Lorentzian curve straightening and analytic continuation

Purdue

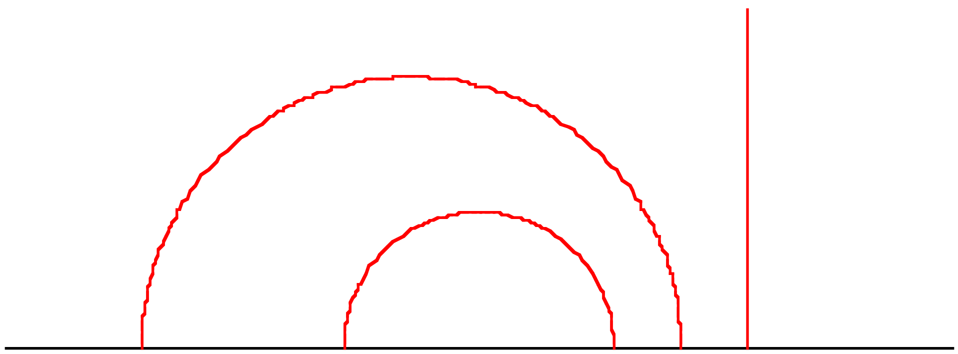
8 April 2002

Plan of talk:

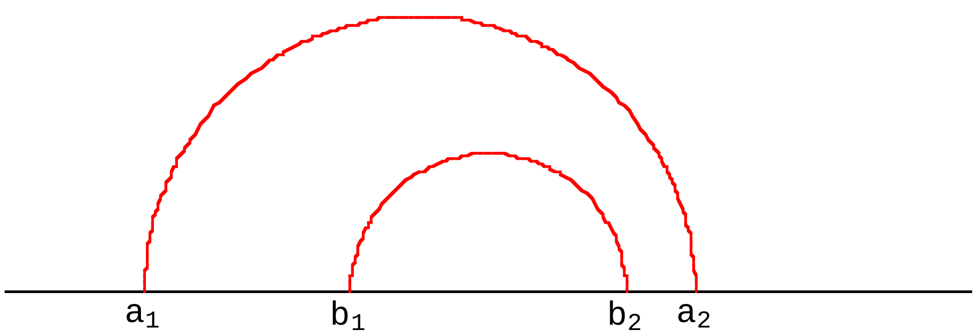
1) **Poincaré $\longrightarrow$ Lorentz**
2) **Straightening**
3) **The (2+1)-dimensional case**
4) **Analytic continuation**
5) **Hartogs problems**
6) **Complexified independent variable**

Starting point: Poincaré half-plane

- metric: $\frac{dx^2+dy^2}{y^2}$
- geodesics: circles $\perp \mathbb{R}$; vertical lines



Let's look for geometry on $\mathcal{G} \stackrel{\text{def}}{=} \{$ geodesics $\}$

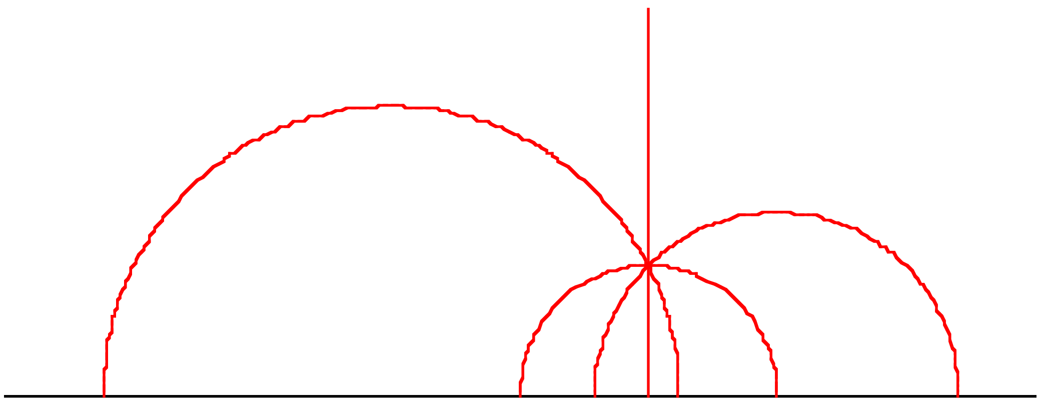


Natural invariant: cross-ratio $\gamma = \frac{(a_1-b_1)(a_2-b_2)}{(a_1-b_2)(a_2-b_1)}$
(Label so that $|\gamma| < 1$)

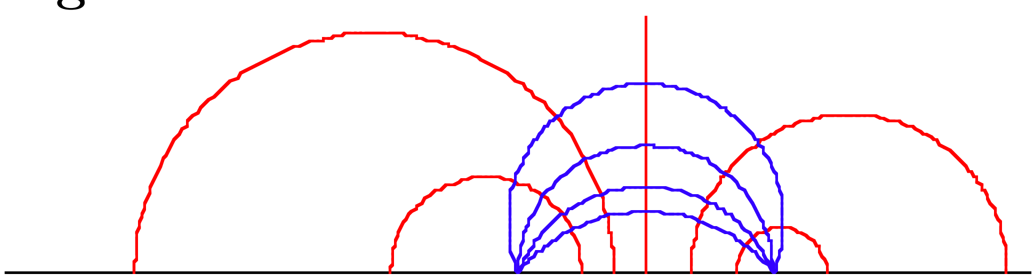
Use $\rho \stackrel{\text{def}}{=} -\left(\log\left(\frac{1-\sqrt{\gamma}}{1+\sqrt{\gamma}}\right)\right)^2 \in \mathbb{R}$

Hessian of ρ at $b_j = a_j$ yields $4\dfrac{da_1\, da_2}{(a_2 - a_1)^2}$, a Lorentz metric!

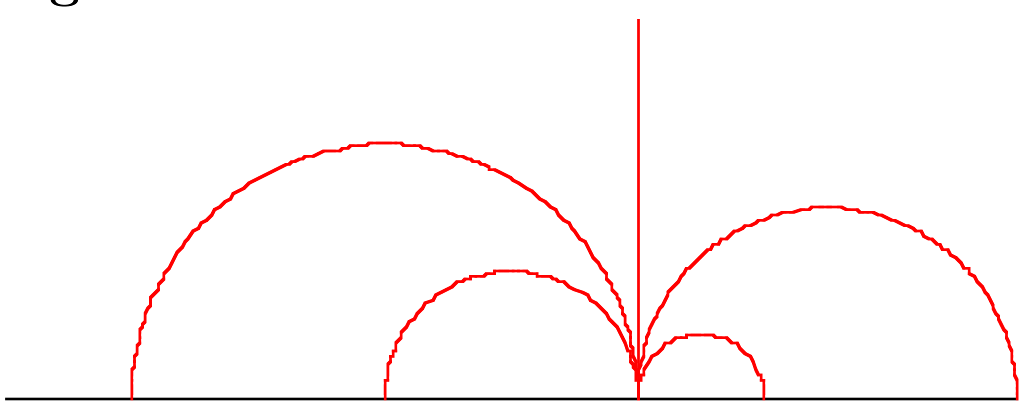
- spacelike geodesics:



- timelike geodesics:



- lightlike geodesics:



Facts about $\mathcal{G}$:

- **Geodesically complete**
- $a, b, \in \mathcal{G} \Rightarrow a, b$ **joined by geodesic (unique if timelike/lightlike)**
- $\rho(a,b) = \left(\int_{\text{geod. from } a \text{ to } b} \sqrt{4 \frac{da_1 da_2}{(a_2 - a_1)^2}} \right)^2$
- $\mathcal{G} = \{\text{unordered pairs in } S^1\} \overset{\text{top.}}{\sim}$ **Möbius band**
- $\mathcal{G}$ **not time-orientable**
- $\pi_1(\mathcal{G}) = \mathbb{Z}$
- **Do *not* get geodesic in each homotopy class (except in spacelike case)**
- **Geodesics don't separate** $\mathcal{G}$
- $\mathcal{G}$ **(or double cover) = 2-dim. deSitter space**

$$\begin{aligned} \{\text{geodesics in } \mathcal{G}\} &\sim \text{ Poin. half-plane} \cup \mathcal{G} \cup S^1 \\ &\sim \text{ real projective plane} \end{aligned}$$

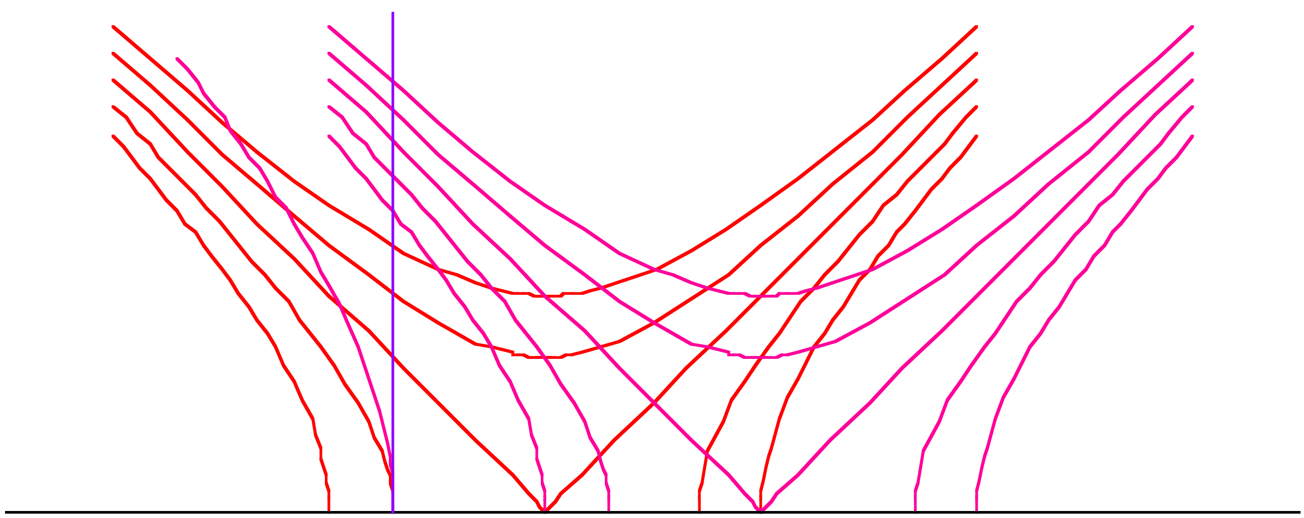
Let $\mathcal{G}' = \mathcal{G} \setminus \{$ vertical lines $\}$.

New coordinates:

$$(x, y) = \text{ top of semi-circle}$$

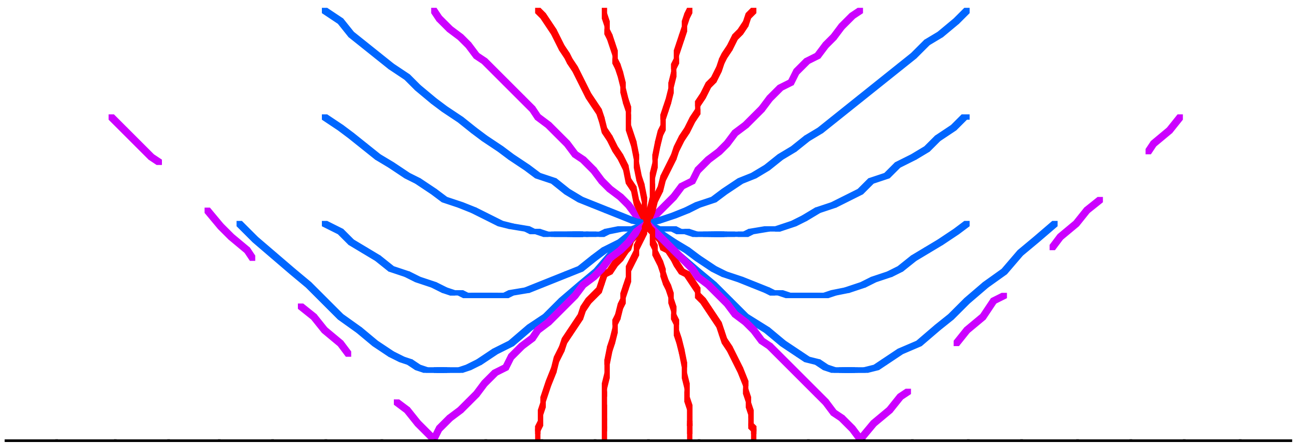
Get:

- **metric** $\frac{dx^2 - dy^2}{y^2}$
- **geodesics:**
  equilateral hyperbolas $\perp \mathbb{R}$; vertical lines



Metric is:

- **incomplete "as $y \to \infty$"**
- **complete "for y bounded"**
- **time-oriented (forward = up)**



$(x_1, y_1), (x_2, y_2)$ **joined by**

- **unique timelike geodesic if** $|x_2 - x_1| < |y_2 - y_1|$
- **unique lightlike geodesic if** $|x_2 - x_1| = |y_2 - y_1|$
- **unique spacelike geodesic if** $|y_2 - y_1| < |x_2 - x_1| < y_1 + y_2$
- ***no* geodesic if** $|x_2 - x_1| \geq y_1 + y_2$

Geodesics *do* separate $\mathcal{G}'$ into "half-spaces"

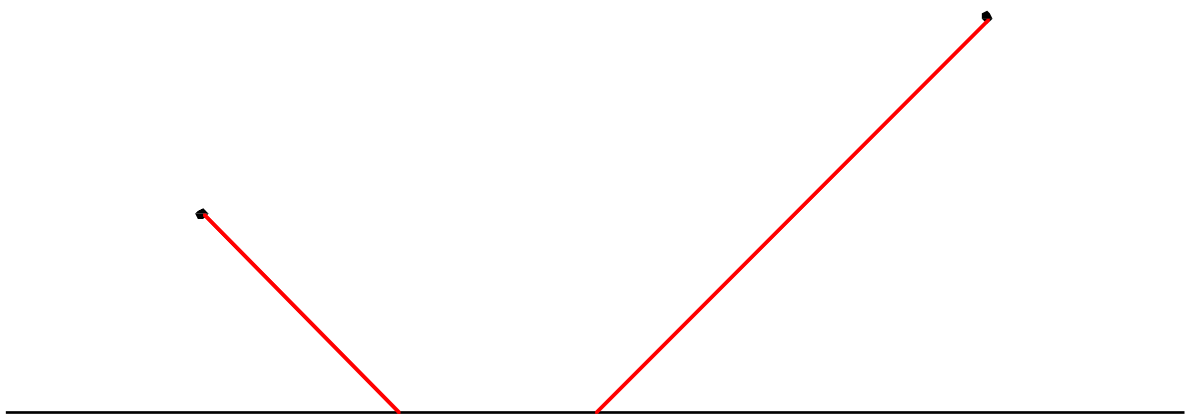
"Hahn-Banach Theorem." **Let $S \subset\subset \mathcal{G}'$. Then the following are equivalent:**

(A) S **is an intersection of half-spaces**
(B) S **contains geodesic joining any two of its points**

$\widehat{S} \stackrel{\text{def}}{=} \bigcap$ of half-spaces containing S

Example. Hull of two points.

$\widehat{\{(x_1, y_1), (x_2, y_2)\}} =$

- geodesic segment if $|x_2 - x_1| < y_1 + y_2$
- 

if $|x_2 - x_1| \geq y_1 + y_2$

Next topic:
"straightening" of (parameterized) curves

I) Smooth curves in $\mathbb{R}^2$

Could call a smooth (endpoint-fixing) homotopy $\{\gamma_t\}$ a *straightening* if

$$\frac{\partial}{\partial t}\gamma_t(s) = \mu(s,t)\left(\frac{\partial}{\partial s}\right)^2 \gamma_t(s), \quad \mu(s,t) \geq 0.$$

Proposition. ***Every smooth curve can be straightened to an affine map.***

Proof. **Use $\mu(s,t) = \frac{1}{1-t}$.** □

II) Polygonal arcs in $\mathbb{R}^2$

Say that a straightening is a sequence of moves in which two adjacent segments are replaced by a single segment.

Of course all polygonal arcs can be straightened to line segments.

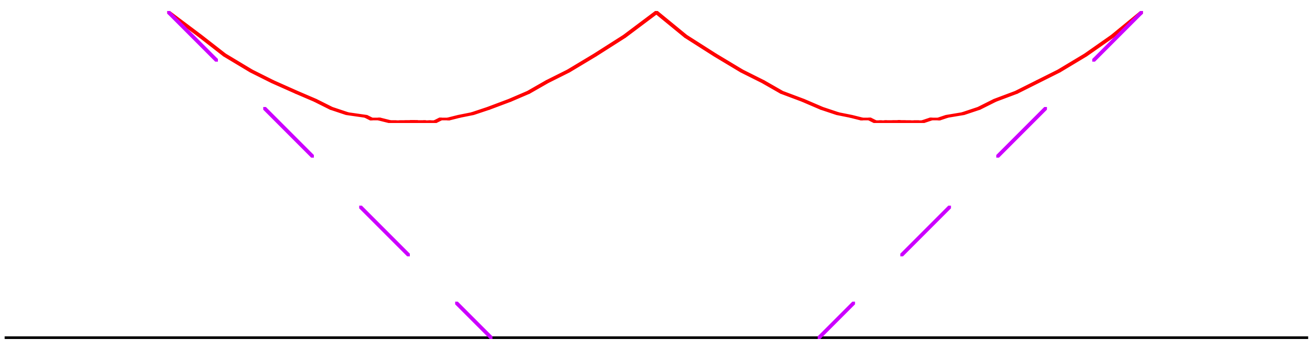
III) Broken geodesics in Riemannian manifold

Same idea, but insist on staying in same homotopy class.

All broken geodesics can be straightened *if* ambient manifold is complete.

IV) Broken geodesics in $\mathcal{G}'$

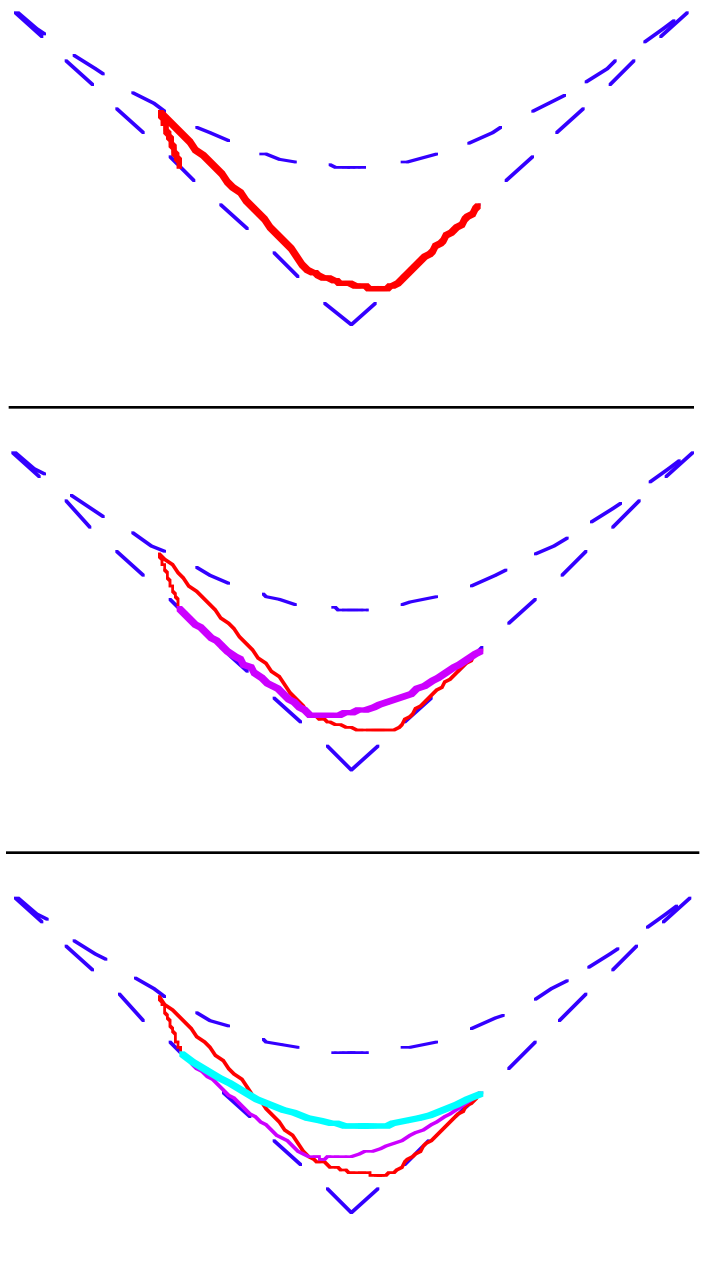
Clearly cannot straighten γ if there is no geodesic joining the endpoints.



What positive results *can* we obtain?

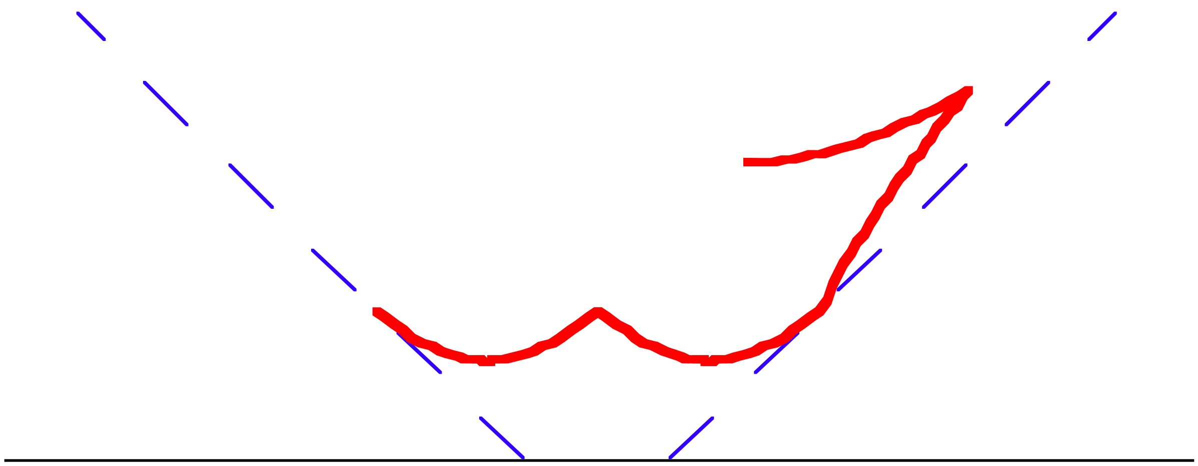
THEOREM A. ***Suppose γ takes values in $K^{compact\ convex} \subset \mathcal{G}'$. Then γ can be straightened.***

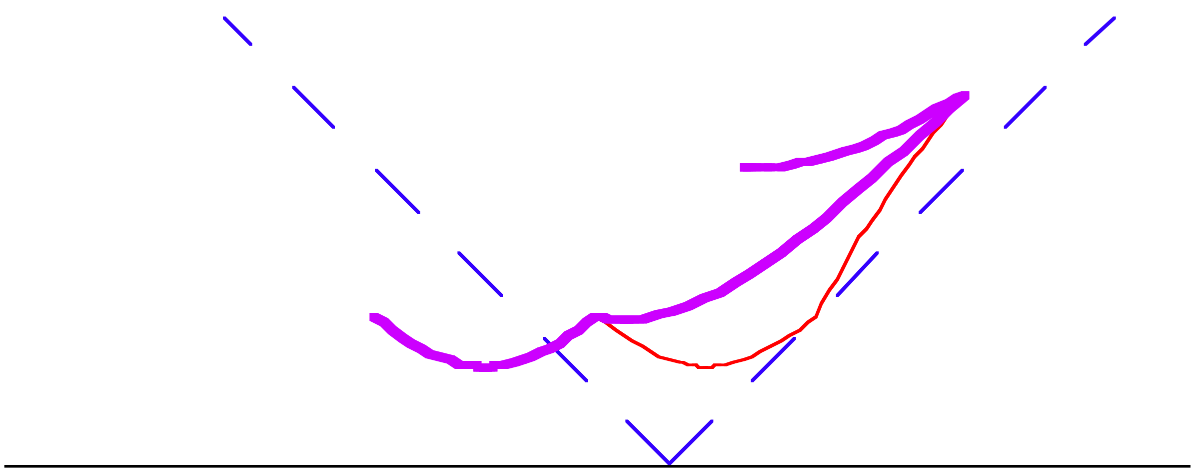
Proof.

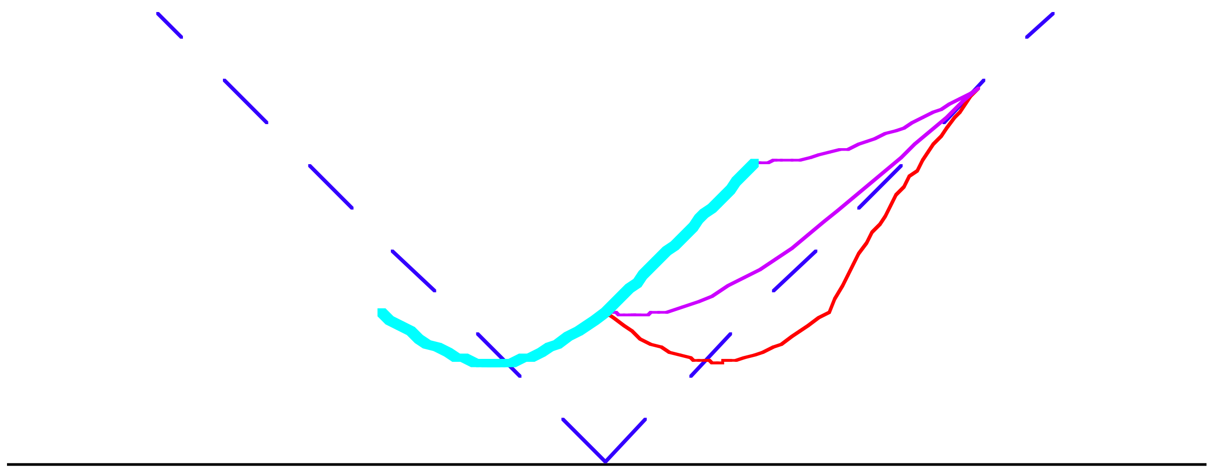


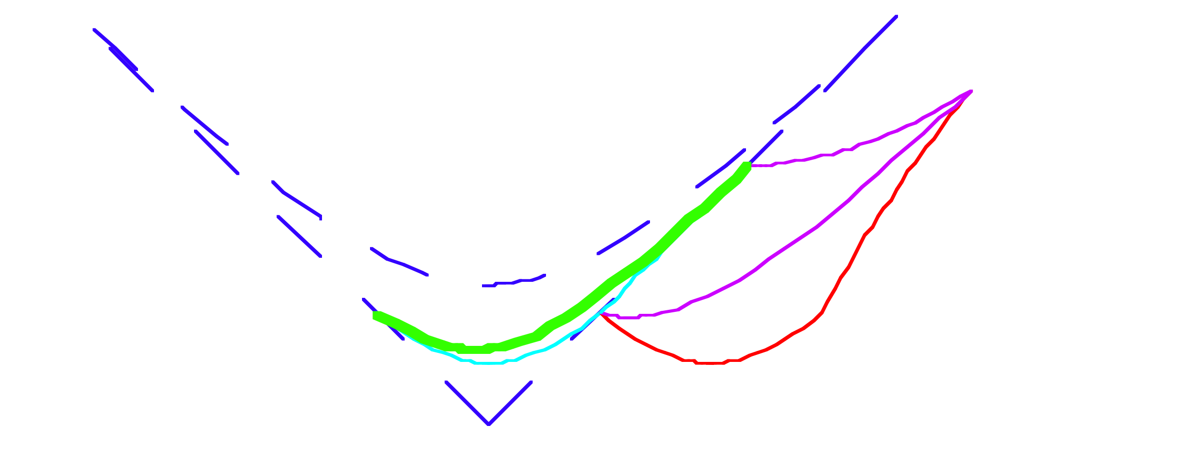
Theorem B. ***Suppose endpoints of γ can be joined by geodesic (i.e., endpoints have compact hull). Then γ can be straightened.***

Proof.









Up one dimension:

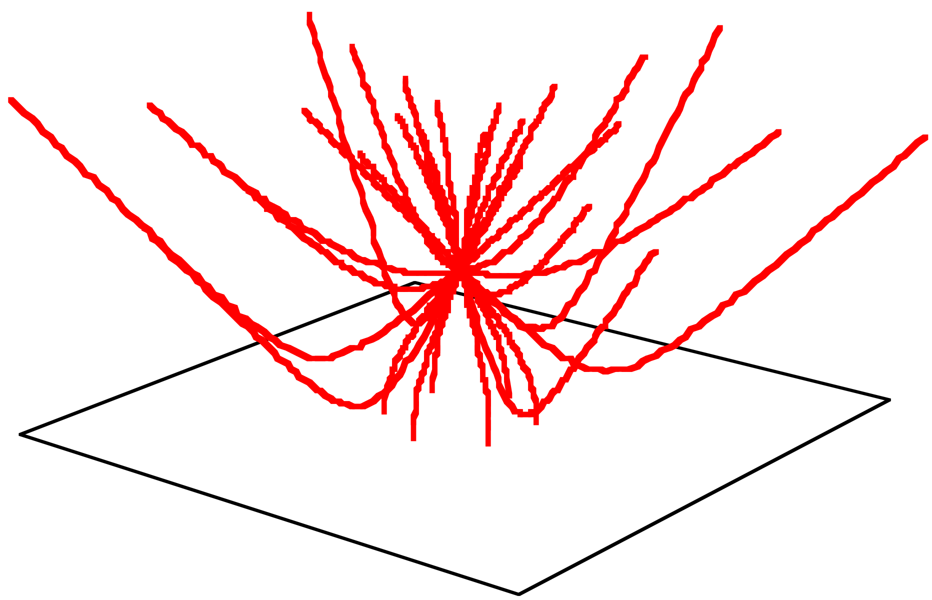
$\mathcal{C} = \{$ circles/lines in $\mathbb{C}\}$

$\mathcal{C}' = \{$ genuine circles in $\mathbb{C}\}$

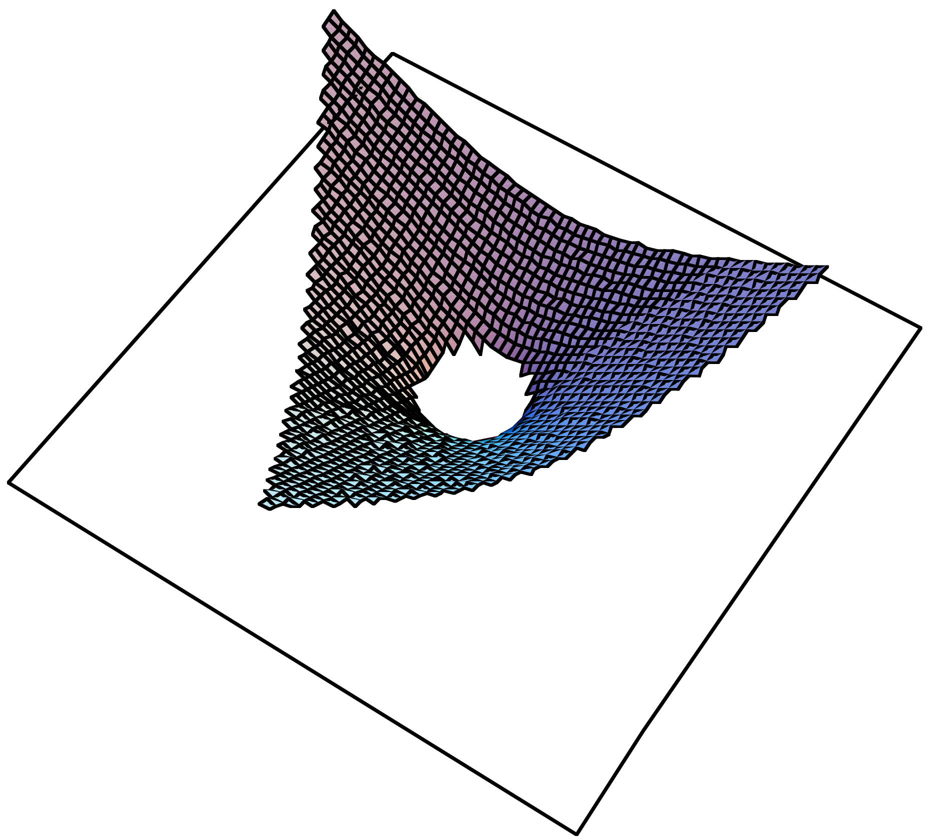
coordinates on $\mathcal{C}'$: (x, y, r)

metric $\frac{dx^2+dy^2-dr^2}{r^2}$ on $\mathcal{C}'$ extends to Möbius-invariant metric on $\mathcal{C}$

Geodesics through $(0, 0, 1)$:



An important difference from 2-dim.:
hull of triangle may not be compact.



Insist that each straightening move involve three consecutive vertices with compact hull.

Theorem A carries over; Theorem B does not.

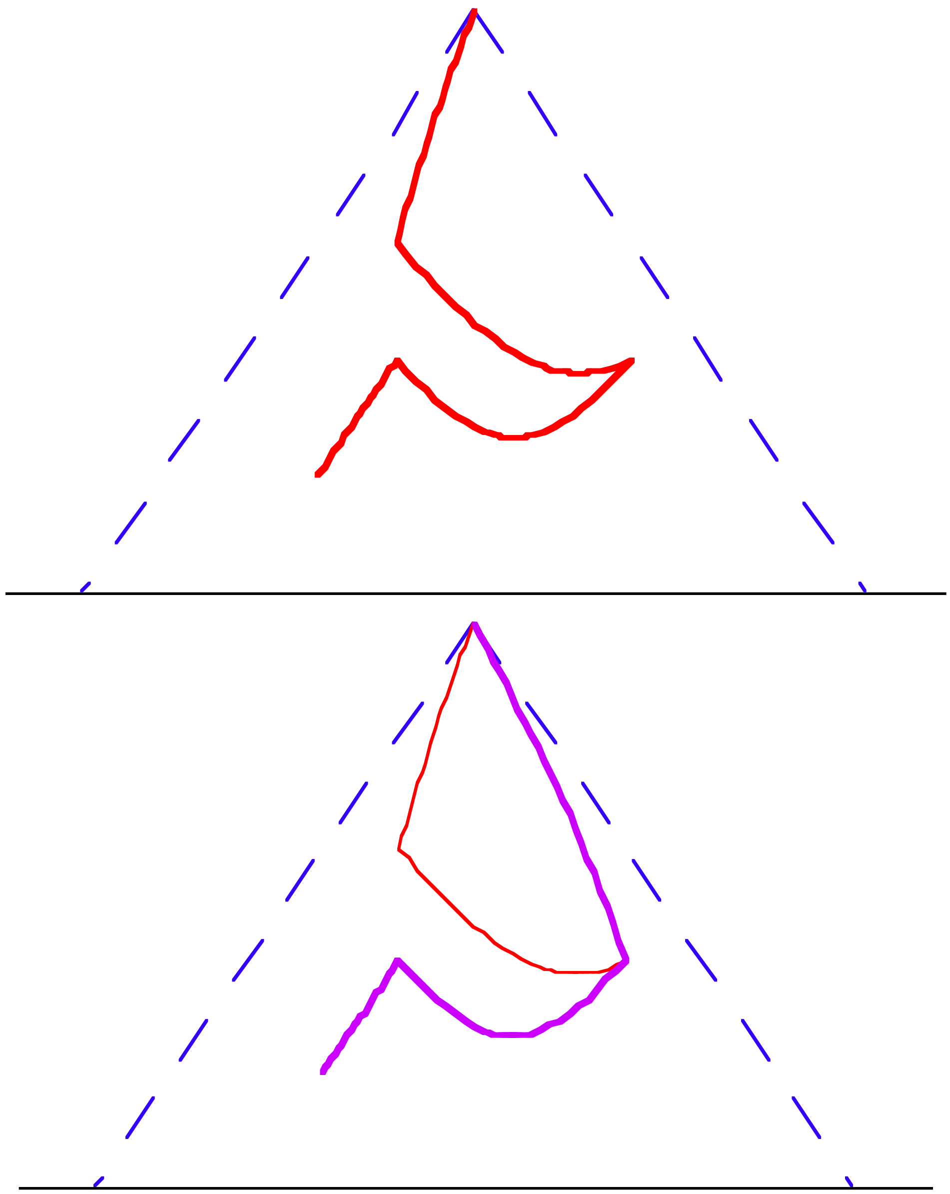
Combinatorial hypothesis on

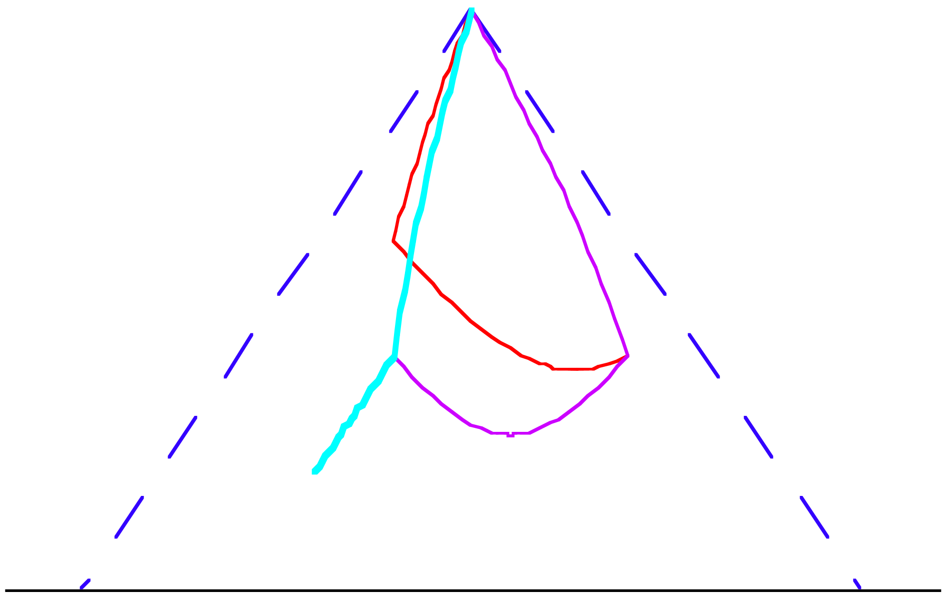
$$\mathcal{S} = \{S \subset \{v_0, \ldots, v_N\} : \widehat{S} \text{ compact}\}$$

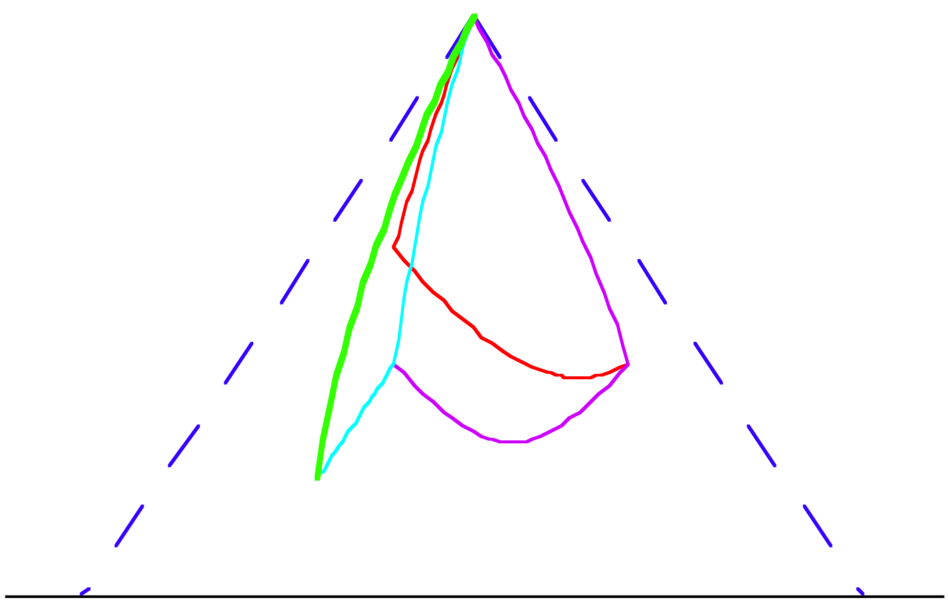
will guarantee straightening.

Theorem C. ***Suppose that γ lies in backwards light cone of initial vertex. Then γ can be straightened.***

Proof. **Always eliminate second vertex.**







Analytic continuation (finally):

Given $\gamma = (\gamma_X, \gamma_Y, \gamma_R) : [\alpha, \beta] \to \mathfrak{C}'$, let

$$\Omega_\gamma = \{(z_1, z_2) \in \mathbb{C}^2 : \alpha < \operatorname{Re} z_1 < \beta, \\ |z_2 - (\gamma_X + i\gamma_Y)(\operatorname{Re} z_1)| < \gamma_R(\operatorname{Re} z_1)\}.$$

PROPOSITION. ***If γ_1 can be straightened to γ_2 then every holomorphic function on Ω_{γ_1} can be analytically continued to Ω_{γ_2}.***

QUESTION: ***Is there a converse?***

If acceleration at corner is $\left\{\begin{array}{c}\text{forward timelike}\\ \text{backwards timelike}\\ \text{spacelike}\end{array}\right\}$

get $\left\{\begin{array}{c}\text{pseudoconcavity}\\ \text{pseudoconvexity}\\ \text{mixture}\end{array}\right\}$.

Straightening arguments $\rightsquigarrow$ analytic continuation theorems.

“Hartogs problem”

Given: $f : \overline{\Delta} \to \mathbb{C}$ cont., $f(\mathrm{b}\Delta) \subset \overline{\Delta}$

Ask: do all functions holo. in nbhd. of

$$\left(\mathrm{b}\Delta \times \overline{\Delta}\right) \cup \mathrm{graph}\, f$$

have analytic cont. to nbhd. of $\overline{\Delta} \times \overline{\Delta}$?

- (Hartogs)
  Yes for $f \equiv 0$, f holo.
- (Chirka(-Rosay))
  Yes for $f : \overline{\Delta} \to \overline{\Delta}$
- (Thm C. $+\epsilon$)
  Yes for $f(re^{i\theta}) = g(r)e^{ik\theta}$, $k \geq 0$
- (Thm B. $+\epsilon$)
  Yes for $f(re^{i\theta}) = g(r)e^{-ik\theta}$, g $\mathbb{R}$-valued
- (Thm C. $+\epsilon$)
  Yes for $f(re^{i\theta}) = g(r)e^{-ik\theta}$, $r^k|g(r)| \leq 1$
- (Alexander-Wermer)
  No for $f(re^{i\theta}) = g(r)e^{-i\theta}$,
  $g(r) = Mr(r^2 - 1)e^{ir^2}$, $M \geq 26$

To get more flexible results, complexify independent variable, study maps

$$\gamma : D_{(s,t)}^{\text{planar domain}} \to \mathcal{C}'$$

replacing

$$\Delta_t^{\mathcal{C}'}$$

by

$$\Delta_{s,t}^{\mathcal{C}'} + \frac{2}{r} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{\partial \gamma}{\partial s} \times \frac{\partial \gamma}{\partial t}$$

Issues:

- **Discretization**
- **Need revised ideas for straightening priorities**

EXTRA
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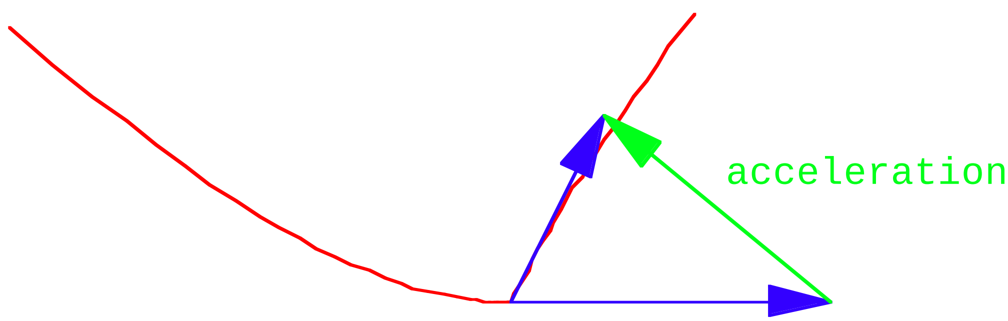
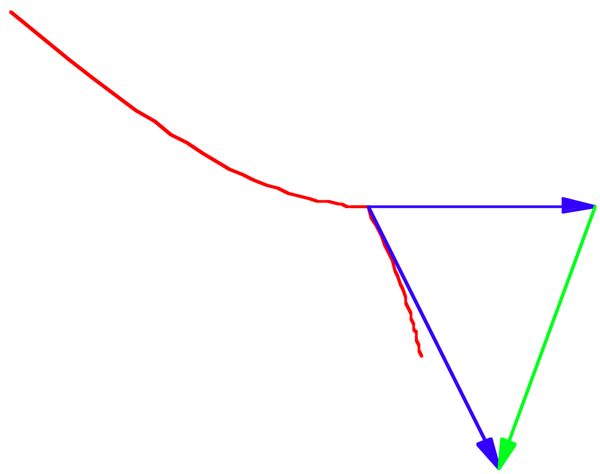
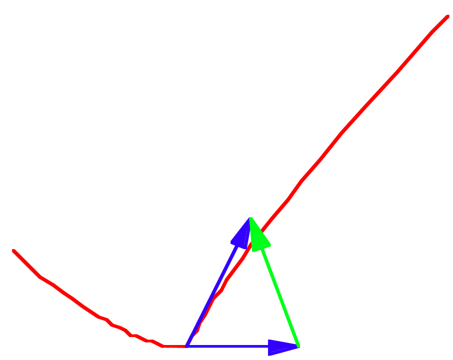
If γ is 2-piece broken geodesic then ψconvexity properties of Ω_γ depend on "acceleration":



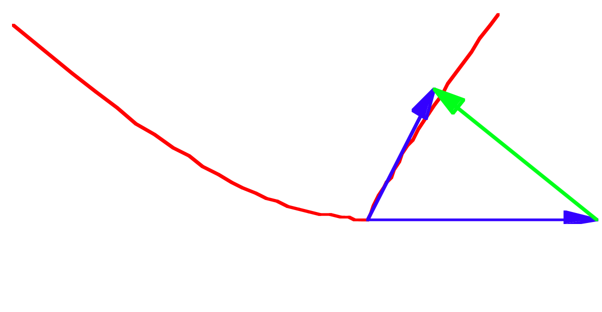
- **backward timelike/lightlike acceleration ⇒ ψconvex**



- **forward timelike/lightlike acceleration ⇒ ψconcave**

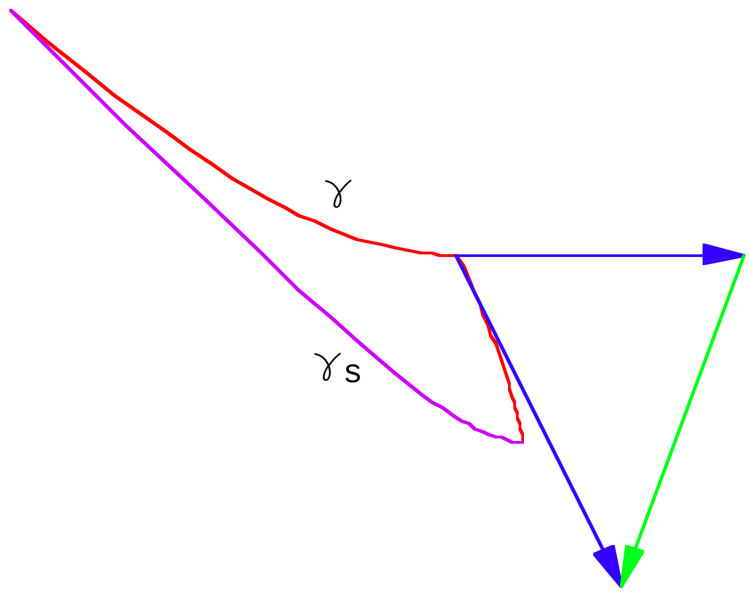


- **spacelike acceleration ⇒ mixed**

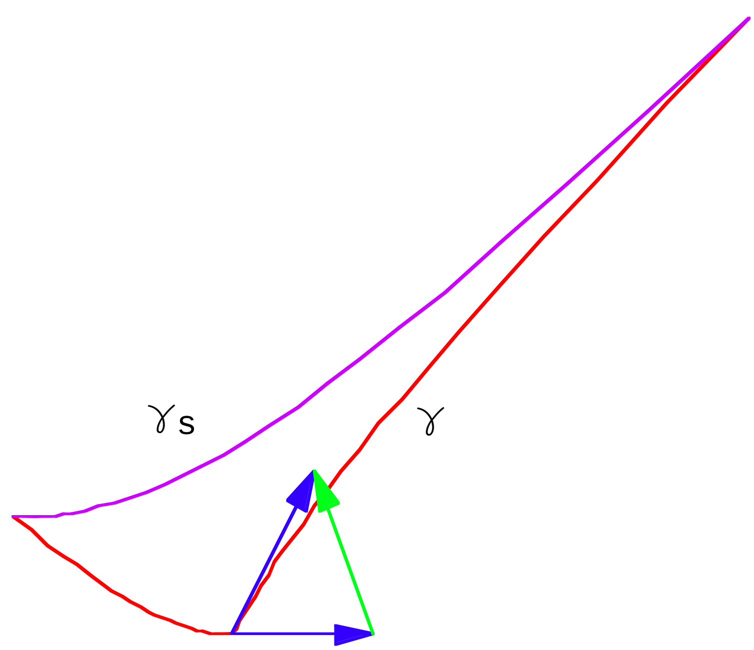


Straightening:

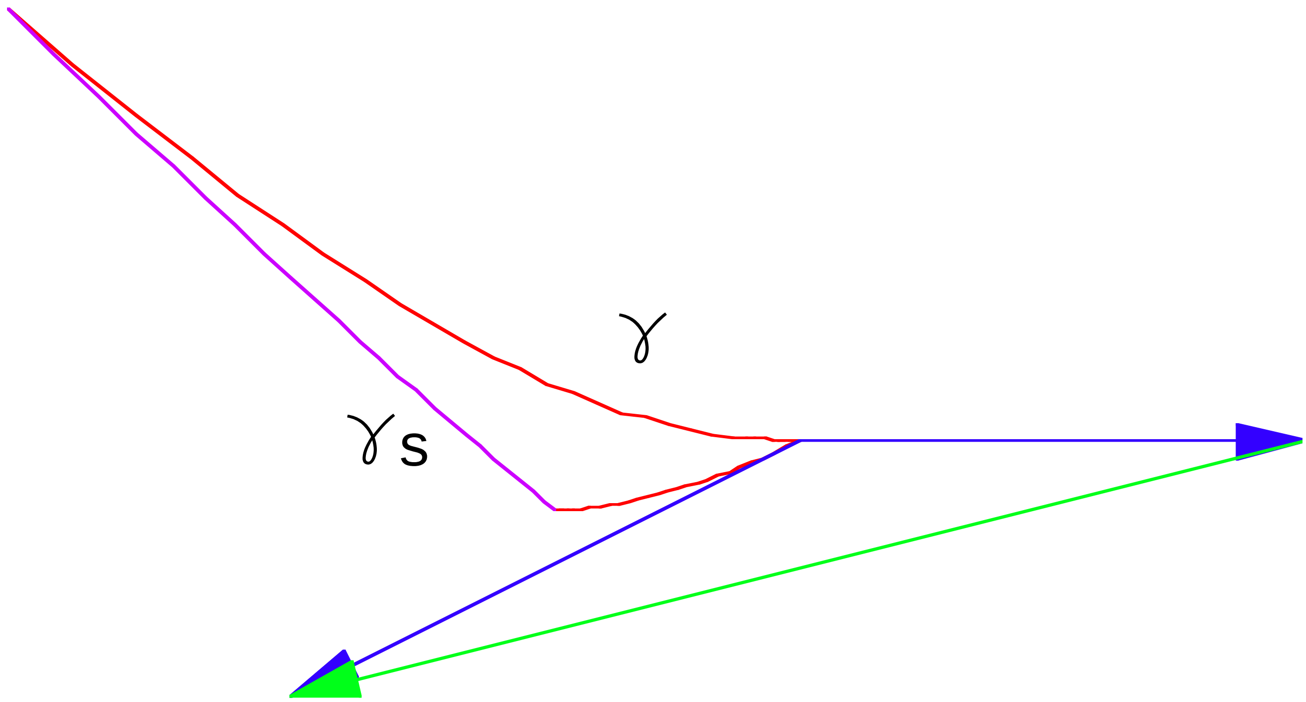
- **ψconvex corner: $\boldsymbol{\Omega}_{\gamma_s} \subset \boldsymbol{\Omega}_\gamma$**



- **ψconcave corner: $\boldsymbol{\Omega}_\gamma \subset \boldsymbol{\Omega}_{\gamma_s} = \widetilde{\boldsymbol{\Omega}_\gamma}$**



• **mixed corner:** $\mathbf{\Omega}_\gamma \overset{\not\subseteq}{\underset{\not\supseteq}{}} \mathbf{\Omega}_{\gamma_s} \subsetneq \widetilde{\mathbf{\Omega}_\gamma}$



In some very special cases: $\widetilde{\mathbf{\Omega}_\gamma} = \mathbf{\Omega}_\gamma \cup \mathbf{\Omega}_{\gamma_s}$