HOLOMORPHIC MOTION OF CIRCLES THROUGH AFFINE BUNDLES

DAVID E. BARRETT

1. Introduction

The pivotal topic of this paper is the study of Levi-flat real hypersurfaces S with circular fibers in a rank 1 affine bundle A over a Riemann surface X. (To say that S is Levi-flat is to say that S admits a foliation by Riemann surfaces; equivalently, in the language of [SuTh], S may be said to prescribe a holomorphic motion of circles through A.)

After setting notation and terminology in §2 we proceed in §3 to examine the Levi-form of a general real hypersurface with circular fibers, emphasizing the connection with curvature considerations.

In §4 we focus on the Levi-flat case. In Theorems 5 and 6 we construct moduli spaces for Levi-flat S attached to a fixed underlying line bundle L in the compact and non-compact cases, respectively. In particular, when X is compact we show that the existence of a Levi-flat S implies that $0 \leq \deg L \leq 2\,\mathrm{genus}(X) - 2$. (The bound is sharp.)

Theorem 7 in §7 states that when S is Levi-flat, the Levi-foliation on S extends to a holomorphic foliation of the $\mathbb{CP}^1$ bundle obtained from A by compactifying the fibers. In the general case, the extended foliation in constructed by looking for holomorphic sections of A whose distance from the center is harmonic with respect to the appropriate metric. In §7 we show that this construction produces a foliation even in some cases where S "disappears into the recomplexification of A."

§6 looks at general holomorphic foliations (transverse to fibers) of compactified rank 1 affine bundles; in particular, it is shown that such foliations are classified up to equivalence by a "Schwarzian derivative" and a "curvature function." An Addendum to Theorem 7 shows how to recognize when such a foliation arises from a Levi-flat hypersurface.

The remaining sections contain postponed proofs.

2. Notation and terminology

2.1. **Affine bundles.** Let L be a holomorphic line bundle over a Riemann surface X; L can be defined by local trivializations with transition functions of the form

$$(z, w) \mapsto (\phi_{\alpha,\beta}(z), \chi_{\alpha,\beta}(z) \cdot w)\,. \tag{2.1}$$

An affine bundle A over X associated to L can be defined by local trivializations with transition functions of the form

$$(z, w) \mapsto (\phi_{\alpha,\beta}(z), \chi_{\alpha,\beta}(z) \cdot w + \sigma_{\alpha,\beta}(z))$$

Supported in part by the National Science Foundation.

satisfying the appropriate cocycle condition. Over each point $\zeta \in X$ we have a well-defined subtraction operation $A_\zeta \times A_\zeta \to L_\zeta$ defined in local bundle coordinates by

$$((z(\zeta), w_1), (z(\zeta), w_2) \mapsto (z(\zeta), w_1 - w_2).$$

We will use the term *L-shear* to refer to a biholomorphic map between affine bundles A and A' over X associated to L taking each fiber A_ζ to the corresponding fiber A'_ζ and preserving the subtraction operation.

Let γ be a smooth section of an affine bundle A associated to L. Then $\overline{\partial}\gamma$ defines a section of $L \otimes T^{*(0,1)}(X)$.

The following result follows easily from the definitions.

Proposition 1. *Let A_1 and A_2 be affine bundles over X associated to a fixed line bundle L, and let γ_j be a smooth section of A_j. Then the following conditions are equivalent:*

(1) *there is an L-shear from A_1 to A_2 carrying γ_1 to γ_2;*
(2) $\overline{\partial}\gamma_1 = \overline{\partial}\gamma_2$.

Conversely, let ω be a smooth section of $L \otimes T^{*(0,1)}(X)$. Then using a system of local solutions of $\overline{\partial}u = \omega$ we may construct an affine bundle A associated to L and a section γ of A satisfying $\overline{\partial}\gamma = \omega$. Alternatively we may accomplish the same end by taking the total space of A to be the total space of L equipped with the unique complex structure J_ω satisfying:

- J_ω coincides with the standard structure J_0 on vectors tangent to fibers;
- a (local) section γ of L is J_ω-holomorphic if and only if it solves $\overline{\partial}\gamma = -\omega$ (with respect to the standard structure J_0).

Using local coordinates (z, w) coming from a local trivialization of L we find that the (1,0) tangent vector fields for the structure J_ω are spanned by $\frac{\partial}{\partial w}$ and $\frac{\partial}{\partial z} - \overline{\langle \omega, \frac{\partial}{\partial \overline{z}} \rangle \frac{\partial}{\partial \overline{w}}}$. The integrability of J_ω can be checked directly; alternately we may note that a solution of $\overline{\partial}\gamma = -\omega$ on a open set U induces a biholomorphic map

$$\begin{aligned} \left(L\big|_U, J_0\right) &\to \left(L\big|_U, J_\omega\right) \\ (z, w) &\mapsto (z, w + \gamma(z)) \,. \end{aligned}$$

It follows that (L, J_ω) is an affine bundle over X associated to L; since the induced Cauchy-Riemann operator $\overline{\partial}_\omega$ satisfies $\overline{\partial}_\omega = \overline{\partial}_0 + \omega$ we find that the zero section of L provides a distinguished smooth section of (L, J_ω) satisfying $\overline{\partial}_\omega 0 = \omega$.

2.2. **Bundle metrics; hypersurfaces with circular fibers.** Let A be an affine bundle over X associated to the line bundle L and let γ be a smooth section of A. Suppose now that the line bundle L is equipped with a Hermitian metric h. Then we may consider the real hypersurface $S = S_{\gamma,h} \subset A$ whose fiber over $\zeta \in X$ is the circle centered at $\gamma(\zeta)$ with unit radius with respect to h. Using bundle coordinates (z, w) and writing $h = e^{u(z)}\,|dw|$ we find that $S_{\gamma,h}$ is given by the equation $|w - \gamma(z)| = e^{-u(z)}$.

Proposition 2. *The map*

$$[S_{\gamma,h}] \mapsto \left(\overline{\partial}\gamma, h\right)$$

is a bijection between

$$\frac{\{\textit{real hypersurfaces with circular fibers in affine bundles associated to } L\}}{\{L\textit{-shears}\}}$$

and

$$\{\textit{smooth sections of } L \otimes T^{*(0,1)}(X)\} \times \{\textit{Hermitian metrics on } L\}.$$

In particular, $S_{\gamma,h}$ is equivalent up to L-shears to the unit circle bundle $\Sigma_{\overline{\partial}\gamma,h} \subset \left(L, J_{\overline{\partial}\gamma}\right)$.

3. Levi-form computations

Remarks on terminology. *If L is a (not necessarily holomorphic) line bundle given by transition functions* (2.1)*, we let $|L|$ denote the line bundle given by transition functions*

$$(z, w) \mapsto (\phi_{\alpha,\beta}(z), |\chi_{\alpha,\beta}(z)| \cdot w)\,.$$

We will frequently find it convenient to identify $|L|^2$ with $L \otimes \overline{L}$.

A metric on L may be viewed as a section of $|L|^{-1}$.

Proposition 3. *Let L be a line bundle over a Riemann surface X, let ω be a smooth section of $L \otimes T^{*(0,1)}(X)$, and let h be a Hermitian metric on L. Then the unit circle bundle $\Sigma_{\omega,h}$ in*

$$(L, J_\omega) \textit{ is } \left\{\begin{matrix}\textit{strictly pseudoconcave}\\ \textit{pseudoconcave}\\ \textit{Levi-flat}\\ \textit{pseudoconvex}\\ \textit{strictly pseudoconvex}\end{matrix}\right\} \textit{ if and only if } \left\{\begin{matrix}\frac{i}{2}\Theta > ih^2\omega \wedge \overline{\omega} + h^{-1}\left|\partial\left(h^2\omega\right)\right|\\ \frac{i}{2}\Theta \ge ih^2\omega \wedge \overline{\omega} + h^{-1}\left|\partial\left(h^2\omega\right)\right|\\ \frac{1}{2}\Theta = h^2\omega \wedge \overline{\omega},\ \partial\left(h^2\omega\right) = 0\\ \frac{i}{2}\Theta \le ih^2\omega \wedge \overline{\omega} - h^{-1}\left|\partial\left(h^2\omega\right)\right|\\ \frac{i}{2}\Theta < ih^2\omega \wedge \overline{\omega} - h^{-1}\left|\partial\left(h^2\omega\right)\right|\end{matrix}\right\}; \textit{ here}$$

$\Theta = -2\partial\overline{\partial}\log h$ is the curvature $(1,1)$-form for the metric h, and the inequalities are taken with respect to the standard orientation on X.

To further explain the above equations, note that

- $h^2\omega \wedge \overline{\omega}$ is a section of $|L|^{-2} \otimes \left(L \otimes T^{*(0,1)}\right) \otimes \left(\overline{L} \otimes T^{*(1,0)}\right) = T^{*(1,1)}$;
- $h^2\omega$ is a section of the anti-holomorphic bundle $|L|^{-2} \otimes L \otimes T^{*(0,1)} = \overline{L}^{-1} \otimes T^{*(0,1)}$;
- $\partial\left(h^2\omega\right)$ is a section of $\overline{L}^{-1} \otimes T^{*(1,1)}$;
- $\left|\partial\left(h^2\omega\right)\right|$ is a section of $|L|^{-1} \otimes T^{*(1,1)}$;
- $h^{-1}\left|\partial\left(h^2\omega\right)\right|$ is a section of $T^{*(1,1)}$.

By Proposition 2, the results of Proposition 3 also describe the pseudoconvexity properties of hypersurfaces $S_{\gamma,h}$ with $\overline{\partial}\gamma = \omega$. Passing to local coordinates as in §2.2 this translates to the statement that the hypersurface $|w - \gamma(z)| = e^{-u(z)}$ is $\left\{\begin{matrix}\text{strictly pseudoconcave}\\ \text{pseudoconcave}\\ \text{Levi-flat}\\ \text{pseudoconvex}\\ \text{strictly pseudoconvex}\end{matrix}\right\}$ if and only if $\left\{\begin{matrix} -u_{z\overline{z}} > e^{2u}\left|\gamma_{\overline{z}}\right|^2 + e^u\left|\gamma_{z\overline{z}} + 2u_z\gamma_{\overline{z}}\right| \\ -u_{z\overline{z}} \geq e^{2u}\left|\gamma_{\overline{z}}\right|^2 + e^u\left|\gamma_{z\overline{z}} + 2u_z\gamma_{\overline{z}}\right| \\ -u_{z\overline{z}} = e^{2u}\left|\gamma_{\overline{z}}\right|^2,\ \gamma_{z\overline{z}} + 2u_z\gamma_{\overline{z}} = 0 \\ -u_{z\overline{z}} \leq e^{2u}\left|\gamma_{\overline{z}}\right|^2 - e^u\left|\gamma_{z\overline{z}} + 2u_z\gamma_{\overline{z}}\right| \\ -u_{z\overline{z}} < e^{2u}\left|\gamma_{\overline{z}}\right|^2 - e^u\left|\gamma_{z\overline{z}} + 2u_z\gamma_{\overline{z}}\right| \end{matrix}\right\}$. For a proof in this framework see [Ber, Prop. 2.3] and the references cited there.

For $\omega \equiv 0$ Proposition 3 reduces to the following (quite classical) result.

Corollary 4. *Let L be a line bundle over a Riemann surface X equipped with a Hermitian metric h. Then the unit circle bundle $\Sigma_{0,h}$ in L is* $\left\{\begin{matrix} \textit{strictly pseudoconcave} \\ \textit{pseudoconcave} \\ \textit{Levi-flat} \\ \textit{pseudoconvex} \\ \textit{strictly pseudoconvex} \end{matrix}\right\}$ *if and only if the curvature form Θ satisfies* $\left\{\begin{matrix} i\Theta > 0 \\ i\Theta \geq 0 \\ \Theta = 0 \\ i\Theta \leq 0 \\ i\Theta < 0 \end{matrix}\right\}$.

For vector bundles of higher dimension, curvature conditions for Hermitian and Finsler metrics are related to the theory of interpolation of norms [Roc].

4. The Levi-flat case

Recall that a real hypersurface in a complex manifold is said to be *Levi-flat* if its Levi-form vanishes identically. A real hypersurface is Levi-flat if and only if it admits a (uniquely-determined) codimension-one foliation with complex leaves [Kra, p. 308].

In the situation of Proposition 3, if $\Sigma_{\omega,h} \subset (L, J_\omega)$ is Levi-flat then $h^2\overline{\omega}$ is a holomorphic section of $L^{-1} \otimes T^{*(1,0)}(X)$.

If $h^2\overline{\omega} \equiv 0$ then by Proposition 1 we may take $A = L$; also the curvature Θ of the metric h vanishes identically so that h is flat. In the case where X is compact we thus have $\deg L = 0$ [GrHa, §1.1]. (Recall that every degree 0 line bundle admits a flat metric, unique up to scalar multiples [GrHa, §1.2].)

To analyze the case where $h^2\overline{\omega}$ does not vanish identically, note that $2h^{-1}\left|h^2\overline{\omega}\right| = 2h|\omega|$ is a non-negative section of $\left|T^{*(1,0)}(X)\right|$; it may be viewed as a conformal metric on X with a so-called conical singularity of total angle $2\pi(j+1)$ at any point where $h^2\overline{\omega}$ has a zero of order j (see for example [HuTr, §2]). The corresponding area form is $2ih^2\omega \wedge \overline{\omega}$, and away from the degeneracies the scalar curvature is given by

$$\begin{aligned} -\frac{2i\partial\overline{\partial}\log\left(2h^{-1}\left|h^2\overline{\omega}\right|\right)}{2ih^2\omega\wedge\overline{\omega}} &= \frac{2i\partial\overline{\partial}\log h}{2ih^2\omega\wedge\overline{\omega}} \\ &= -\frac{\Phi/2}{h^2\omega\wedge\overline{\omega}} \\ &= -1. \end{aligned}$$

To take proper account of the degeneracies we may compute the Gauss-Bonnet form in the sense of distributions:

$$-2i\partial\overline{\partial}\log\left(2h^{-1}\left|h^2\overline{\omega}\right|\right) = -2ih^2\omega\wedge\overline{\omega} - 2\pi\sum_{\{\zeta\in X:\omega(\zeta)=0\}}\left(\text{order of vanishing of } h^2\overline{\omega} \text{ at } \zeta\right)\cdot\delta_\zeta,$$

where δ_ζ denotes a unit point mass at ζ. (See for example [Bar, Lemma 11].)

In the case where X is compact, invocation of the Gauss-Bonnet theorem yields

$$\begin{aligned} 4\pi(1-\text{genus}(X)) &= -\int_X 2i\partial\overline{\partial}\log\left(2h^{-1}\left|h^2\overline{\omega}\right|\right) \\ &= -\int_X 2ih^2\omega\wedge\overline{\omega} - 2\pi\sum_{\{\zeta\in X:\omega(\zeta)=0\}}\left(\text{order of vanishing of } h^2\overline{\omega} \text{ at } \zeta\right) \\ &= -\int_X 2ih^2\omega\wedge\overline{\omega} - 2\pi\deg\left(L^{-1}\otimes T^{*(1,0)}(X)\right) \\ &= -\int_X 2ih^2\omega\wedge\overline{\omega} - 2\pi\left(2(\text{genus}(X)-1)-\deg L\right) \end{aligned}$$

so that

$$\deg L = \frac{1}{\pi}\int_X ih^2\omega\wedge\overline{\omega} > 0.$$

Conversely, suppose that X is compact and that $\deg L > 0$. Then for any non-trivial holomorphic section f of $L^{-1}\otimes T^{*(1,0)}(X)$ the results of [HuTr, Thm. B] allow us to construct (uniquely) a conformal metric $\widetilde{h}$ on X of curvature -1 with conical singularities of total angle $2\pi(j+1)$ at points where f has a zero of order j. Setting $h = 2\widetilde{h}^{-1}|f|, \omega = \dfrac{\widetilde{h}^2}{4f}$ we find that $\Sigma_{\omega,h}\subset(L,J_\omega)$ is Levi-flat.

Let $LFC(X,L)$ denote the space

$$\frac{\{\text{Levi-flat hypersurfaces with circular fibers in affine bundles associated to } L\}}{\{L\text{-shears}\}}.$$

Then we may sum up the preceding discussion as follows.

Theorem 5. *Let X be a compact Riemann surface and let L be a holomorphic line bundle on X.*

If $\deg L < 0$ then $LFC(X,L) = \emptyset$.

If $\deg L = 0$ *then the map* $[S_{\gamma,h}] \mapsto h$ *is a bijection between* $LFC(X,L)$ *and the one-dimensional space of flat metrics on* L.

If $\deg L > 0$ *then the map* $[S_{\gamma,h}] \mapsto h^2\partial\overline{\gamma}$ *is a bijection between* $LFC(X,L)$ *and the space of non-trivial holomorphic sections of* $L^{-1} \otimes T^{*(1,0)}(X)$.

Note that if $\deg L > 2\,\mathrm{genus}(X) - 2$ then $\deg\left(L^{-1} \otimes T^{*(1,0)}(X)\right) < 0$ so that there are no non-trivial holomorphic sections of $L^{-1} \otimes T^{*(1,0)}(X)$; thus in this case we again have $LFC(X,L) = \emptyset$.

Note also that for $\deg L > 0$ we never have $A = L$, else we would have

$$\begin{aligned} 0 &= -\int_X \overline{\partial}\left(h^2\,\partial\overline{\gamma}\right)\gamma \\ &= \int_X h^2\left|\overline{\partial}\gamma\right|^2, \end{aligned}$$

forcing $h^2\partial\overline{\gamma} \equiv 0$.

To treat the case of non-compact X we will get a simpler-to-state result by working modulo not just shears but arbitrary fiber-preserving biholomorphic maps – following terminology in dynamics [AnLe] we will refer to such maps as *overshears.*

Theorem 6. *If* A *is a rank 1 affine bundle over a noncompact Riemann surface* X *then the map* $[S_{\gamma,h}] \mapsto 2h\left|\overline{\partial}\gamma\right|$ *is a bijection from*

$$\frac{\{\text{Levi-flat } S_{\gamma,h} \subset A\text{: } \gamma \text{ not holomorphic}\}}{\{\text{overshears}\}}$$

to the space

$$\{\text{conformal metrics on } X \text{ of curvature } -1 \text{ with all total angles} \in 2\pi\mathbb{N}\}.$$

Proof. Recall from Proposition 2 that S_{γ_1,h_1} and S_{γ_2,h_2} are equivalent modulo shears if and only if $h_1 = h_2$ and $\overline{\partial}\gamma_1 = \overline{\partial}\gamma_2$. Similarly, it is straightforward to check that S_{γ_1,h_1} and S_{γ_2,h_2} are equivalent modulo overshears if and only if there is $g : X \to \mathbb{C}\setminus\{0\}$ holomorphic with $h_1 = |g|h_2, \overline{\partial}\gamma_1 = g^{-1}\overline{\partial}\gamma_2$; the latter condition implies that $2h_1|\overline{\partial}\gamma_1| = 2h_2|\overline{\partial}\gamma_2|$, showing that our map is well-defined.

To check injectivity, note if S_{γ_1,h_1} and S_{γ_2,h_2} are Levi-flat then equality of $2h_1|\overline{\partial}\gamma_1|$ and $2h_2|\overline{\partial}\gamma_2|$ forces the holomorphic sections $h_1^2\,\partial\overline{\gamma}_1$ and $h_2^2\,\partial\overline{\gamma}_2$ to have the same zeros (counting multiplicities) so that $h_1^2\,\partial\overline{\gamma}_1 = gh_2^2\,\partial\overline{\gamma}_2$ for some holomorphic $g : X \to \mathbb{C}\setminus\{0\}$. Thus

$$\begin{aligned} h_1 &= \frac{2\left|h_1^2\,\partial\overline{\gamma}_1\right|}{2h_1\left|\overline{\partial}\gamma_1\right|} = |g|h_2, \\ \overline{\partial}\gamma_1 &= \frac{\left(2h_1\left|\overline{\partial}\gamma_1\right|\right)^2}{4h_1^2\,\partial\overline{\gamma}_1} = g^{-1}\,\overline{\partial}\gamma_2, \end{aligned}$$

showing that S_{γ_1,h_1} and S_{γ_2,h_2} are equivalent modulo overshears, as required.

To prove surjectivity, let $\widetilde{h}$ be a conformal metric on X of curvature -1 with all total angles $\in 2\pi\mathbb{N}$. By the Weierstraß Product Theorem [For, Thm. 26.5] and the triviality of $L^{-1}\otimes T^{*(1,0)}(X)$ [For, Thm. 30.3] we may pick a holomorphic section f of $L^{-1}\otimes T^{*(1,0)}(X)$ so that f vanishes to order j at ζ if and only if $\widetilde{h}$ has total angle $2\pi(j+1)$ at ζ. Let $h = 2\widetilde{h}^{-1}|f|, \omega = \dfrac{\widetilde{h}^2}{4f}$. Then $\Sigma_{\omega,h}\subset(L,J_\omega)$ is Levi-flat, and our mapping takes $[\Sigma_{\omega_h}]$ to $\widetilde{h}$, as required. □

Turning to the flat case, a standard argument shows that when X is non-compact, $S_{\gamma,h}$ with γ holomorphic are classified up to overshears by the associated monodromy homomorphism $\pi_1(X)\to S^1$ [CaLN, Chap. V].

We close this section with consideration of the special case where $L = T^{*(1,0)}(X)$ (with X not necessarily compact) and the (1,1)-form ω is positive. Then $h^2\overline{\omega}$ is both holomorphic and positive, hence equal to a constant $C/2$; it follows that h is $\sqrt{C}$ times the metric on $T^{*(1,0)}(X)$ induced by the conformal metric on X with area form ω. Moreover, $2h|\omega| = Ch^{-1}$ has curvature -1, so the conformal metric on X with area form ω has curvature $-C$.

5. Extension of Levi foliations

If A is an affine bundle over X we will denote by $\widehat{A}$ the $\mathbb{CP}^1 = \widehat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ bundle over X obtained by adding a point at infinity to each fiber A_ζ.

Theorem 7. *Let $S = S_{\gamma,h}$ be a Levi-flat hypersurface with circular fibers in an affine bundle A over a Riemann surface X. Then the Levi-foliation of S extends uniquely to a holomorphic foliation of $\widehat{A}$.*

The extended foliation $\mathcal{F}_S$ is transverse to the fibers $\widehat{A}_\zeta$.

If the corresponding line bundle metric h is flat then $\mathcal{F}_S$ is described by the condition

(CM) *the graph of a local holomorphic section ν of A lies in a leaf if and only if $\|\nu-\gamma\|$ is constant.*

If the corresponding line bundle metric h is not flat then $\mathcal{F}_S$ is described by the condition

(LHM) *the graph of a local holomorphic section ν of A lies in a leaf if and only if $\log\|\nu-\gamma\|$ is harmonic on $X\setminus(\nu-\gamma)^{-1}(0)$.*

Theorem 7 will be proved in §9.

6. Foliations of compactified affine bundles

6.1. **Residues.** Let A be an affine bundle over X associated to a line bundle L and let $\mathcal{F}$ be a holomorphic foliation of $\widehat{A}$ tranvserse to fibers $\widehat{A}_\zeta$. For $\zeta\in X$ let ν_ζ be the unique (germ of a) meromorphic section of A with graph contained in a leaf of $\mathcal{F}$ satisfying $\nu_\zeta(\zeta)=\infty$.

If ν_ζ has a simple pole at ζ then the residue $\operatorname{Res}_\zeta \nu_\zeta$ defines an element of $T^{(1,0)}_\zeta \otimes L_\zeta$. (To be specific, we may choose a small loop C_ζ about ζ and a local holomorphic section γ of A; we then define a functional Υ_ζ on $T^{*(1,0)}_\zeta \otimes L^{-1}_\zeta$ by the formula

$$\Upsilon_\zeta(\omega(\zeta)) = \frac{1}{2\pi i}\int\limits_{C_\zeta} (\nu_\zeta - \gamma)\cdot\omega$$

for ω a local holomorphic section of $T^{*(1,0)}(X) \otimes L^{-1}$. Υ_ζ is clearly linear, and it is easy to check that the right hand side depends only on $\omega(\zeta)$ and in particular does not depend on the choice of C_ζ or γ. Then we can define $\operatorname{Res}_\zeta \nu_\zeta$ to be the element of $T^{(1,0)}_\zeta(X) \otimes L_\zeta$ representing Υ_ζ.)

We may define a section $\kappa_{\mathcal{F}}$ of $T^{*(1,0)}(X) \otimes L^{-1}$ by setting $\kappa_{\mathcal{F}}(\zeta) = -2i\,(\operatorname{Res}_\zeta \nu_\zeta)^{-1}$ when ν_ζ has a simple pole at ζ and $\kappa_{\mathcal{F}}(\zeta) = 0$ when ν_ζ has a multiple pole at ζ.

Proposition 8. *For $\mathcal{F}$ as above, $\kappa_{\mathcal{F}}$ is a holomorphic section of $T^{*(1,0)}(X) \otimes L^{-1}$.*

Proof. Since the fibers of $\widehat{A}$ are compact, the transversality hypothesis guarantees that $\mathcal{F}$ is locally equivalent to a product foliation on $X \times \widehat{\mathbb{C}}$ [CaLN, Chap. V]. Thus, choosing bundle coordinates (z, w) for the restriction of A to a small open set $U \subset X$ we find that there are holomorphic functions $a(z), b(z), c(z), d(z)$ with $ad - bc \equiv 1$ so that leaves of $\mathcal{F}$ are given by equations of the form

$$\frac{a(z)w + b(z)}{c(z)w + d(z)} = C \tag{6.1}$$

or

$$w = \frac{d(z)C - b(z)}{-c(z)C + a(z)}, \tag{6.2}$$

C constant. To get $w = \infty$ when $z = \zeta$ we must set $C = C_\zeta = a(\zeta)/c(\zeta)$ so that

$$\nu_\zeta(z) = \frac{a(\zeta)d(z) - b(z)c(\zeta)}{a(z)c(\zeta) - a(\zeta)c(z)}.$$

A short computation reveals that

$$\kappa_{\mathcal{F}} = 2i\mathcal{W}(a, c) \tag{6.3}$$

where $\mathcal{W}(a, c)$ denotes the Wronskian $a\,dc - c\,da$; it follows immediately that $\kappa_{\mathcal{F}}$ is holomorphic. ∎

6.2. **Schwarzians.** Let L be a holomorphic line bundle over a Riemann surface X and let Λ be a non-vanishing section of $T^{*(1,0)}(X) \otimes \overline{L}$. With respect to a local coordinate z and a corresponding local non-vanishing holomorphic section η of L we may write $\Lambda = \lambda(z)\,dz \otimes \overline{\eta}$. Replacing z and η by $\widetilde{z} = \phi(z)$ and $\widetilde{\eta} = f\eta$ we find that

$$\lambda(z) = \widetilde{\lambda}(\widetilde{z})\phi'(z)\overline{f}$$

and

$$(\log \lambda(z))_{zz} - \frac{1}{2}(\log \lambda(z))_z^2 = \left(\left(\log \widetilde{\lambda}(\widetilde{z})\right)_{\widetilde{z}\widetilde{z}} - \frac{1}{2}\left(\log \widetilde{\lambda}(\widetilde{z})\right)_{\widetilde{z}}^2\right) \cdot (\phi'(z))^2 + (\log \phi'(z))_{zz} - \frac{1}{2}(\log \phi'(z))_z^2 .$$

Thus $\mathcal{R}\Lambda \stackrel{\text{def}}{=} (\log \lambda(z))_{zz} - \frac{1}{2}(\log \lambda(z))_z^2$ defines a section of the affine line bundle $A_{\mathcal{S}}(X)$ with transition functions

$$(z, w) \mapsto \left(\phi(z), \frac{w - \left((\log \phi'(z))_{zz} - \frac{1}{2}(\log \phi'(z))_z^2\right)}{(\phi'(z))^2}\right).$$

If ω is the area form of a conformal metric then $\mathcal{R}\omega$ is holomorphic if and only if that metric has constant curvature. (See Lemma 15 in §9.)

If f is meromorphic and non-constant then the Schwarzian derivative $\mathcal{S}f \stackrel{\text{def}}{=} \mathcal{R}(df)$ defines a meromorphic section of $A_{\mathcal{S}}(X)$. The standard transformation law [Leh, II.1.1] may be written

$$\mathcal{S}(T \circ f) - \mathcal{S}f = (\mathcal{S}T \circ f)\,(df)^2\,;$$

here T is a non-constant meromorphic function on a domain containing the range of f, the subtraction of two sections of $A_{\mathcal{S}}(X)$ on the left-hand side results in a section of the associated line bundle $\left(T^{*(1,0)}(X)\right)^2$ of quadratic differentials, and $\mathcal{S}T$ is the "classical" (scalar-valued) Schwarzian derivative of T. Note that $\mathcal{S}(T \circ f) \equiv \mathcal{S}f$ if and only if $\mathcal{S}T \equiv 0$ if and only if T is a linear fractional transformation.

A result of Laine and Sorvali [LaSo, Cor. 4.8] states that if X is simply-connected then a meromorphic section τ of $A_{\mathcal{S}}(X)$ is the Schwarzian derivative of a non-constant meromorphic function on X if and only if the following condition holds:

(LS) at each pole ζ of τ there is a holomorphic coordinate z vanishing at ζ and an integer $k > 1$ so that the representation of τ with respect to z takes the form

$$\frac{1-k^2}{2} z^{-2} + c_{-1}z^{-1} + c_0 + \cdots + c_{k-2}z^{k-2} + \cdots \tag{6.4}$$

with

$$\det \begin{pmatrix} 2(1-k) & 0 & 0 & \cdots & 0 & c_{-1} \\ c_{-1} & 2(4-2k) & 0 & \cdots & 0 & c_0 \\ c_0 & c_{-1} & 2(9-3k) & \cdots & 0 & c_1 \\ c_1 & c_0 & c_{-1} & \cdots & 0 & c_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{k-3} & c_{k-4} & c_{k-5} & \cdots & 2((k-1)^2-(k-1)k & c_{k-3} \\ c_{k-2} & c_{k-3} & c_{k-4} & \cdots & c_{-1} & c_{k-2} \end{pmatrix} = 0.$$

If the condition in (LS) holds at ζ for a fixed coordinate z then it will hold (with the same value of k) for any other holomorphic coordinate vanishing at ζ; k is in fact the multiplicity at ζ of any solution f of $\mathcal{S}f = \tau$.

Returning now to the notation of the proof of Proposition 8 let us examine the function $C_z = a(z)/c(z)$. It is easy to check that replacing w by $M(z)w + B(z)$ in (6.1) induces no change in $a(z)/c(z)$, whereas changing the representation (6.1) by replacing C by $T(C)$ for some fixed linear fractional transformation T has the effect of replacing $a(z)/c(z)$ by $T\,(a(z)/c(z))$. Thus a/c is determined up to post-composition with a linear fractional transformation by the foliation $\mathcal{F}$.

In view of (6.1) and (6.3), a/c is constant if and only if $\kappa_{\mathcal{F}} \equiv 0$ if and only if the infinity-section $w \equiv \infty$ is a leaf of $\mathcal{F}$. If this does not occur then the Schwarzian derivative $\mathcal{S}(a/c)$ gives rise to a global meromorphic section $\mathcal{S}_{\mathcal{F}}$ of $A_{\mathcal{S}}(X)$ satisfying the condition (LS).

Lemma 9. *Let ν_1, ν_2, ν_3 be distinct local sections of A with graphs contained in leaves of $\mathcal{F}$. Then $\mathcal{S}_{\mathcal{F}} = \mathcal{S}\left(\frac{\nu_2-\nu_3}{\nu_1-\nu_3}\right)$.*

Proof. We may assume that ν_1, ν_2, ν_3 are defined by (6.2) with $C = 0, \infty, 1$, respectively. Then $\frac{\nu_2-\nu_3}{\nu_1-\nu_3} = \frac{a}{c}$. □

Lemma 10. *If $\kappa_{\mathcal{F}}$ is not $\equiv 0$ then the only overshear from A to A taking $\mathcal{F}$ to $\mathcal{F}$ is the identity map.*

Proof. Choose ζ_1, ζ_2 so that ν_{ζ_1} and ν_{ζ_2} are distinct meromorphic sections of A defined on a open set U containing ζ_1 and ζ_2.

Then for generic $\zeta \in U$ the overshear in question must fix the two distinct finite points $\nu_{\zeta_1}(\zeta)$ and $\nu_{\zeta_2}(\zeta)$, forcing the overshear to be the identity map. □

Proposition 11. *If X is a Riemann surface then the map $[\mathcal{F}] \mapsto \mathcal{S}_{\mathcal{F}}$ is a bijection from*

$$\frac{\bigcup_{A \text{ affine over } X} \{\textit{holomorphic foliations } \mathcal{F} \textit{ of } \widehat{A} \textit{ transverse to fibers with } \kappa_{\mathcal{F}} \not\equiv 0\}}{\{\textit{overshears}\}}$$

to

$$\{\textit{meromorphic sections of } A_{\mathcal{S}}(X) \textit{ satisfying (LS)}\}.$$

Proof. The preceding discussion shows that our map is well-defined.

To prove injectivity, note that if $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are two candidate foliations with $\mathcal{S}_{\widetilde{\mathcal{F}}} = \mathcal{S}_{\mathcal{F}}$ then for a pair of representations of the form (6.1) on the same coordinate patch we have $\mathcal{S}(\widetilde{a}/\widetilde{c}) = \mathcal{S}(a/c)$, so that $\widetilde{a}/\widetilde{c} = T \circ (a/c)$ for some linear fractional transformation T; changing the representation of $\widetilde{\mathcal{F}}$ by replacing $\widetilde{C}$ by $T\widetilde{C}$ we may arrange that $\widetilde{a}/\widetilde{c} = a/c$. Then an elementary calculation shows that the overshear $(z, w) \mapsto \left(z, \frac{a\widetilde{d}-c\widetilde{b}}{d\widetilde{a}-b\widetilde{c}}w + \frac{b\widetilde{d}-d\widetilde{b}}{d\widetilde{a}-b\widetilde{c}}\right)$ maps $\mathcal{F}$ to $\widetilde{\mathcal{F}}$. By Lemma 10 there are no nontrivial locally defined overshears mapping $\mathcal{F}$ to $\mathcal{F}$, so the overshears mapping $\mathcal{F}$ to $\widetilde{\mathcal{F}}$ are locally unique and therefore patch together to define a global overshear.

To prove surjectivity, recall from §6.2 that given any meromorphic section τ of $A_{\mathcal{S}}(X)$ satisfying (LS) and any $\zeta \in X$ we may pick f holomorphic and non-constant on a neighborhood V of ζ with $\mathcal{S}f = \mathcal{S}f^{-1} = \tau$ on V. Then the foliation $\mathcal{F}_f$ on $V \times \mathbb{C}$ defined by $\frac{w}{f(z)w+1} = C$ satisfies $\mathcal{S}_{\mathcal{F}} = \tau$. The argument of the preceding paragraph shows that $\mathcal{F}_f$ is determined by τ up to overshears, so choosing a family of local solutions covering X the overshears defined on overlaps can be used to construct the desired affine bundle A and foliation $\mathcal{F}$. □

Proposition 12. *For a fixed line bundle L over a Riemann surface X and a fixed meromorphic section τ of $A_{\mathcal{S}}(X)$, the map $[\mathcal{F}] \mapsto \kappa_{\mathcal{F}}$ is a bijection from*

$$\frac{\bigcup_{A \text{ associated to } L} \{\text{holomorphic foliations } \mathcal{F} \text{ of } \widehat{A} \text{ tranvserse to fibers with } \mathcal{S}_{\mathcal{F}} = \tau\}}{\{L\text{-shears}\}}$$

to

$$\left\{f \in H\left(T^{*(1,0)}(X) \otimes L^{-1}\right) : f \text{ vanishes to order } k-1 \text{ at } \zeta \Leftrightarrow (6.4) \text{ holds at } \zeta\right\}.$$

Proof. The map in question is well-defined since L-shears do not affect the construction of $\operatorname{Res}_\zeta \nu_\zeta$.

To see that our map takes its values in the prescribed space, note that the quotient rule $d(a/c) = \frac{\mathcal{W}(a,c)}{c^2}$ in conjunction with (6.3) shows that $\kappa_{\mathcal{F}}$ must vanish to order $k-1$ when a/c has multiplicity k, and (as mentioned earlier) this in turn will happen precisely when (6.4) holds at the point in question.

Using the notation of §2.1 it is easy to see that any overshear between affine bundles A_1 and A_2 associated to L is equivalent modulo L-shears to a map $F_g : (L, J_\omega) \to (L, J_{g\omega})$ that dilates each fiber L_ζ by the factor $g(\zeta)$; here g is a holomorphic map from X into $\mathbb{C} \setminus \{0\}$. The bijectivity claimed in Proposition 12 now follows easily from Proposition 11 and the transformation law $\kappa_{F_g^*\mathcal{F}} = g^{-1}\kappa_{\mathcal{F}}$. □

6.3. **Recognizing extended Levi-foliations.** In §9 we will prove the following.

Addendum to Theorem 7. *If $\mathcal{F}$ is the extended Levi-foliation of a Levi-flat hypersurface $S_{\gamma,h}$ then $\kappa_{\mathcal{F}} = -2ih^2\overline{\omega}$ and $\mathcal{S}_{\mathcal{F}} = \mathcal{R}(2ih^2\omega \wedge \overline{\omega}) = \mathcal{R}(\overline{\omega})$, where $\omega = \overline{\partial}\gamma$.*

Suppose we are given a rank 1 affine bundle A over a Riemann surface X and a holomorphic foliation $\mathcal{F}$ of $\widehat{A}$ transverse to fibers. How can we determine whether or not $\mathcal{F}$ is the extended Levi-foliation $\mathcal{F}_{S_{\gamma,h}}$ for some Levi-flat $S_{\gamma,h}$?

If X is non-compact then Theorem 6, Proposition 11, and the Addendum to Theorem 7 combine to yield the (somewhat tautological) conclusion that $\mathcal{F}$ is an extended Levi-foliation if and only if

- $\kappa_{\mathcal{F}} \equiv 0$ and $\mathcal{F}$ has a leaf with unitary holonomy projecting bijectively onto X,

or

- $\mathcal{S}_{\mathcal{F}} = \mathcal{R}(\Lambda)$ where Λ is the area form of a conformal metric on X of curvature -1 with all total angles $\in 2\pi\mathbb{N}$.

If X is compact then Theorem 5 shows that $\mathcal{F}$ will not be an extended Levi-foliation unless the degree of the corresponding line bundle L is ≥ 0.

If $\deg L = 0$ then the only possible extended Levi-foliation is that on $\widehat{A} = \widehat{L}$ induced via condition (CM) by the unique (up to positive constants) flat metric on L.

If $\deg L > 0$ then Theorem 5, Proposition 12 and the Addendum to Theorem 7 combine to show that $\mathcal{F}$ is an extended Levi-foliation if and only if $\mathcal{S}_{\mathcal{F}} = \mathcal{R}(\Lambda)$, where Λ is the area form of the metric $\widetilde{h}$ constructed from the divisor of κ_ω in the proof of Theorem 5.

7. Foliations from "phantom hypersurfaces"

Let A be an affine bundle associated to the cotangent bundle $T^{*(1,0)}(\mathbb{CP}^1)$ of the Riemann sphere $\mathbb{CP}^1$. Since $\deg T^{*(1,0)}(\mathbb{CP}^1) = -2$, Theorem 5 shows that A does not contain a Levi-flat hypersurface with circular fibers. On the other hand, if ω is the area form for the usual spherical metric on $\mathbb{CP}^1$ then taking $A = (L, J_\omega)$, $\gamma = 0$, and h to be the metric on $T^{*(1,0)}(\mathbb{CP}^1)$ induced by the spherical metric on $\mathbb{CP}^1$, it turns out that the condition (LHM) from Theorem 7 still defines a holomorphic foliation $\mathcal{F}$ on $\widehat{A}$ transverse to fibers. Comparing with the last paragraph of §4 we see that $\mathcal{F}$ is formally $\mathcal{F}_{\Sigma_{\omega,ih}}$ – but of course the radius of the fibers is not allowed to be imaginary!

More generally we have the following.

Theorem 13. *Let A be an affine bundle over X associated to a line bundle L equipped with metric h, and let γ be a smooth section of A. Suppose that $\omega = \overline{\partial}\gamma$ is nowhere-vanishing. Then the following conditions are equivalent:*

1. *Condition (LHM) of Theorem 7 describes a holomorphic foliation $\mathcal{F}$ of $\widehat{A}$ with leaves transverse to fibers.*
2. *$\kappa_\omega \overset{def}{=} -\frac{i\partial\overline{\partial}\log\omega}{\omega}$ is a holomorphic section of $L^{-1} \otimes T^{*(1,0)}(X)$ and $h^2|\omega|$ is a flat metric on $L \otimes T^{(1,0)}(X)$.*

The notation κ_ω is motivated by the fact that if ω is a positive (1,1)-form then κ_ω is the curvature of the conformal metric with ω as area form. The notation $\kappa_{\mathcal{F}}$ from §6.1 was motivated by the Addendum to Theorem 14 found at the end of this section.

Theorem 13 will be proved in §9 essentially as a special case of the following result which allows for zeroes of ω.

Theorem 14. *Let A be an affine bundle over X associated to a line bundle L equipped with metric h, let γ be a smooth section of A, and let $\omega = \overline{\partial}\gamma$. Assume that ω is not $\equiv 0$. Then the following conditions are equivalent:*

1. *Condition (LHM) of Theorem 7 describes a holomorphic foliation $\mathcal{F}$ of $\widehat{A}$ with leaves transverse to fibers.*
2. *There is an open set $U \subset A$ with $\pi(U) = X$ such that condition (LHM) of Theorem 7 describes a holomorphic foliation $\mathcal{F}$ of U with leaves transverse to fibers.*
3. *For each $\zeta \in X$ there is a neighborhood V of p together with*
   (a) *a holomorphic section ν of A on V with $\nu - \gamma$ non-vanishing*

and

(b) *a non-vanishing holomorphic section* η *of* L *on* V

such that $\log\|\nu-\gamma\|$ *and* $\eta/(\nu-\gamma)$ *are harmonic. (Note that* $\eta/(\nu-\gamma)$ *will in general be* $\mathbb{C}$*-valued.)*

(4) ω *and* h *admit local representations of the form*

$$\omega = \frac{\eta\, d\overline{g}}{(f-\overline{g})^2}$$
$$h = |f-\overline{g}|\,\mu;$$

here f *and* g *are holomorphic functions with* $f-\overline{g}$ *non-vanishing,* η *is a non-vanishing holomorphic section of* L *and* μ *is a flat metric on* L*.*

(5) (a) $h^2|\omega|$ *is a flat metric on* $L\otimes T^{(1,0)}(X)$ *off of the zero set of* ω*;*

(b) $\mathcal{R}\overline{\omega}$ *is a meromorphic section of* $A_{\mathcal{S}}(X)$ *satisfying condition (LS) from §6.2;*

(c) *if* $\mathcal{R}\overline{\omega}$ *has a pole at* ζ *and* z *is a local coordinate vanishing at* ζ *then*

$$\omega = \overline{z}^{k-1}\varphi \tag{7.1}$$

for some smooth φ *defined near* ζ *with* $\phi(\zeta)\neq 0$*.*

Remark. *In condition* (5) *above, the values of* k *in* (7.1) *and condition (LS) will coincide wherever* ω *vanishes.*

When $\omega\equiv 0$, the existence of section ν (not $\equiv\gamma$) satisfying the condition in (LHM) implies that h is flat. In this case (CM) defines a foliation but (LHM) does not.

Addendum to Theorem 14. *For* $\mathcal{F}$ *as in Theorem 14 we have*

$$\kappa_{\mathcal{F}} = \kappa_\omega,$$
$$\mathcal{S}_{\mathcal{F}} = \mathcal{R}\left(\partial\overline{\partial}\log\omega\right)$$

on $\{\zeta\in X : \omega(\zeta)\neq 0\}$*.*

Theorem 14 is proved in the next section; the Addendum will be proved in §9.

Remark. *A Levi-flat* $S\subset A$ *given by the equation* $h^2\,|w-\gamma|^2 = 1$ *is the pullback via the map* $\mathrm{Id}\times_X\overline{\mathrm{Id}} : A\to A\times_X\overline{A}$ *of the hypersurface* $h^2(w-\gamma)(\widetilde{w}-\overline{\gamma}) = 1$*. In view of the Addendum to Theorem 7, a foliation* $\mathcal{F}$ *constructed from the condition (LHM) may be viewed as stemming from the hypersurface* $i\kappa_{\mathcal{F}}(w-\gamma)(\widetilde{w}-\overline{\gamma}) = 2\,\partial\overline{\gamma}$*, though this hypersurface may not intersect* $\left(\mathrm{Id}\times_X\overline{\mathrm{Id}}\right)(A)$*.*

8. Proof of Theorem 14

(1)⇒(2): Take $U = X$. ☐

(2)⇒(3): According to (2), for $\zeta\in X$ we may find a neighborhood V of ζ and a biholomorphic map Ψ from $V\times\Delta$ to an open subset of A with $\pi\circ\Psi=\pi$ such that for fixed ξ, $\Psi(\cdot,\xi)$ is a holomorphic section with $\log\|\Psi(\cdot,\xi)-\gamma(\cdot)\|$ harmonic. (Here Δ is the unit disk in $\mathbb{C}$.)

Let $\nu(\cdot) = \Psi(\cdot, 0)$ and let η be the non-vanishing holomorphic section of L over V given by $z \mapsto \frac{d}{d\xi}\Psi(z,\xi)\big|_{\xi=0}$.

Working with bundle coordinates (z, w) on $\pi^{-1}(V)$ we may write $\Psi(z,\xi) = (z, \psi(z,\xi))$, $h = e^{u(z)}\,|dw|$ to obtain

$$\begin{aligned} 0 &= \partial\overline{\partial}\log\left(e^{2u(z)}\,|\psi(z,\xi)-\gamma(z)|^2\right) \\ &= 2\,\partial\overline{\partial}u + \partial\overline{\partial}\log(\psi-\gamma) + \partial\overline{\partial}\log\overline{(\psi-\gamma)} \end{aligned}$$

for each fixed ξ. It follows that

$$\begin{aligned} 0 &= 2\,\partial\overline{\partial}\left(\frac{d}{d\xi}u\big|_{\xi=0}\right) + \partial\overline{\partial}\left(\frac{d}{d\xi}\log(\psi-\gamma)\big|_{\xi=0}\right) + \partial\overline{\partial}\left(\frac{d}{d\xi}\log\overline{(\psi-\gamma)}\big|_{\xi=0}\right) \\ &= 0 + \partial\overline{\partial}(\eta/(\nu-\gamma)) + 0, \end{aligned}$$

as required. □

(3)⇒(4): We may locally represent $\eta/(\nu-\gamma)$ as $f-\overline{g}$ with f, g holomorphic, $f-\overline{g}$ non-vanishing. Thus

$$\nu - \gamma = \frac{\eta}{f-\overline{g}}$$

and

$$\omega = -\overline{\partial}(\nu-\gamma) = \frac{\eta\,d\overline{g}}{(f-\overline{g})^2}.$$

Moreover,

$$v \stackrel{\text{def}}{=} \log\|\nu-\gamma\| = \log h|\nu-\gamma| = \log\frac{h|\eta|}{|f-\overline{g}|}$$

is harmonic, so

$$h = |f-\overline{g}|\mu$$

where $\mu \stackrel{\text{def}}{=} e^{v}|\eta|^{-1}$ is a flat metric on L. □

(4)⇒(1): Suppose that in one of the hypothetical local representations the function g is constant. Then $\omega \equiv 0$ on the open set in question, and an analytic continuation argument shows $\omega \equiv 0$ globally, contrary to hypothesis. So g must be non-constant in each of the hypothetical local representations.

Let ν be a holomorphic section of A with $\log\|\nu-\gamma\|$ harmonic on a connected open set $V \subset X$ on which ω and h admit the prescribed representations. Since

$$\overline{\partial}\frac{-\eta}{f-\overline{g}} = \omega = \overline{\partial}\gamma$$

we have that

$$\rho \stackrel{\text{def}}{=} \nu - \gamma - \frac{\eta}{f-\overline{g}}$$

is a holomorphic section of L.

Thus

$$\begin{aligned}\|\nu-\gamma\| &= h\cdot|\nu-\gamma| \\ &= |f-\overline{g}|\mu\cdot\left|\rho+\frac{\eta}{f-\overline{g}|}\right| \\ &= \mu\,|(f-\overline{g})\rho+\eta|\end{aligned}$$

On the set

$$V' = \{\zeta\in V : \rho(\zeta)\neq 0\}$$

we have

$$\|\nu-\gamma\| = \mu|\rho|\,|(f-\overline{g})+\eta/\rho|\,.$$

Since g is non-constant, the set

$$V'' \stackrel{\text{def}}{=} \{\zeta\in V' : \frac{\eta}{\rho}(\zeta)+f(\zeta)\neq\overline{g(\zeta)}\}$$

is dense in V'. On V'' we have

$$\begin{aligned}0 &= \partial\overline{\partial}\log\|\nu-\gamma\| \\ &= \frac{1}{2}\partial\overline{\partial}\log\left(\frac{\eta}{\rho}+f-\overline{g}\right)+\frac{1}{2}\partial\overline{\partial}\log\overline{\left(\frac{\eta}{\rho}+f-\overline{g}\right)} \qquad \text{(since } \overline{\partial}\rho=0 \text{ and } \mu \text{ is flat)} \\ &= \frac{d\left(\frac{\eta}{\rho}+f\right)\wedge\overline{dg}}{2\left(\frac{\eta}{\rho}+f-\overline{g}\right)^2}+\frac{dg\wedge\overline{d\left(\frac{\eta}{\rho}+f\right)}}{2\left(\overline{\frac{\eta}{\rho}+f}-g\right)^2}\end{aligned}$$

so

$$\frac{d\left(\frac{\eta}{\rho}+f\right)\wedge\overline{dg}}{\left(\frac{\eta}{\rho}+f-\overline{g}\right)^2} = -\frac{dg\wedge\overline{d\left(\frac{\eta}{\rho}+f\right)}}{\left(\overline{\frac{\eta}{\rho}+f}-g\right)^2} \tag{8.1}$$

If $\frac{\eta}{\rho}+f$ is non-constant then we may apply $\partial\overline{\partial}\log$ to both sides of (8.1) to obtain

$$\frac{d\left(\frac{\eta}{\rho}+f\right)\wedge\overline{dg}}{\left(\frac{\eta}{\rho}+f-\overline{g}\right)^2} = \frac{dg\wedge\overline{d\left(\frac{\eta}{\rho}+f\right)}}{\left(\overline{\frac{\eta}{\rho}+f}-g\right)^2} \tag{8.2}$$

on a dense subset V''' of V''.

Combining (8.1) and (8.2) we find that

$$\frac{d\left(\frac{\eta}{\rho}+f\right)\wedge\overline{dg}}{\left(\frac{\eta}{\rho}+f-\overline{g}\right)^2}\equiv 0$$

on V''', hence on V''. Since g is non-constant, $\frac{\eta}{\rho}+f$ must be constant on V'', hence also on V'. Thus $\rho=\frac{\eta}{C-f}$ on V'; if ρ does not vanish identically then $\rho=\frac{\eta}{C-f}$ also on V (so in fact $V'=V$). Thus

$$\nu=\gamma+\frac{\eta}{f-\overline{g}}+\frac{\eta}{C-f}; \tag{8.3}$$

taking $C=\infty$ we obtain the remaining case $\rho\equiv 0, \nu=\gamma+\frac{\eta}{f-\overline{g}}$.

This describes the required foliation on $\widehat{A}\big|_V$, and the local uniqueness shows that these foliations patch together to give the required foliation on $\widehat{A}$. □

(4)⇒(5): $h^2|\omega|=\mu^2|\eta||d\overline{g}|$ is flat off of the zero set of dg. Straightforward computation (see (9.1) below) shows that $\mathcal{R}\overline{\omega}=\mathcal{R}(dg)=\mathcal{S}g$ so that [LaSo, Cor. 4.8] shows that $\mathcal{R}\overline{\omega}$ satisfies (LS). Condition (5c) holds by inspection. □

(5)⇒(4): Let $\zeta\in X$. By [LaSo, Cor. 4.8] there is a meromorphic function g defined near ζ with

$$\mathcal{R}\overline{\omega}=\mathcal{R}(dg)=\mathcal{S}g; \tag{8.4}$$

if $\mathcal{R}\overline{\omega}$ has a pole at ζ then the multiplicity of g at ζ is the integer k from (6.2). Post-composing g with a fractional linear transformation we may assume that $g(\zeta)=0$.

We focus first on the case where $\mathcal{R}\overline{\omega}$ has no pole at ζ. Then we may take g to be our local coordinate z. Fixing a local non-vanishing holomorphic section ξ of L we set $\omega=\lambda(z)\,d\overline{z}\otimes\xi$.

We wish to arrange that $(\log\overline{\lambda})_z(\zeta)\neq 0$. If this is not true we may replace g by $\frac{g}{1+g}$. We find then that $\lambda(z)$ is replaced by $(1+\overline{z})^{-2}\lambda(z)$ and that $(\log\overline{\lambda})_z$ is replaced by $-2(1+z)+(1+z)^2(\log\overline{\lambda})_z$ so that $(\log\overline{\lambda})_z(\zeta)$ is now non-zero.

With our choice of coordinate now fixed we have

$$\mathcal{R}\overline{\omega}=\mathcal{S}g=\mathcal{S}z\equiv 0$$

and so

$$\left(\frac{1}{(\log\overline{\lambda})_z}\right)_z=-\frac{(\log\overline{\lambda})_{zz}}{(\log\overline{\lambda})_z^2}=-\frac{1}{2}.$$

Thus

$$\frac{1}{(\log\overline{\lambda})_z}=-\frac{z-\overline{f}}{2}$$

with f holomorphic, $f(p)\neq\overline{g}(p)$. This yields

$$(\log\overline{\lambda})_z=-\frac{2}{z-\overline{f}}$$

and hence

$$\log\overline{\lambda}=-2\log(z-\overline{f})+\overline{h}, \tag{8.5}$$

h holomorphic. Then

$$\omega = \frac{e^h \xi \, \overline{dg}}{(f - \overline{g})^2};$$

setting $\eta = e^h \xi$ we have the desired local representation for ω, and condition (5a) sets up the corresponding representation for h.

We turn now to the case where $\mathcal{R}\overline{\omega}$ has a pole at ζ, recalling that in this case g has a zero of multiplicity k at ζ, where k is the integer from (6.2).

Here g cannot serve as a coordinate at ζ but using g as a (non-univalent) coordinate z in a punctured neighborhood of ζ and replacing z by $\sqrt[k]{z}$ in (5c) to obtain

$$\lambda(z) = k^{-1} \varphi(z^{1/k}), \varphi(0) \neq 0 \tag{8.6}$$

we have

$$(\log \lambda)_{\overline{z}} = \mathcal{O}\left(z^{\frac{1}{k}-1}\right). \tag{8.7}$$

If $(\log \lambda)_{\overline{z}} \to 0$ as we approach ζ then replacing g by $\frac{g}{1+g}$ as before we may arrange that $(\log \lambda)_{\overline{z}}$ tends to a non-zero limit as we approach ζ; thus we may assume that

$$(\log \lambda)_{\overline{z}} \text{ does not approach } 0 \text{ at } \zeta. \tag{8.8}$$

Following our previous computation we set

$$f(z) = \begin{cases} -\overline{z} - \frac{2}{(\log \lambda)_{\overline{z}}} & \text{for } (\log \lambda)_{\overline{z}} \neq 0 \\ \infty & \text{for } (\log \lambda)_{\overline{z}} = 0. \end{cases}$$

so that

$$\frac{1}{f(z)} = \begin{cases} -\frac{(\log \lambda)_{\overline{z}}}{2 + \overline{z}(\log \lambda)_{\overline{z}}} & \text{for } (\log \lambda)_{\overline{z}} \neq 0 \\ 0 & \text{for } (\log \lambda)_{\overline{z}} = 0. \end{cases}$$

Our earlier work shows that the continuous function $1/f$ is holomorphic where it is non-zero, so Radó's Theorem [Nar, 11.8] shows that $1/f$ is in fact holomorphic in a deleted neighborhood of p. In view of (8.7) and (8.8), after shrinking our neighborhood we may assume that f is holomorphic in a neighborhood of ζ with $f \sim cg^{j/k}$ for some $c \neq 0$ and some integer $0 \leq j \leq k-1$. Consequently we may also assume that $f \neq \overline{g}$ except perhaps at ζ.

Continuing on to (8.5) we find that we must replace h by $h + \frac{2j}{k} \log g$ to ensure that h is single-valued near ζ. We thus have

$$\omega = \frac{g^{2j/k} e^h \xi \, \overline{dg}}{(f - \overline{g})^2}$$

with h holomorphic in a punctured neighborhood of ζ. In view of (8.6), h must have a removable singularity at ζ. Letting $\eta = e^h\xi$ as before we find that

$$\omega = \frac{g^{2j/k}\eta\,\overline{dg}}{(f-\overline{g})^2}$$

The smoothness of φ in condition (5c) implies that of

$$\frac{f}{g^{j/k}} - \sqrt{\frac{\eta\,d\overline{g}}{\omega}} = \frac{\overline{g}}{g^{j/k}};$$

differentiating k times it follows that $j = 0$; thus $f \neq \overline{g}$ at ζ.

As before, the local representation for ω and condition (5a) induces the corresponding representation for h. □

9. More proofs

Proof of Addendum to Theorem 14. Comparing the formula (8.3) describing $\mathcal{F}$ to the definitions in §§6.1 and 6.2 we find that

$$\begin{aligned}\nu_\zeta &= \gamma + \frac{\eta}{f-\overline{g}} + \frac{\eta}{f(\zeta)-f}\\ \operatorname*{Res}_\zeta \nu_\zeta &= -\frac{\eta}{df}\\ \kappa_{\mathcal{F}} &= \frac{2i\,df}{\eta}\\ \mathcal{S}_{\mathcal{F}} &= \mathcal{S}f.\end{aligned}$$

On the other hand, from the representation in condition (4) of Theorem 14 we have

$$\begin{aligned}\partial\overline{\partial}\log\omega &= -2\frac{df\wedge d\overline{g}}{(f-\overline{g})^2}\\ \kappa_\omega &= 2i\frac{df}{\eta}\\ \mathcal{R}(\partial\overline{\partial}\log\omega) &= \mathcal{S}f.\end{aligned}$$

(The computation of $\mathcal{R}(\partial\overline{\partial}\log\omega)$ is facilitated by taking f to be the coordinate function z – permissible away from critical points of f – leading to

$$\mathcal{R}(\partial\overline{\partial}\log\omega) = \frac{2}{(f-\overline{g})^2} - \frac{1}{2}\left(\frac{-2}{f-\overline{g}}\right)^2 = 0 = \mathcal{S}f \tag{9.1}$$

as claimed.)

Thus everything matches. □

Lemma 15. *If ω is a non-vanishing section of $L\otimes T^{*(0,1)}(X)$ then $\mathcal{R}\overline{\omega}$ is holomorphic if and only if κ_ω is holomorphic.*

Proof. This follows from the easily-checked identity

$$\left(\overline{\mathcal{R}\overline{\omega}}\right)_z = i\omega\left(\kappa_\omega\right)_{\overline{z}}. \tag{9.2}$$

□

Proof of Theorem 13. This is now immediate from conditions (1) and (5) from Theorem 14 combined with Lemma 15. □

Proof of Theorem 7 and Addendum to Theorem 7. If h is flat then in local coordinates we have $h = e^{\mathrm{Re}\, f(z)}\,|dw|$, f holomorphic. The hypersurface is given by $e^{\mathrm{Re}\, f(z)}|w-\gamma(z)| = 1$, the leaves of the Levi-foliation are defined by equations of the form $e^{f(z)}\left(w-\gamma(z)\right) = e^{i\theta_0}$, and leaves of the extended foliation are given by equations of the form $e^{f(z)}\left(w-\gamma(z)\right) = C$. If ν is holomorphic then the holomorphic function $e^{f(z)}\left(\nu(z)-\gamma(z)\right)$ will be constant if and only if $\|\nu-\gamma\| = \left|e^{f(z)}\left(\nu(z)-\gamma(z)\right)\right|$ is constant.

If h is not flat then $h^2\overline{\omega}$ is holomorphic so that condition (5a) of Theorem 14 holds. Also, away from zeros of ω we have $0 = \partial\overline{\partial}\log h^2\omega$ so that

$$\begin{aligned}\partial\overline{\partial}\log\omega &= -2\partial\overline{\partial}\log h\\ &= \Theta\\ &= 2h^2\omega\wedge\overline{\omega}\end{aligned}$$

and $\kappa_\omega = -2ih^2\overline{\omega}$ is holomorphic. By Lemma 15, $\mathcal{R}\overline{\omega}$ is holomorphic away from zeroes of ω.

The fact that $h^2\overline{\omega}$ is holomorphic also easily implies that condition (5c) holds and that $\mathcal{R}\overline{\omega}$ has poles at zeroes of ω.

Consulting Theorem 13 we see that the condition (LHM) provides the desired extended foliation away from zeroes of ω. By the Addendum to Theorem 14 we have $\kappa_{\mathcal{F}} = \kappa_\omega = -2ih^2\overline{\omega}$ and $\mathcal{S}_{\mathcal{F}} = \mathcal{R}\left(\partial\overline{\partial}\log\omega\right) = \mathcal{R}\left(2h^2\omega\wedge\overline{\omega}\right) = \mathcal{R}\left(\overline{\omega}\right)$.

If $\omega(z_0) = 0$ then picking distinct sections ν_1, ν_2, ν_3 near z_0 whose graphs are leaves of the original Levi-foliation we have from Lemma 9 that $\mathcal{R}\left(\overline{\omega}\right) = \mathcal{S}_{\mathcal{F}} = \mathcal{S}\left(\frac{\nu_2-\nu_3}{\nu_1-\nu_3}\right)$ so that $\mathcal{R}\left(\overline{\omega}\right)$ satisfies condition (LS) from §6.2. Thus condition (5) of Theorem 14 holds, so condition (1) must hold as well. □

References

[AnLe] E. Andersén and L. Lempert, *On the group of holomorphic automorphisms of* $\mathbb{C}^n$, Invent. Math. **110** (1992), 371–388.

[Bar] D. Barrett, *On a class of conformal metrics arising in work of Seiberg and Witten,* Math. Zeit. **233** (2000), 149-164.

[Ber] B. Berndtsson, *Levi-flat surfaces with circular sections*, Several Complex Variables, Mathematical Notes Vol. 38, Princeton University Press, 1993, pp. 136-159.

[CaLN] C. Camacho and A. Lins Neto, *Geometric theory of foliations,* Birkhäuser, 1985.

[For] O. Forster, *Lectures on Riemann surfaces,* Springer-Verlag, 1981.

[GrHa] P. Griffiths and J. Harris, *Principles of algebraic geometry,* John Wiley & Sons, 1978.

[HuTr] D. Hulin and M. Troyanov, *Prescribing curvature on open surfaces,* Math. Ann. **293** (1992), 277-315.

[Kra] S. Krantz, *Function theory of several complex variables (2nd ed.)*, Wadsworth & Brooks/Cole, 1992.

[LaSo] I. Laine and T. Sorvali, *Local solutions of* $w'' + A(z)w = 0$ *and branched polymorphic functions,* Results in Mathematics **10** (1986), 107-129.

[Leh] O. Lehto, *Univalent functions and Teichmüller spaces,* Springer-Verlag, 1987.

[Nar] R. Narasimhan, *Complex analysis in one variable,* Birkhäuser, 1985.

[Roc] R. Rochberg, *Interpolation of Banach spaces and negatively curved vector bundles*, Pacific J. Math. **110** (1984), 355-376.

[SuTh] D. Sullivan and W. Thurston, *Extending holomorphic motions,* Acta Math. **157** (1986), 243-257.

Dept. of Mathematics, University of Michigan, Ann Arbor, MI 48109-1109 USA
E-mail address: barrett@umich.edu