Population Models. Stability

Logistic Equation

\[
\frac{dP}{dt} = k(M-P)P, \quad t>0, \quad M>0
\]

\[
\frac{dP}{(M-P)P} = kdt
\]

\[
\frac{1}{(M-P)P} = \frac{1}{M} \left( \frac{1}{M-P} + \frac{1}{P} \right): \text{partial fraction decomposition}
\]

(see previous lecture)

\[
\int \frac{dP}{(M-P)P} = \int kdt
\]

\[
\frac{1}{M} \int \left( \frac{1}{M-P} + \frac{1}{P} \right) dP = \int k dt
\]

\[
\frac{1}{M} \left( - \ln |M-P| + \ln |P| \right) = kt + \bar{C}
\]

\[
\frac{1}{M} \ln \left| \frac{P}{M-P} \right| = kt + \bar{C}
\]

\[
\ln (a \cdot b) = \ln a + \ln b
\]

\[
\ln \frac{a}{b} = \ln a - \ln b
\]
\[ \frac{P}{M-P} = e^{-uM} + c \quad \exp \]

\[ \frac{P}{M-P} = e^{-uM+c} \]

\[ \frac{P}{M-P} = C e^{-uM} \quad (M-P) \quad (1) \]

\[ P = C(M-P)e^{-uM} \]

\[ P = C Me^{-uM} - C Pe^{-uM} \Rightarrow P + C Pe^{-uM} = CM e^{-uM} \]

\[ P(1 + Ce^{-uM}) = CM e^{-uM} \]

Solve for \( P \)

\[ \Rightarrow P(t) = \frac{CM e^{-uM}}{1 + Ce^{-uM}} = \frac{M}{1 + \frac{1}{C} e^{-uM}} \quad \lambda, M > 0 \]

\[ \lim_{t \to \infty} P(t) = \lim_{t \to \infty} \frac{M}{1 + \frac{1}{C} e^{-uM}} = M: \]

\[ \frac{CM e^{-uM}}{1 + Ce^{-uM}} = \frac{M}{\frac{1}{C} e^{-uM} + 1} \]

\[ P = M \text{ is stable equil. solution} \]
To find $C$, we use (1):

$$\frac{P}{M-P} = Ce^{-\lambda t}$$

$P(0) = P_0 \Rightarrow \frac{P_0}{M-P_0} = C$

$$P(t) = \frac{M}{1 + \frac{M-P_0}{P_0}e^{-\lambda t}} = \frac{MP_0}{P_0 + (M-P_0)e^{-\lambda t}} = P(t)$$

**Note**

$P_0 = 0 \Rightarrow P(t) = 0$ for all $t$ \quad \{ p=0 \text{ and } p=M \text{ are two equil. solutions} \}

$P_0 = M \Rightarrow P(t) = M$ for all $t$ \quad \{ p=0 \text{ and } p=M \text{ are two equil. solutions} \}

Let $0 < P_0 < M$ or $P_0 > M \Rightarrow \lim_{t \to \infty} P(t) = M$

$\Rightarrow P(t) = 0$ is unstable equil. solution

$P(t) = M$ is stable
This confirms what we already knew from stability analysis.

![Solution curves](image)

Q: What happens when \( P_0 < 0 \)?

\[
P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-\lambda Mt}}
\]

at some finite time \( t^* \), \( P_0 + (M-P_0)e^{-\lambda M t^*} = 0 \)

\[
\Rightarrow \lim_{t \to t^*} P(t) = -\infty
\]
Doomsday / Extinction Model

\[ \frac{dP}{dt} = kP(P-M), \quad k, M > 0 \]

There are two equil. solutions: \( P = 0, \ P = M \)

\( P^* \)

\[ \text{Solution: } P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}} \]

if \( 0 < P_0 < M \), \( \lim_{t \to \infty} P(t) = 0 : \text{extinction} \)

if \( P_0 > M \) \( \Rightarrow \) there is some finite \( + \) at which

denominator \( \frac{P_0 + (M-P_0)e^{-kMt}}{P_0 + (M-P_0)e^{-kMt}} = 0 \) and \( \lim_{t \to +} P(t) = + \infty : \text{doomsday} \)
Logistic Equation with Harvesting

\[ x = x(t) \]

\[ \frac{dx}{dt} = ax - bx^2 - h \]

logistic equation w/ harvesting

where \( a, b, h > 0 \)

\[ \frac{dx}{dt} = bx(M-x) - h \]

RHS does not depend on \( t \) explicitly

\( \Rightarrow \) this is an autonomous DE

Def: A differential equation \( \frac{dx}{dt} = f(x) \) is called autonomous.

\( \Rightarrow \) does not depend on \( t \) explicitly
Consider

\[ \frac{dx}{dt} = k x (M-x) - h \]

\[ \frac{dx}{dt} = -k(x-H)(x-N) \]

if \( (kM)^2 - 4kh > 0 \)

or \( M^2 - \frac{4hl}{k} > 0 \)

\[ H, N = \frac{1}{2} \left( M \pm \sqrt{M^2 - 4 \frac{hl}{k}} \right) \]

\[ a x^2 + b x + c = a(x-x_1)(x-x_2) \]

\[ x_1, x_2 : \text{roots} \]

\[ x_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ -k x^2 + kMx - h = 0 \]

\[ x = \frac{-kM \pm \sqrt{(kM)^2 - 4kh}}{-2k} \]

de note these roots by \( H \) and \( N \)

\( w/ \) "+" sign

\( w/ \) "-" sign

\[ H, N: 2 \text{ equil. sols} \]

\( x = H \) is unstable equil. sol

\( x = N \) is stable
\[ x = N: \text{ new limiting population size} \]  
\[ \text{(due to harvesting)} \]

Solution (by separation of variables and partial fraction decomposition):
\[
x(t) = \frac{N(x_0 - H) - H(x_0 - N) e^{-k(N-H)t}}{(x_0 - H) - (x_0 - N) e^{-k(N-H)t}}
\]

where \( x(0) = x_0 \).

If \( x_0 < H \), then there is a finite time \( t_1 \):
\[
\lim_{t \to t_1} x(t) = -\infty \quad \text{but before } t_1, \ x(t) \text{ becomes zero: extinction}
\]
$x(t) = H$: threshold equilibrium solution that separates two different solution behaviour:

- If $x_0 > H \Rightarrow \lim_{t \to \infty} x(t) = N$
- If $x_0 < H \Rightarrow$ population becomes extinct in a finite time (due to harvesting)
Logistic Equation w/ Harvesting (Cont'd)

Ex: \[ \frac{dx}{dt} = x(4-x) - h \]: logistic equation w/ harvesting rate \( h \)

\( x(t) \): population size (hundreds) measured in \( t \): measured in years

If \( h = 0 \), we have logistic equation: \[ \frac{dx}{dt} = x(4-x) \]

Then \( \lim_{t \to \infty} x(t) = 4 = M \) (hundred)

Now let \( h = 3 \).

\[ \frac{dx}{dt} = x(4-x) - 3 \]

\[ x(4-x) - 3 = 4x - x^2 - 3 = -(x^2 - 4x + 3) = -(x-1)(x-3) \]

\( x_1 = 1 \), \( x_2 = 3 \)

\( x_1 \cdot x_2 = 3 \), \( x_1 + x_2 = 4 \)
Two critical points: \( x = 1 \) and \( x = 3 \)

\[
\frac{dx}{dt} = -(x-1)(x-3)
\]

\( x^1 \uparrow \)

1 \rightarrow 3 \rightarrow x

\( x = 1 \) : unstable equilibrium solution
\( x = 3 \) : stable equilibrium solution

[Phase Diagram]

\[ ax^2 + bx + c = 0 \]
\[ x_1, x_2: \text{ roots} \]
\[ x_1 \cdot x_2 = \frac{c}{a} \]
\[ x_1 + x_2 = -\frac{b}{a} \]

Vietta's Theorem

If lake is stocked initially with 100 fish, then \( x(t) \rightarrow 300 \) fish as \( t \rightarrow \infty \).

If lake is stocked initially with fewer than 100 fish, then lake will be "fished out" due to excessive harvesting within finite time.
Consider again \( \frac{dx}{dt} = x(4-x) - h \)

\[ x(4-x) - h = 0 \]

\[ -(x^2 - 4x + h) = 0 \]

Recall, critical points of \( \frac{dx}{dt} = f(x) \) are solutions \( x(t) = c \) for which \( f(c) = 0 \)

\[ x_{1,2} = \frac{4 \pm \sqrt{16 - 4h}}{2} = 2 \pm \sqrt{4 - h} \]

if \( h < 4 \Rightarrow \) discriminant \( 4 - h > 0 \Rightarrow \) there are two real distinct roots

if \( h = 4 \Rightarrow \) discriminant \( 4 - h = 0 \Rightarrow \) one real repeated root \( x_{1,2} = 2 \)

if \( h > 4 \Rightarrow 4 - h < 0 \Rightarrow \) there are no real roots \( h > 4 \)

\( x' \)

\( h < 4 \)

\( h = 4 \)

phase diagram

\[ \hline \]

\[ \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3}
\end{array} \]
\[
\frac{dx}{dt} = x(y-x) - h
\]

Crit. pts: \(x(y-x) - h = 0\)

\[-x^2 + xy - h = 0\]

\[x^2 - xy + h = 0\]

\[x_{1,2} = \frac{y \pm \sqrt{y^2 - 4h}}{2}\]

\[ax^2 + bx + c = 0\]

\[x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\]

\[= \frac{y \pm \sqrt{4h-y}}{2} = 2 \pm \sqrt{4h-y} \quad (This \ is \ for \ y-h > 0 \ or \ h < y)\]

If \(h = y \Rightarrow x_{1,2} = 2 \pm 0 = 2\)

If \(h < y \Rightarrow \) roots are complex \(\Rightarrow\) no real roots

We don't consider the case when \(h = 0\) on pg. 3.

We consider all possible cases as \(h\) varies.

If \(h = 0\), then

\[x_{1,2} = 2 \pm \sqrt{y} = 2 \pm 2 = 0 \pm y\]

and they depend on \(h\).

We have 3 cases: \(h < y\) (two distinct real roots), \(h = y\) (one root), \(h > y\) (no real roots)
\( h < y \): two distinct equil. sol.\(^{1,5}\)

\( h = y \): one equil. sol.\(^{1,5}\)

\( h > y \): no equil. sol.\(^{1,5}\)

\( x_1 \) is unstable equil. sol.\(^{1,5}\)

\( x_2 \) is stable \(-1\)

2 equil. sol.\(^{1,5}\)

\( x = 2 \) is unstable equil. sol.\(^{1,5}\)

(semi-stable)

1 equil. sol.\(^{1,5}\)

no equil. solution

Summary

\( h < y \): two distinct equil. sol.\(^{1,5}\)

\( h = y \): one equil. sol.\(^{1,5}\)

\( h > y \): no equil. sol.\(^{1,5}\)
Def: The value \((h=r)\) for which qualitative behaviour of solutions of DE with parameter \(k\) changes as \(k\) increases is called a bifurcation point.

One of the ways to visualize this change in solution behaviour is to plot bifurcation diagram.

\[
\frac{dx}{dt} = x(y-x)-h
\]

\(f(x)\)

\(x(c)=c: \text{critical point} \Rightarrow f(c)=0\)

\(c(y-c)-h=0\)

\[
(c-2)^2 = 4-h
\]

\(h=0 \Rightarrow (c-2)^2 = 4\)

\(c = 2 \pm 2 = 4, 0\)