M(0) = J \end{cases}$, we obtain $M(t) = J\cdot e^{(\lambda-\mu)t}$.

(b) $G(s,t) = \sum_{i=0}^{\infty} s^i \cdot p_i(t)$.

$$\frac{\partial G}{\partial t} = \sum_{i=0}^{\infty} s^i\cdot p_i' = p_0' + \sum_{i=1}^{\infty} s^i\cdot p_i' = \mu\cdot p_1 + \sum_{i=1}^{\infty} s^i\left[(i-1)\lambda p_{i-1} - i(\lambda+\mu)p_i + (i+1)\mu p_{i+1}\right]$$

$$= \lambda\left(\sum_{j=0}^{\infty} j\cdot s^{j+1} p_j - \sum_{j=1}^{\infty} j s^j p_j\right) + \mu\cdot\left(p_1 - \sum_{j=1}^{\infty} j s^j p_j + \sum_{j=2}^{\infty} j\cdot s^{j-1} p_j\right)$$

$$= \lambda(s-1)\cdot\sum_{j=1}^{\infty} j s^j p_j - \mu(s-1)\cdot\sum_{j=1}^{\infty} j s^{j-1} p_j$$

But $\frac{\partial G}{\partial s} = \sum_{i=1}^{\infty} i s^{i-1} p_i$.

Hence $\frac{\partial G}{\partial t} = \lambda(s-1)s\frac{\partial G}{\partial s} - \mu(s-1)\frac{\partial G}{\partial s}$.

Thus $G(s,t)$ solves

$$(*)\begin{cases} \frac{\partial G}{\partial t} = (s-1)(\lambda s-\mu)\frac{\partial G}{\partial s} \\ G(s,0) = s^J \end{cases}.$$

Now let's check that the function $G(s,t)$ given in the problem,

$$G(s,t) = \begin{cases} \left(\dfrac{\mu(1-s) - (\mu - \lambda s)\, e^{-(\lambda-\mu)t}}{\lambda(1-s) - (\mu-\lambda s)\, e^{-(\lambda-\mu)t}} \right)^J, & \lambda \neq \mu \\[2ex] \left(\dfrac{\lambda(1-s)t + s}{\lambda(1-s)t + 1} \right)^J, & \lambda = \mu \end{cases}$$

solves (*) above. This is a direct computation.

(c)

$$q = \lim_{t\to\infty} \mathbb{P}\{X_t = 0\} = \lim_{t\to\infty} G(0,t) = \begin{cases} \lim_{t\to\infty} \left(\dfrac{\mu - \mu e^{-(\lambda-\mu)t}}{\lambda - \mu e^{-(\lambda-\mu)t}} \right)^J, & \lambda \neq \mu. \\[2ex] \lim_{t\to\infty} \left(\dfrac{\lambda t}{\lambda t + 1} \right)^J, & \lambda = \mu \end{cases}$$

$$= \begin{cases} \left(\dfrac{\mu}{\lambda}\right)^J, & \lambda > \mu \\ 1, & \lambda < \mu \\ 1, & \lambda = \mu. \end{cases}$$

3.(1) From $\mathrm{Var}(Y) = \phi''(1) + \phi'(1) - (\phi'(1))^2$ and $\mathbb{E}[Y] = \phi'(1)$,

we have $\sigma^2 = \phi''(1) + \mu - \mu^2$

Setting $V_n \equiv \mathrm{Var}(X_n)$,

$V_{n+1} = \phi''_{n+1}(1) + \phi'_{n+1}(1) - (\phi'_{n+1}(1))^2$.

From $\phi_{n+1}(s) = \phi(\phi_n(s))$, $\phi'_{n+1}(s) = \phi'(\phi_n(s)) \cdot \phi'_n(s)$, and

$$\phi''_{n+1}(s) = \phi''(\phi_n(s)) \cdot (\phi'_n(s))^2 + \phi'(\phi_n(s)) \cdot \phi''_n(s).$$

Using $\phi_n(1) = 1$ (true for all generating ftn's),

$\phi'_n(1) = \mathbb{E}[X_n] = \mu^n\, \mathbb{E}[X_0] = \mu^n$ (from class),

we have

$$\phi''_{n+1}(1) = \phi''(1) \cdot (\mu^n)^2 + \phi'(1) \cdot \phi''_n(1)$$

$$= (\sigma^2 - \mu + \mu^2) \cdot \mu^{2n} + \underbrace{\phi'(1)}_{=\,\mu} \cdot \phi''_n(1)$$

Plug this into $V_{n+1} = \phi''_{n+1}(1) + \phi'_{n+1}(1) - (\phi'_{n+1}(1))^2$;

$$V_{n+1} = \left[(\sigma^2 - \mu + \mu^2)\mu^{2n} + \mu\cdot\phi''_n(1)\right] + \mu^{n+1} - \mu^{2(n+1)}$$

$$= \mu\cdot\phi''_n(1) + \sigma^2\mu^{2n} - \mu^{2n+1} + \mu^{n+1} .$$

But $V_n = \phi''_n(1) + \phi'_n(1) - (\phi'_n(1))^2 = \phi''_n(1) + \mu^n - \mu^{2n}$.

Hence

$$V_{n+1} = \mu\cdot(V_n - \mu^n + \mu^{2n}) + \sigma^2\cdot\mu^{2n} - \mu^{2n+1} + \mu^{n+1}$$

$$= \mu\cdot V_n + \sigma^2\cdot\mu^{2n} .$$

(2) $V_n = \mu\cdot V_{n-1} + \sigma^2\cdot\mu^{2n-2} = \mu\cdot(\mu V_{n-2} + \sigma^2\cdot\mu^{2n-4}) + \sigma^2\cdot\mu^{2n-2}$

$$= \mu^2 V_{n-2} + \sigma^2(\mu^{2n-3} + \mu^{2n-2}) = \mu^2\cdot(\mu V_{n-3} + \sigma^2\cdot\mu^{2n-6}) + \sigma^2(\mu^{2n-3} + \mu^{2n-2})$$

$$= \mu^3\cdot V_{n-3} + \sigma^2(\mu^{2n-4} + \mu^{2n-3} + \mu^{2n-2})$$

$$= \cdots$$

$$= \mu^{n-1}\cdot V_1 + \sigma^2(\mu^n + \mu^{n+1} + \cdots + \mu^{2n-3} + \mu^{2n-2})$$

Note that as X_1 is distributed as Y, $V_1 = Var(X_1) = \sigma^2$.

$$\Rightarrow V_n = \sigma^2\{\mu^{n-1} + \mu^n + \mu^{n+1} + \cdots + \mu^{2n-2}\}$$

$$= \sigma^2\mu^{n-1}\{1 + \mu + \mu^2 + \cdots + \mu^{n-1}\}$$

If $\mu \neq 1$, $V_n = \sigma^2\mu^{n-1}\cdot\dfrac{1-\mu^n}{1-\mu}$.

If $\mu = 1$, $V_n = n\sigma^2$.

(3) $\lim_{n\to\infty} Var(X_n) \underset{\text{using (2)}}{=} \begin{cases} \infty, & \mu \geq 1 \\ 0, & \mu < 1 . \end{cases}$

4. Set $q_n \equiv \mathbb{P}\{X_n = 0\}$. In the proof of Thm 1 in the class, we obtained that $q_n = \phi(q_{n-1})$.

Here $\phi(s) = \mathbb{E}[s^Y]$. In this case,

$$\phi(s) = \sum_{k=0}^{\infty} s^k \cdot a_k = \sum_{k=0}^{\infty} (s(1-p))^k \cdot p = \frac{p}{1-s(1-p)}.$$

Hence $$q_n = \frac{p}{1-(1-p)q_{n-1}}. \qquad n=2,3,4,\dots$$

Observe that $\{X_n = 0\} \supset \{X_{n-1} = 0\}$ since if the population is zero at the $(n-1)^{th}$ generation, then so is at the n^{th} generation.

Thus $\mathbb{P}\{T=n\} = \mathbb{P}\{X_n = 0, X_{n-1} \neq 0\} = q_n - q_{n-1}$. $, n=2,3,\dots$

Note that $q_1 = \mathbb{P}\{X_1 = 0\} = a_0 = p = \mathbb{P}\{T=1\}$

$$\mathbb{P}\{T=1\} = p \qquad (=q_1)$$

$$\mathbb{P}\{T=2\} = q_2 - q_1 = \frac{p}{1-(1-p)q_1} - q_1 = \frac{p}{1-(1-p)p} - p = \frac{(1-p)p^2}{1-(1-p)p}$$

$$\mathbb{P}\{T=3\} = q_3 - q_2 = \frac{p}{1-(1-p)q_2} - q_2 = \frac{p}{1-\frac{(1-p)^2p^2}{1-(1-p)p}} - \frac{(1-p)p^2}{1-(1-p)p}$$

5. Let $\hat{Y}$ be the # of population after 1 generation from one individual.

$\hat{a}_k \equiv \mathbb{P}\{\hat{Y} = k\} = \mathbb{P}\{Y=k, \text{the parent dies}\} + \mathbb{P}\{Y=k-1, \text{the parent does not die}\}$

$= a_k \cdot r + a_{k-1}\cdot(1-r)$, $k=1,2,\dots$

$$\Rightarrow \quad \hat{a}_k = \begin{cases} a_k r + a_{k-1}\cdot(1-r), & k=1,2,3,\dots \\ a_0 \cdot r & , k=0 \end{cases}$$

The new process is again a branching process with $\hat{Y}$ as the distribution of the # of "offsprings" from an individual.

The extinction occurs with prob. 1

$\Leftrightarrow \quad \hat{\mu} \le 1$.

But
$$\hat{\mu} = \mathbb{E}[\hat{Y}] = \sum_{k=0}^{\infty} k \cdot \hat{a}_k = \sum_{k=1}^{\infty} k \cdot (a_k r + a_{k-1}(1-r))$$
$$= \left(\sum_{k=1}^{\infty} k a_k\right) r + \left(\sum_{k=1}^{\infty} k a_{k-1}\right)(1-r)$$
$$= \mu r + \left(\sum_{i=0}^{\infty} (i+1) a_i\right) \cdot (1-r)$$
$$= \mu r + (\mu+1) \cdot (1-r) = \mu + 1 - r.$$

Hence $\hat{\mu} \le 1 \Leftrightarrow \mu \le r$.

$\Updownarrow$

extinction occurs with prob. 1.