Ex Solve

\[ y'' - 6y' + 11y = 0 \]
\[ (D^3 - 6D^2 + 11D - 6) y = 0 \]
\[ (D - 1)(D - 2)(D - 3) y = 0 \]

Since we can permute the order of \(D - 1, D - 2\) and \(D - 3\), we can use a function that to find all three solutions.

Recall

\[ (D - a)e^{ax} = 0 \]

\[ (D - 1)(D - 2)(D - 3)e^{3x} = (D - 1)(D - 2) 0 = 0 \Rightarrow e^{3x} \text{ is a solution} \]

\[ (D - 2)(D - 3)(D - 1)e^x = (D - 2)(D - 3) 0 = 0 \Rightarrow e^x \text{ is a solution} \]
\[(D-3)(D-1)(D-2)e^{2x} = (D-3)(D-1)0 = 0 \Rightarrow e^{2x} \text{ is a sol}^{\frac{1}{2}}\]

The general solution is

\[y(x) = C_1e^x + C_2e^{2x} + C_3e^{3x}\]

where \(C_1, C_2, C_3\) are arbitrary constants and function \(e^x, e^{2x}, e^{3x}\) are linearly independent (more later)

Ex (from last time). Solve

\[y'' - 4y' - 5y = 0\]

\[(D^2 - 4D - 5)y = 0\]

\[(D+1)(D-5)y = 0\]

Solutions: \(e^{-x}, e^{5x}\)

General solution: \(y(x) = C_1e^x + C_2e^{5x}\)
Alternative approach

Assume that a solution has the form: $y = e^{rx}$

$y' = re^{rx}, \quad y'' = r^2e^{rx}$

$y'' - 4y' - 5y = 0 \Rightarrow r^2e^{rx} - 4re^{rx} - 5e^{rx} = 0$

$e^{rx}(r^2 - 4r - 5) = 0$

$\neq 0 \Rightarrow [r^2 - 4r - 5 = 0]$

characteristic equation

Roots: $r_1 = -1, \quad r_2 = 5$

Solutions: $y_1(x) = e^{r_1x} = e^{-x}, \quad y_2(x) = e^{r_2x} = e^{5x}$

compare with $(D^2 - 4D - 5) \cdot y = 0$

same as above
Ex: Solve \( y'' - 4y' - 5y = 0 \) subject to ICs \( y(0) = 5, \ y'(0) = 7 \).

The general solution is

\[ y(x) = C_1 e^x + C_2 e^{5x} \]

\( y(0) = 5 \Rightarrow [C_1 + C_2 = 5] \)

\( y'(x) = -C_1 e^x + 5C_2 e^{5x} \)

\( y'(0) = 7 \Rightarrow [-C_1 + 5C_2 = 7] \)

\[
\begin{cases}
  C_1 + C_2 = 5 \\
  -C_1 + 5C_2 = 7
\end{cases}
\]

System of 2 equations

Add eqs:

\( 6C_2 = 12 \Rightarrow C_2 = 2 \)

\( C_1 = 5 - C_2 = 5 - 2 = 3 \)

\[ y(x) = 3e^x + 2e^{5x} \]

Linear dependence and independence, Wronskian

Recall definitions of hyperbolic functions: hyperbolic cosine and hyperbolic sine.

\[
\cosh(ax) = \frac{e^{ax} + e^{-ax}}{2}
\]

\[
\sinh(ax) = \frac{e^{ax} - e^{-ax}}{2}
\]
\[ i^2 = -1 \]

\[
\begin{align*}
\cosh(ax) &= \frac{e^{ix} + e^{-ix}}{2} \\
\sinh(ax) &= \frac{e^{ix} - e^{-ix}}{2i}
\end{align*}
\]

Note:

**Example:** Consider DE

\[
\frac{d^2y}{dx^2} - y = 0
\]

**Solutions:** \( e^x, e^{-x} \)

Can we write the general solution as \( y(x) = c_1 e^x + c_2 e^{-x} \)?
How about
\[ y(x) = C_1e^x + C_2e^{-x} + C_3 \tanh x \, ? \quad \text{No} \]
\[ y(x) = C_1e^x + C_2e^{-x} + C_3 \frac{1}{2}(e^x - e^{-x}) = \left( C_1 + \frac{C_3}{2} \right)e^x + \left( C_2 - \frac{C_3}{2} \right)e^{-x} \]

Only two arbitrary constants, not three! \( K_1, K_2 \)

**Def**: A linear combination of functions \( f_1(x), f_2(x), \ldots, f_n(x) \)

is
\[ C_1f_1(x) + C_2f_2(x) + \ldots + C_n f_n(x) \]
where \( C_1, C_2, \ldots, C_n \): arbitrary constants,

E.g. \( \tanh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x} \)

**Def**: A set of functions \( f_1(x), f_2(x), \ldots, f_n(x) \), defined on a common interval \( I \), is said to be linearly dependent (LD)
if there exists a set of constants $C_1, C_2, \ldots, C_n$, not all being 0, such that the linear combination

$$C_1 f_1(x) + C_2 f_2(x) + \ldots + C_n f_n(x) \equiv 0 \quad \text{for all } x \in I$$

Note: If at least one constant, say $C_i \neq 0$, then we can write

$$f_i(x) = -\frac{C_i}{C_i} f_i(x) - \frac{C_2}{C_i} f_2(x) - \ldots - \frac{C_{i-1}}{C_i} f_{i-1}(x) - \frac{C_{i+1}}{C_i} f_{i+1}(x) - \ldots$$

ie. $f_i(x)$ is a linear combination of the rest of the functions.
Is the set of functions $2e^x, e^{-x}, \sin x, \cos x$ linearly dependent?

To answer this question, we form a linear combination of the given functions and equate it to zero. Then we try to find constants $c_1, c_2, c_3, c_4$, not all being zero, that the linear combination is zero for all $x$.

Thus,

$$c_1e^x + c_2e^{-x} + c_3\sin x + c_4\cos x = 0 \quad \forall x \in \mathbb{R}$$

$$c_1e^x + c_2e^{-x} + c_3\sin x + c_4\cos x \cdot \frac{1}{2}(e^x + e^{-x}) = 0$$

$$\overbrace{(c_1 + 2c_4)}^{= 0} e^x + \underbrace{(c_2 + 2c_4)}_{= 0} e^{-x} + c_3\sin x = 0$$
\[ \begin{align*}
C_1 + 2C_4 &= 0 \\
C_2 + 2C_4 &= 0 \\
C_3 &= 0 \\
\end{align*} \]

\[ \begin{align*}
C_1 = -2, \\
C_2 = -1, \\
C_3 = 0, \\
C_4 = \frac{1}{2} \\
\end{align*} \]

Let \[ C_1 = 0, \quad C_2 = 0. \]

Since we were able to find constants \( C_1, C_2, C_3, C_4 \), at least one of them is non-zero, functions \( e_x, e_y, e_\theta \), are LD.

The choice of constants is not unique.

Note:
\[ C_1 = -1, \quad C_2 = -2, \quad C_3 = 0, \quad C_4 = 10 \]

Def. A set of functions is linearly independent \((LI)\) if it is not linearly dependent.
Ex : $2e^x, xe^x, x^j$ L Di or L I?

\[ C_1 e^x + C_2 xe^x + C_3 x = 0 \quad \forall x \]

Since it is true for all $x$, it should be true for $x = 0$.

$x = 0$:

\[ C_1 + C_2 \cdot 0 + C_3 \cdot 0 = 0 \quad \Rightarrow \quad C_1 = 0 \]

\[ C_2 x e^x + C_3 x = 0 \quad \bigg\vert \frac{1}{x} \quad x \neq 0 \]

\[ C_2 e^x + C_3 = 0 \]

$x = 1$:

\[ C_2 e^1 + C_3 = 0 \quad \bigg\vert \quad C_2 (e^1 - e^0) = 0 \quad \Rightarrow \quad C_2 = 0 \]

$x = -1$:

\[ C_2 e^{-1} + C_3 = 0 \quad \bigg\vert \quad \neq 0 \]

\[ \Rightarrow \quad C_3 = 0 \]

\[ \therefore \quad e^x, xe^x, x \text{ are L I.} \]