A “floating body” approach to Fefferman’s surface area

David Barrett

Chapel Hill

25 October 2003

Plan of talk

(1) Real affine geometry

(2) Complex analogue

(3) My contribution

(4) Questions / further directions

Plane curves

$\gamma : I \longrightarrow \mathbb{R}^2$ smooth

$$ds = |\gamma'(t)|\, dt$$

$$\kappa = \frac{\omega^{\text{area}}\,(\gamma'(t),\gamma''(t))}{|\gamma'(t)|^3}$$

$$\int_\gamma \kappa^p\, ds = \dots$$

$$\int_\gamma \sqrt[3]{\kappa}\, ds = \int_I \sqrt[3]{\omega(\gamma'(t), \gamma''(t))}\, dt \overset{\text{def}}{=} \text{Aff}(\gamma)$$

$=$ affine arc length of γ

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ affine

$$\Rightarrow \text{Aff}(T(\gamma)) = \det{}^{1/3}(T) \cdot \text{Aff}(\gamma)$$

Convex hypersurfaces in $\mathbb{R}^n$

$S^{\text{smooth}} \subset \mathbb{R}^n$

$$\text{Aff}(S) = \int_S \kappa^{\frac{1}{n+1}}\, dS = \text{aff. surf. area of } S$$

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ affine

$$\Rightarrow \text{Aff}(T(S)) = \det^{\frac{n-1}{n+1}}(T) \cdot \text{Aff}(S)$$

What if S convex but not smooth?

Use "convex floating bodies" –

$K \subset \mathbb{R}^n$ convex body; $\delta > 0$

$$K_\delta \stackrel{\text{def}}{=} \bigcup_{\substack{H \text{ open half-space} \\ \text{vol}(K \cap H) \leq \delta}} K \cap H$$

$K \setminus K_\delta$ = convex floating body

Theorem [Blaschke / Leichtweiss / Schütt-Werner] ***If bK is C^2 then***

$$\mathrm{Aff}(\mathrm{b}K) = \lim_{\delta \searrow 0} c_n \frac{\mathrm{vol}(K_\delta)}{\delta^{2/(n+1)}}.$$

... **but how does this help with non-smooth case?**

Definition/Theorem [Schütt-Werner] ***For K a general convex body,***

$$\mathrm{Aff}(\mathrm{b}K) = \lim_{\delta \searrow 0} c_n \frac{\mathrm{vol}(K_\delta)}{\delta^{2/(n+1)}}$$
$$= \int_{\mathrm{b}K} \kappa_{\text{non-sing.}}^{\frac{1}{n+1}} \, dS.$$

What does Aff(bK) tell us?

- **Theorem** [Gruber]. *Let*

$$F_K(\nu) = \inf_{\substack{P^{polyhedron} \subset K \\ \#\, faces \leq \nu}} \mathrm{vol}(K \setminus P).$$

Then

$$F_K(\nu) \underset{\nu \to \infty}{\sim} c_n \,\mathrm{Aff}(bK)^{\frac{n+1}{n-1}} \cdot \nu^{-\frac{2}{n-1}}.$$

Theorem [Blaschke / Hug].
$\mathrm{Aff}(bK)^{\frac{n+1}{n-1}} \cdot \mathrm{vol}(K)^{-1}$ *maximal for ellipsoids.*

Corollary. *Round potatoes are hardest to peel.*

- **Theorem** [Bárány / Schütt].

$$\mathrm{Prob}\,(x_{\nu+1} \notin \mathrm{Hull}\{x_1, \ldots, x_\nu\}) \underset{\nu \to \infty}{\sim} c_n \,\mathrm{Aff}(\mathrm{b}K) \cdot \mathrm{vol}(K)^{\frac{1-n}{n+1}} \cdot \nu^{-\frac{2}{n+1}}.$$

Complex analysis

Is there a complex analogue of affine surface area?

$S \subset \mathbb{C}^n$, smooth, (str.) pseudoconvex

<u>Definition</u> [Fefferman, Adv. Math. '79].

$$\operatorname{Fef}(S) = c_n \int_S |\det \mathcal{L}|^{\frac{1}{n+1}}\, dS$$

T biholomorphic

$\Rightarrow$ integrand for $\operatorname{Fef}(T(S))$

$= \det^{\frac{2n}{n+1}}(T') \cdot$ integrand for $\operatorname{Fef}(S)$

$\Rightarrow$ corresponding Szegö kernels satisfy

$$S_\Omega(z,\zeta) = S_{\Phi(\Omega)}(\Phi(z),\Phi(\zeta)) \cdot \det^{\frac{n}{n+1}}\Phi'(z) \cdot \overline{\det^{\frac{n}{n+1}}\Phi'(\zeta)}$$

What about non-smooth pseudconvex domains?

$\longrightarrow$ alt. def's in the smooth case?

<u>Theorem</u> [B.]. *Let $\Omega \subset\subset \mathbb{C}^n$ be str. ψ-convex, $b\Omega$ C^3. For $M > 0$ let $P_M(\Omega)$ be the set of holo. fcns. h on Ω s.t.*

(1) $\|h\|_{C^3(\overline{\Omega})} \leq M$;

(2) $\emptyset \neq \overline{\Omega} \cap h^{-1}(0) \subset b\Omega$;

(3) $|dh| \geq M^{-1}$ *on* $\overline{\Omega} \cap h^{-1}(0)$.

Let $\Omega_{M,\delta} = \bigcup_{\substack{h \in P_M(\Omega),\, \eta > 0 \\ \mathrm{vol}\{|h| < \eta\} < \delta}} \{|h| < \eta\}$.

Then for M large we have

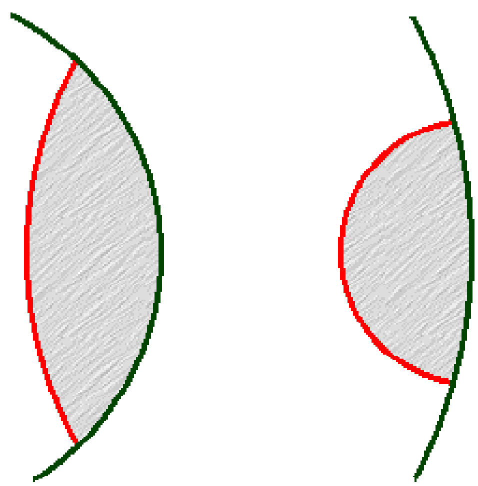
$$\mathrm{Fef}(b\Omega) = c_n \lim_{\delta \to 0} \frac{\mathrm{vol}(\Omega_{M,\delta})}{\delta^{1/(n+1)}}.$$

Quasi-invariance of $P_M(\Omega)$:

if $G : \overline{\Omega}_1 \longrightarrow \overline{\Omega}_2$ a C^3 holo. diffeo. $\Rightarrow$ for $M > 0$ there are $M_\sharp > M_\flat > 0$ s.t. $P_{M_\flat}(\Omega_2) \circ G \subset P_M(\Omega_1) \subset P_{M_\sharp}(\Omega_2) \circ G$.

Why $|h|$ rather than $\operatorname{Re} h$?

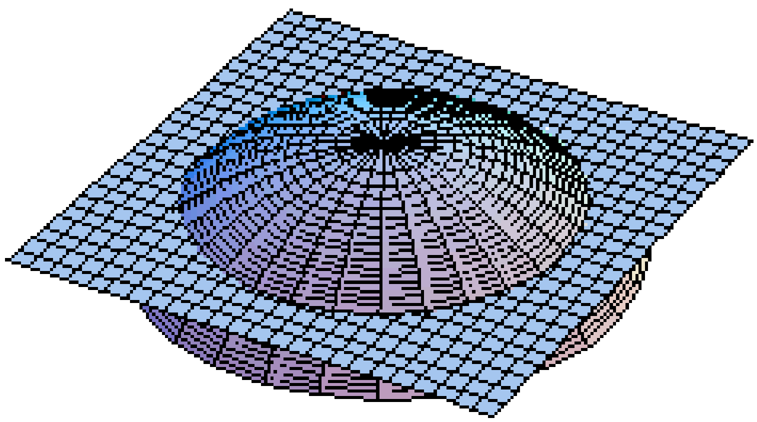
- **$n = 1$:**

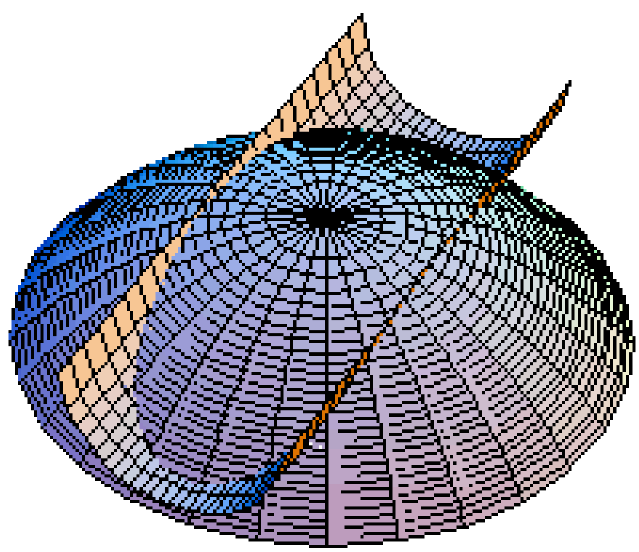


$\Omega_{M,\delta} \approx \sqrt{\frac{2\delta}{\pi}}$–collar of b$\Omega$, so

$$\frac{\mathrm{vol}(\Omega_{M,\delta})}{\sqrt{\delta}} \longrightarrow \sqrt{\frac{2}{\pi}}\,\ell(\mathrm{b}\Omega) = c_1\,\mathrm{Fef}(\mathrm{b}\Omega)$$

- $n > 1$:





Sketch of proof (n=2):

- After vol.-pres. change of coords to $(z, w) = (z, u + iv)$, have

$$\begin{aligned} h(z, w) &= w \\ \Omega &= \{v > \lambda|z|^2 + \operatorname{Re} \mu z^2 + \dots\} \end{aligned}$$

- $\operatorname{vol}(\Omega \cap \{|w| < \eta\}) = \frac{2\pi}{3} \frac{\eta^3}{\sqrt{\lambda^2 - |\mu|^2}} + O(\eta^{\frac{7}{2}})$

- The case $\mu = 0$ occurs

- $\Omega_{M,\delta}$ has thickness $\approx C \sqrt[3]{\delta|\mathcal{L}|}$ $\square$

Questions / further directions

(1) More general domains

(2) Approx. by analytic polyhedra

(3) Random holomorphic hulls

(4) "Hausdorff-Levi" dimension

Example: $\Omega = \Delta \times \Delta$. **Then** $\lim_{\delta \to 0} \frac{\mathrm{vol}(\Omega_{M,\delta})}{\sqrt[3]{\delta}} = 0$ **but** $0 < \lim_{\delta \to 0} \frac{\mathrm{vol}(\Omega_{M,\delta})}{\sqrt{\delta}} < \infty$ **leads to "correct" Hardy space.**

(5) Isoperimetric inequality:

Is Fef $b\Omega \cdot \mathrm{vol}(\Omega)^{-n/(n+1)}$ **maximized by affine balls?**