AN ASYMPTOTIC FORMULA FOR THE NUMBER OF NON-NEGATIVE INTEGER MATRICES WITH PRESCRIBED ROW AND COLUMN SUMS

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Abstract. We count $m \times n$ non-negative integer matrices (contingency tables) with prescribed row and column sums (margins). For a wide class of *smooth* margins we establish a computationally efficient asymptotic formula approximating the number of matrices within a relative error which approaches 0 as m and n grow.

1. Introduction and main results

Let $R = (r_1, \ldots, r_m)$ and $C = (c_1, \ldots, c_n)$ be positive integer vectors such that

$$r_1 + \ldots + r_m = c_1 + \ldots + c_n = N.$$

We are interested in the number $\#(R, C)$ of $m \times n$ non-negative integer matrices $D = (d_{ij})$ with row sums R and column sums C. Such matrices D are often called *contingency tables* with *margins* (R, C). The problem of computing or estimating $\#(R, C)$ efficiently has attracted considerable attention, see, for example, [B+72], [Be74], [DE85], [DG95], [D+97], [Mo02], [CD03], [C+05], [CM07], [GM07], [B+08] and [Ba09].

Asymptotic formulas for numbers $\#(R, C)$ as m and n grow are known in sparse cases, where the average entry N/mn of the matrix goes to 0, see [B+72], [Be74], [GM07] and in the case when all row and all column sums are equal, $r_1 = \ldots = r_m$ and $c_1 = \ldots = c_n$, [CM07]. In [Ba09] an asymptotic formula for $\log \#(R, C)$ is established under quite general circumstances.

In this paper, we prove an asymptotic formula for $\#(R, C)$ for a reasonably wide class of *smooth* margins (R, C).

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(1.1) The typical matrix and smooth margins. The typical matrix was introduced in [Ba09] and various versions of smoothness for margins were introduced in [B+08] and in [Ba08]. The following function

$$g(x) = (x+1)\ln(x+1) - x\ln x \quad \text{for} \quad x \geq 0$$

plays the crucial role. It is easy to see that g is increasing and concave with $g(0) = 0$. For an $m \times n$ non-negative matrix $X = (x_{kj})$ we define

$$g(X) = \sum_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} g(x_{kj}) = \sum_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} \Big((x_{kj}+1)\ln(x_{kj}+1) - x_{kj}\ln x_{kj} \Big).$$

Given margins (R, C), let $P(R, C)$ be the polytope of all real non-negative $m \times n$ matrices $X = (x_{kj})$ with row sums R and column sums C, also known as the *transportation polytope*. We consider the following optimization problem:

$$\text{Find} \quad \max_{X \in P(R,C)} g(X). \tag{1.1.1}$$

Since g is strictly convex, the maximum is attained at a unique matrix $Z = (\zeta_{kj})$, which we call the *typical* matrix with margins (R, C). One can show that $\zeta_{kj} > 0$ for all k and j, see [B+08] and [Ba08]. In [Ba08] it is shown that a random contingency table, sampled from the uniform distribution on the set of all non-negative integer matrices with row sums R and column sums C is, in some rigorously defined sense, likely to be close to the typical matrix Z. In [BH09] we give the following probabilistic interpretation of Z. Let us consider the family of all probability distributions on the set $\mathbb{Z}_+^{m \times n}$ of all non-negative $m \times n$ integer matrices with the expectations in the affine subspace $\mathcal{A}(R, C)$ of the $m \times n$ matrices with row sums R and column sums C. In this family there is a unique distribution of the maximum entropy and Z turns out to be the expectation of that distribution. The maximum entropy distribution is necessarily a distribution on $\mathbb{Z}_+^{m \times n}$ with independent geometrically distributed coordinates, which, conditioned on $\mathcal{A}(R, C)$, results in the uniform distribution on the set of contingency tables with margins (R, C). Function $g(X)$ turns out to be the entropy of the multivariate geometric distribution on $\mathbb{Z}_+^{m \times n}$ with the expectation X.

Let us fix a number $0 < \delta < 1$. We say that margins (R, C) are δ*-smooth* provided the following conditions (1.1.2)–(1.1.4) are satisfied:

$$m \ \geq \ \delta n \quad \text{and} \quad n \ \geq \ \delta m, \tag{1.1.2}$$

so the dimensions of the matrix are of the same order;

$$\delta \tau \ \leq \ \zeta_{kj} \ \leq \ \tau \quad \text{for all} \quad k \quad \text{and} \quad j, \tag{1.1.3}$$

for some τ such that

$$\delta \leq \tau \leq (m+n)^{1/\delta}. \tag{1.1.4}$$

We note that δ-smooth margins are also δ'-smooth for any $0 < \delta' < \delta$.

Condition (1.1.3) requires that the entries of the typical matrix are of the same order and it plays a crucial role in our proofs. Often, one can show that margins are smooth by predicting what the solution to the optimization problem (1.1.1) will look like. For example, if all row sums r_k are equal, symmetry requires that we have $\zeta_{kj} = c_j/m$ for all k and j, so the entries of the typical matrix are of the same order provided the column sums c_j are of the same order. On the other hand, (1.1.3) is violated in some curious cases. For example, if $m = n$ and $r_1 = \ldots = r_{n-1} = c_1 = \ldots = c_{n-1} = n$ while $r_n = c_n = 3n$, the entry ζ_{nn} of the typical matrix is linear in n, namely $\zeta_{nn} > 0.58n$, while all other entries of Z remain bounded by a constant, see [Ba08]. If we change r_n and c_n to $2n$, the entry ζ_{nn} becomes bounded by a constant as well. In [B+08] it is proven that if the ratio of the maximum row sum $r_+ = \max_k r_k$ to the minimum row sum $r_- = \min_k r_k$ and the ratio of the maximum column sum $c_+ = \max_j c_j$ to the minimum column sum $c_- = \min_j c_j$ do not exceed a number $\beta < (1+\sqrt{5})/2 \approx 1.618$, then (1.1.3) is satisfied with some $\delta = \delta(\beta) > 0$. The bound $(1+\sqrt{5})/2$ is not optimal, apparently it can be increased to 2, see [Lu08]. It looks plausible that if the margins are of the same order and sufficiently generic then the entries of the typical table are of the same order as well.

The lower bound in (1.1.4) requires that the density N/mn of the margins, that is the average entry of the matrix, remains bounded away from 0. This is unavoidable as our asymptotic formula does not hold for sparse cases where $N/mn \longrightarrow 0$, see [GM07]. It is less clear whether the upper bound in (1.1.4) is indeed needed. We were unable to do without it, although it seems plausible that at the very least the polynomial upper bound can be replaced by a weaker mildly exponential bound.

We proceed to define various objects needed to state our asymptotic formula.

(1.2) Quadratic form q and related quantities. Let $Z = (\zeta_{kj})$ be the typical matrix defined in Section 1.1. We consider the following quadratic form $q : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}$:

$$\begin{gathered} q(s,t) = \frac{1}{2} \sum_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} \left(\zeta_{kj}^2 + \zeta_{kj}\right)\left(s_k + t_j\right)^2 \quad \text{where} \\ s = (s_1, \ldots, s_m) \quad \text{and} \quad t = (t_1, \ldots, t_n). \end{gathered} \tag{1.2.1}$$

Thus q is a positive semidefinite quadratic form. It is easy to see that the null-space of q is spanned by vector

$$u = \Big(\underbrace{1, \ldots, 1}_{m \text{ times}}; \underbrace{-1, \ldots, -1}_{n \text{ times}}\Big).$$

Let $H = u^\perp$, $H \subset \mathbb{R}^{m+n}$, be the orthogonal complement to u. Then the restriction $q|H$ is a positive definite quadratic form and hence we can define its determinant $\det q|H$ that is the product of the non-zero eigenvalues of q. Let us define polynomials $f, h : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}$ by

$$f(s,t) = \frac{1}{6} \sum_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} \zeta_{kj} \left(\zeta_{kj}+1\right) \left(2\zeta_{kj}+1\right) \left(s_k + t_j\right)^3$$

(1.2.2) and

$$h(s,t) = \frac{1}{24} \sum_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} \zeta_{kj} \left(\zeta_{kj}+1\right) \left(6\zeta_{kj}^2 + 6\zeta_{kj}+1\right) \left(s_k + t_j\right)^4.$$

We consider the Gaussian probability measure on H with the density proportional to e^{-q} and define

$$\mu = \mathbf{E}\, f^2 \quad \text{and} \quad \nu = \mathbf{E}\, h.$$

Now we have all the ingredients to state our asymptotic formula.

(1.3) Theorem. *Let us fix $0 < \delta < 1$. Let (R, C) be δ-smooth margins, let the function g and the typical matrix Z be as defined in Section 1.1 and let the quadratic form q and values of μ and ν be as defined in Section 1.2. Then the value of*

$$\frac{e^{g(Z)} \sqrt{m+n}}{(4\pi)^{(m+n-1)/2} \sqrt{\det q|H}} \exp\left\{-\frac{\mu}{2} + \nu\right\}$$

approximates $\#(R, C)$ within a relative error which approaches 0 as $m, n \longrightarrow +\infty$. More precisely, for any $\epsilon > 0$ the above expression approximates $\#(R, C)$ within relative error ϵ provided

$$m + n \ \geq \ \left(\gamma + \ln \frac{1}{\epsilon}\right)^{\gamma + \ln \frac{1}{\epsilon}},$$

where $\gamma = \gamma(\delta)$ is a positive constant.

It is quite possible that the rate of convergence of the relative error to 0 that we are able to establish is not optimal. In fact, Canfield and McKay [CM07] establish a stronger convergence in the particular case of all row sums being equal and all column sums being equal. In Section 1.5 below we briefly sketch an argument that our formula indeed becomes the asymptotic formula of [CM07] when $r_1 = \ldots = r_m$ and $c_1 = \ldots = c_n$. In [Ba09] it is proven that the value $g(Z)$ provides an asymptotic approximation to $\ln \#(R, C)$ for a rather wide class of margins (essentially, we need only the density N/mn to be bounded away from 0 but do not need a subtler condition (1.1.3) of smoothness). The first part

(1.3.1)
$$\frac{e^{g(Z)} \sqrt{m+n}}{(4\pi)^{(m+n-1)/2} \sqrt{\det q|H}}$$

of the formula is called the "Gaussian approximation" in [BH09]. It has the following intuitive explanation. Let us consider a random matrix X with the multivariate geometric distribution on the set $\mathbb{Z}_+^{m\times n}$ of all non-negative integer matrices such that $\mathbf{E}\, X = Z$, where Z is the typical matrix with margins (R, C). It follows from the results of [BH09] that the distribution of X conditioned on the affine subspace $\mathcal{A} = \mathcal{A}(R, C)$ of matrices with row sums R and column sums C is uniform with the probability mass function of $e^{-g(Z)}$ for every non-negative integer matrix in $\mathcal{A}$. Therefore,

$$\#(R, C) = e^{g(Z)} \mathbf{P}\left\{X \in \mathcal{A}\right\}.$$

Let $Y \in \mathbb{R}^{m+n}$ be a random vector obtained by computing m row sums and n column sums of X. Then $\mathbf{E}\, Y = (R, C)$ and

$$\mathbf{P}\left\{X \in \mathcal{A}\right\} = \mathbf{P}\left\{Y = (R, C)\right\}.$$

We obtain (1.3.1) if we assume in the spirit of the Local Central Limit Theorem that the distribution of Y in the vicinity of $\mathbf{E}\, Y$ is close to the $(m + n - 1)$-dimensional Gaussian distribution (we lose one dimension since the row and column sums of a matrix are bound by one linear relation: the sum of all row sums is equal to the sum of all column sums). This assumption is not implausible since the coordinates of Y are obtained by summing up of a number of independent entries of X.

The correction factor

$$\exp\left\{-\frac{\mu}{2} + \nu\right\}$$

is, essentially, the *Edgeworth correction* in the Central Limit Theorem. In the course of the proof of Theorem 1.3 we establish a two-sided bound

$$\gamma_1(\delta) \ \leq \ \exp\left\{-\frac{\mu}{2} + \nu\right\} \ \leq \ \gamma_2(\delta)$$

for some constants $\gamma_1(\delta), \gamma_2(\delta) > 0$ as long as the margins (R, C) remain δ-smooth.

De Loera [DL09] ran a range of numerical experiments which seem to demonstrate that already the Gaussian approximation (1.3.1) works reasonably well for contingency tables.

(1.4) Computation. Optimization problem (1.1.1) is convex and can be solved, for example, by interior point methods, see [NN94]. That is, for any $\epsilon > 0$ the entries ζ_{kj} of the typical matrix Z can be computed within relative error ϵ in time polynomial in $\ln(1/\epsilon)$ and $m + n$.

Given Z, quantities $\det q|H$, μ and ν can be computed by linear algebra algorithms. Here is one way to do it.

Let $Q=(q_{kj})$ be the $(m+n)\times(m+n)$ symmetric matrix defined as follows:

$$\begin{aligned}
q_{kk} =&\sum_{j=1}^n \left(\zeta_{kj}^2+\zeta_{kj}\right) \quad \text{for} \quad k=1, \ldots, m, \\
q_{k_1k_2} =&0 \quad \text{for} \quad \text{for} \quad 1 \leq k_1 \neq k_2 \leq m, \\
q_{(j+m)(j+m)} =&\sum_{k=1}^m \left(\zeta_{kj}^2+\zeta_{kj}\right) \quad \text{for} \quad j=1, \ldots, n, \\
q_{(j_1+m)(j_2+m)} =&0 \quad \text{for} \quad 1 \leq j_1 \neq j_2 \leq n, \\
q_{k(j+m)}=q_{(j+m)k} =&\zeta_{kj}^2+\zeta_{kj} \quad \text{for} \quad 1 \leq k \leq m \quad \text{and} \quad 1 \leq j \leq n.
\end{aligned}$$

Hence we can write

$$q(x)=\frac{1}{2}\langle x, Qx\rangle \quad \text{for} \quad x=\left(s_1, \ldots, s_m; t_1, \ldots, t_n\right),$$

where $\langle\cdot,\cdot\rangle$ is the standard scalar product in $\mathbb{R}^{m+n}$. Let Q_0 be the $(m+n-1)\times(m+n-1)$ submatrix of Q consisting of the entries of Q that do not belong to the last row or to the last column. Then

$$\det q|H=(m+n)\det Q_0,$$

cf. Lemma 3.5.

Let $L\subset\mathbb{R}^{m+n}$ be a hyperplane not containing the null-space of q. Then the restriction $q|L$ of q onto L is a positive definite quadratic form and we can define the Gaussian probability measure on L with the density proportional to e^{-q}. It turns out that expectation of any polynomial in (s_k+t_j) does not depend on the choice of L, see Lemma 3.1. We can conveniently choose L to be the coordinate hyperplane $t_n=0$. Then $C=Q_0^{-1}$ is the covariance matrix of $s_1,\ldots,s_m;t_1,\ldots,t_{n-1}$. That is, for $C=(c_{kj})$ we have

$$\begin{aligned}
&\mathbf{E}\, s_{k_1}s_{k_2}=c_{k_1k_2} \quad \text{for} \quad 1\leq k_1,k_2\leq m,\\
&\mathbf{E}\, t_{j_1}t_{j_2}=c_{(j_1+m)(j_2+m)} \quad \text{for} \quad 1\leq j_1,j_2\leq n-1,\\
&\mathbf{E}\, s_kt_j=c_{k(j+m)} \quad \text{for} \quad 1\leq k\leq m \quad \text{and} \quad j=1,\ldots,n-1.
\end{aligned}$$

Combining these identities with the identities

$$\mathbf{E}\, t_jt_n=\mathbf{E}\, s_kt_n=0$$

we compute the covariances

$$\mathbf{E}\, \left(s_{k_1}+t_{j_1}\right)\left(s_{k_2}+t_{j_2}\right).$$

To compute $\mu=\mathbf{E}\, f^2$ and $\nu=\mathbf{E}\, h$ we use Wick's formula, in particular the identities

$$\begin{aligned}
&\mathbf{E}\, u^3v^3=9\left(\mathbf{E}\, u^2\right)\left(\mathbf{E}\, v^2\right)\left(\mathbf{E}\, uv\right)+6\left(\mathbf{E}\, uv\right)^3 \quad \text{and}\\
&\mathbf{E}\, u^4=3\left(\mathbf{E}\, u^2\right)^2
\end{aligned} \tag{1.4.1}$$

for Gaussian random variables $u=(s_{k_1}+t_{j_1})$ and $v=(s_{k_2}+t_{j_2})$, see Section 4.2.

(1.5) Matrices with equal row sums and equal column sums. Here we sketch a computation showing that the asymptotic formula of Theorem 1.3 is equivalent to the formula of Canfield and McKay [CM07] provided all row sums are equal, $R = (\zeta n, \ldots, \zeta n)$ and all column sums are equal, $C = (\zeta m, \ldots, \zeta m)$. By symmetry, for the typical matrix Z we have $\zeta_{kj} = \zeta$ for all k and j. Therefore,

$$e^{g(Z)} = \frac{(\zeta+1)^{(\zeta+1)mn}}{\zeta^{\zeta mn}}.$$

For the quadratic form $q : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}$, we have

$$q(s,t) = \frac{\zeta^2+\zeta}{2} \sum_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} (s_k + t_j)^2 .$$

By Lemma 3.6, apart from a 1-dimensional eigenspace with the eigenvalue 0, the quadratic form q has an $(n-1)$-dimensional eigenspace with the eigenvalue $m\left(\zeta^2+\zeta\right)/2$, an $(m-1)$-dimensional eigenspace with the eigenvalue $n\left(\zeta^2+\zeta\right)/2$ and a 1-dimensional eigenspace with the eigenvalue $(m+n)\left(\zeta^2+\zeta\right)/2$. Therefore, we have

$$\det q|H = 2^{1-m-n}\left(\zeta^2+\zeta\right)^{m+n-1} m^{n-1} n^{m-1}(m+n).$$

Recall that $H \subset \mathbb{R}^{m+n}$ is the orthogonal complement to the null-space of q. We consider the Gaussian probability measure on H with the density proportional to e^{-q} and interpret the coordinate functions s_k and t_j as random variables.

Using the eigenspace decomposition provided by Lemma 3.6, one can show that

$$\begin{aligned}
&\mathbf{E}\,(s_k+t_j)^2 = \frac{m+n-1}{mn\left(\zeta^2+\zeta\right)} \quad \text{for all} \quad k, j \\
&\mathbf{E}\,(s_{k_1}+t_{j_1})(s_{k_2}+t_{j_2}) = -\frac{1}{mn\left(\zeta^2+\zeta\right)} \quad \text{provided} \quad k_1 \neq k_2 \quad \text{and} \quad j_1 \neq j_2, \\
&\mathbf{E}\,(s_k+t_{j_1})(s_k+t_{j_2}) = \frac{m-1}{mn\left(\zeta^2+\zeta\right)} \quad \text{provided} \quad j_1 \neq j_2, \\
&\mathbf{E}\,(s_{k_1}+t_j)(s_{k_2}+t_j) = \frac{n-1}{mn\left(\zeta^2+\zeta\right)} \quad \text{provided} \quad k_1 \neq k_2.
\end{aligned}$$

Note that the random variables $s_{k_1}+t_{j_1}$ and $s_{k_2}+t_{j_2}$ are *weakly correlated* if $k_1 \neq k_2$ and $j_1 \neq j_2$, an example of a general phenomenon, see Section 3, which plays the crucial role in our proof.

Using the first formula of (1.4.1) for Gaussian variables $u = s_{k_1}+t_{j_1}$ and $v = s_{k_2}+t_{j_2}$, we compute

$$\mu = \mathbf{E}\, f^2 \quad \text{for} \quad f = \frac{\zeta(\zeta+1)(2\zeta+1)}{6} \sum_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} (s_k+t_j)^3$$

as

$$\mu \approx \frac{\left(5m^2+6mn+5n^2\right)(2\zeta+1)^2}{12mn\left(\zeta^2+\zeta\right)},$$

where we ignore $O\left(m^{-1}\right)$ and $O\left(n^{-1}\right)$ terms.

Using the second formula of (1.4.1) for Gaussian variables $u=s_k+t_j$, we compute

$$\nu = \mathbf{E}\, h \quad \text{for} \quad h=\frac{\zeta(\zeta+1)\left(6\zeta^2+6\zeta+1\right)}{24}\sum_{\substack{1\leq k\leq m\\ 1\leq j\leq n}}\left(s_k+t_j\right)^4$$

as

$$\nu \approx \frac{\left(6\zeta^2+6\zeta+1\right)(m+n)^2}{8mn\left(\zeta^2+\zeta\right)},$$

where we ignore $O\left(m^{-1}\right)$ and $O\left(n^{-1}\right)$ terms.

Putting everything together, we obtain the following asymptotic formula for $\#(R,C)$:

$$\frac{\left((1+\zeta)^{(1+\zeta)}\zeta^{-\zeta}\right)^{mn}}{(2\pi\zeta(1+\zeta))^{(m+n-1)/2}m^{(n-1)/2}n^{(m-1)/2}} \times \exp\left\{\frac{(6\zeta^2+6\zeta+1)(m+n)^2}{8mn\zeta(1+\zeta)}-\frac{(5m^2+6mn+5n^2)(2\zeta+1)^2}{24mn\zeta(\zeta+1)}\right\}.$$

Let

$$z=\frac{m}{n}+\frac{n}{m} \quad \text{and} \quad A=\frac{\zeta(1+\zeta)}{2}.$$

The formula becomes

$$\frac{\left((1+\zeta)^{(1+\zeta)}\zeta^{-\zeta}\right)^{mn}}{(4A\pi)^{(m+n-1)/2}m^{(n-1)/2}n^{(m-1)/2}}\exp\left\{\frac{1}{2}-\frac{z}{12}-\frac{z}{24A}\right\},$$

which is exactly formula (1.3) from [CM07].

2. An integral representation for the number of contingency tables

In [BH09] we prove the following general result.

(2.1) Theorem. *Let $P\subset\mathbb{R}^p$ be a polyhedron defined by the system of linear equations $Ax=b$, where A is a $d\times p$ integer matrix with columns $a_1,\ldots,a_p\in\mathbb{Z}^d$ and $b\in\mathbb{Z}^d$ is an integer vector, and inequalities $x\geq 0$ (the inequalities are understood as coordinate-wise). Suppose that P is bounded and has a non-empty interior, that is, contains a point $x=(\xi_1,\ldots,\xi_p)$ such that $\xi_j>0$ for $j=1,\ldots,p$. Then the function*

$$g(x)=\sum_{j=1}^{p}\Big((\xi_j+1)\ln(\xi_j+1)-\xi_j\ln\xi_j\Big)$$

attains its maximum on P at a unique point $z = (\zeta_1, \ldots, \zeta_p)$ such that $\zeta_j > 0$ for $j = 1, \ldots, p$.

Let $\Pi \subset \mathbb{R}^d$ be the parallelepiped consisting of the points $t = (\tau_1, \ldots, \tau_d)$ such that

$$-\pi \ \leq \ \tau_k \ \leq \ \pi \quad \textit{for} \quad k = 1, \ldots, d.$$

Then the number $|P \cap \mathbb{Z}^p|$ of integer points in P can be written as

$$|P \cap \mathbb{Z}^p| = \frac{e^{g(z)}}{(2\pi)^d} \int_\Pi e^{-i\langle t, b\rangle} \prod_{j=1}^{p} \frac{1}{1 + \zeta_j - \zeta_j e^{i\langle a_j, t\rangle}} \ dt,$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^d$ and $i = \sqrt{-1}$.

□

The idea of the proof is as follows. Let $X = (x_1, \ldots, x_p)$ be a random vector of independent geometric random variables x_j such that $\mathbf{E}\, x_j = \zeta_j$. Hence values of X are non-negative integer vectors and we show in [BH09] that the probability mass function of X is constant on the set $P \cap \mathbb{Z}^p$ and equals $e^{-g(z)}$ for every integer point in P. Letting $Y = AX$, we obtain

$$|P \cap \mathbb{Z}^p| = e^{g(z)} \mathbf{P}\left\{X \in P\right\} = e^{g(z)} \mathbf{P}\left\{Y = b\right\}$$

and the probability in question is written as the integral of the characteristic function of Y.

Since

$$\sum_{j=1}^{p} \zeta_j a_j = b,$$

in a neighborhood of the origin $t = 0$ the integrand can be written as

$$\begin{aligned}
&e^{-i\langle t, b\rangle} \prod_{j=1}^{p} \frac{1}{1 + \zeta_j - \zeta_j e^{i\langle a_j, t\rangle}} \\
&\quad = \exp\Bigg\{ -\frac{1}{2} \sum_{j=1}^{p} \left(\zeta_j^2 + \zeta_j\right) \langle a_j, t\rangle^2 \\
&\qquad\qquad - \frac{i}{6} \sum_{j=1}^{p} \zeta_j \left(\zeta_j + 1\right)\left(2\zeta_j + 1\right) \langle a_j, t\rangle^3 \\
&\qquad\qquad + \frac{1}{24} \sum_{j=1}^{p} \zeta_j \left(\zeta_j + 1\right)\left(6\zeta_j^2 + 6\zeta_j + 1\right) \langle a_j, t\rangle^4 \\
&\qquad\qquad + O\left(\sum_{j=1}^{p} \left(\zeta_j + 1\right)^5 \langle a_j, t\rangle^5 \right) \Bigg\}.
\end{aligned} \tag{2.2}$$

Note that the linear term is absent in the expansion.

We obtain the following corollary.

(2.3) Corollary. *Let $R=(r_1,\dots,r_m)$ and $C=(c_1,\dots,c_n)$ be margins and let $Z=(\zeta_{kj})$ be the typical matrix defined in Section 1.1. Let*

$$F(s,t)=\exp\left\{-i\sum_{k=1}^m r_k s_k - i\sum_{j=1}^n c_j t_j\right\}\prod_{\substack{1\le k\le m\\ 1\le j\le n}} \frac{1}{1+\zeta_{kj}-\zeta_{kj}e^{i(s_k+t_j)}}.$$

Let $\Pi\subset\mathbb{R}^{m+n}$ be the parallelepiped consisting of the points $(s_1,\dots,s_m;t_1,\dots,t_n)$ such that

$$-\pi \ \le\ s_k,t_j\ \le\ \pi \quad \text{for all} \quad k \quad \text{and} \quad j.$$

Let us identify $\mathbb{R}^{m+n-1}$ with the hyperplane $t_n=0$ in $\mathbb{R}^{m+n-1}$ and let $\Pi_0\subset\Pi$ be the facet of Π defined by the equation $t_n=0$. Then

$$\#(R,C)=\frac{e^{g(Z)}}{(2\pi)^{m+n-1}}\int_{\Pi_0} F(s,t)\ dsdt,$$

where $dsdt$ is the Lebesgue measure in Π_0.

Proof. The number $\#(R,C)$ of non-negative integer $m\times n$ matrices with row sums R and column sums C is the number of integer points in the transportation polytope $P(R,C)$. We can define $P(R,C)$ by prescribing all row sums $r_1,\dots,r_m$ and all but one column sums $c_1,\dots,c_{n-1}$ of a non-negative $m\times n$ matrix. Applying Theorem 2.1, we get the desired integral representation. □

From (2.2) we get the following expansion in the neighborhood of $s_1=\dots=s_k=t_1=\dots=t_n=0$:

$$\begin{aligned}F(s,t)=\exp\Bigg\{&-q(s,t)-if(s,t)+h(s,t)\\ &+O\left(\sum_{kj}\left(1+\zeta_{kj}\right)^5\left(s_k+t_j\right)^5\right)\Bigg\},\end{aligned}\tag{2.4}$$

where q, f, and h are defined by (1.2.1)–(1.2.2).

(2.5) The plan of the proof. We use the integral representation of Corollary 2.3.

In Section 7, we show that the integral outside of a neighborhood U of the origin is asymptotically negligible.

In Section 6, we show that the integral over U produces the asymptotic formula of Theorem 1.3. This requires some preparation. It is easy to show that the contribution of the terms of order 5 and higher in (2.4) is asymptotically negligible in the integral over U. The integral of e^{-q} over U produces the Gaussian term (1.3.1) However, both the cubic term $f(s,t)$ and the fourth-order term $h(s,t)$ contribute substantially to the integral, correcting the Gaussian term (1.3.1) by a constant factor.

In Section 5 we show that $h(s,t)$ remains, essentially, constant in the neighborhood U and hence its contribution e^{ν} can be computed and factored out from the integral.

In Section 4, we show that $f(s,t)$ behaves, essentially, as a Gaussian random variable with respect to the probability measure in $\mathbb{R}^{m+n-1}$ with the density proportional to e^{-q}. This allows us to compute the contribution of the cubic term.

The results of Sections 4 and 5 are based on the analysis in Section 3. In Section 3, we consider coordinate functions s_k and t_j as random variables with respect to the Gaussian probability measure on $\mathbb{R}^{m+n-1}$ with the density proportional to e^{-q}. We show that $s_{k_1}+t_{j_1}$ and $s_{k_2}+t_{j_2}$ are weakly correlated provided $k_1 \neq k_2$ and $j_1 \neq j_2$, cf. the example in Section 1.5, a crucial fact which makes possible the analysis of Sections 4 and 5.

(2.6) Notation. In what follows, we denote by γ, sometimes with an index or a list of parameters, a positive constant depending on the parameters. The actual value of γ may change from line to line.

3. Correlations

Recall (see Section 1.2) that the quadratic form $q : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}$ is defined by

$$q(s,t) = \frac{1}{2} \sum_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} \left(\zeta_{kj}^2 + \zeta_{kj}\right) \left(s_k + t_j\right)^2.$$

Let

$$u = \Bigg(\underbrace{1, \ldots, 1}_{m \text{ times}};\ \underbrace{-1, \ldots, -1}_{n \text{ times}}\Bigg).$$

Let $L \subset \mathbb{R}^{m+n}$ be a hyperplane which does not contain u. Then the restriction $q|L$ of q onto L is a positive definite quadratic form and we can consider the Gaussian probability measure on L with the density proportional to e^{-q}. We consider s_k and t_j as random variables on L and estimate their covariances.

(3.1) Lemma. *For any $1 \leq k_1, k_2 \leq m$ and any $1 \leq j_1, j_2 \leq n$ the covariance*

$$\mathbf{E}\ \left(s_{k_1} + t_{k_1}\right)\left(s_{k_2} + t_{j_2}\right)$$

is independent on the choice the hyperplane L, as long as L does not contain u.

Proof. Let $L_1, L_2 \subset \mathbb{R}^n$ be two hyperplanes not containing u. Then we can define the projection $pr : L_1 \longrightarrow L_2$ along the span of u, so that $pr(x)$ for $x \in L_1$ is the unique $y \in L_2$ such that $y-x$ is a multiple of u. We note that $q(x)=q(x+tu)$ for all $x \in \mathbb{R}^{m+n}$ and all $t \in \mathbb{R}$. Therefore, the push-forward of the Gaussian probability measure on L_1 with the density proportional to e^{-q} is the probability measure on L_2 with the density proportional to e^{-q}. Moreover, the value of $s_k + t_j$ does not change under the projection. Hence the result follows. □

The main result of this section is the following theorem.

(3.2) Theorem. *Let us fix a number $\delta > 0$ and suppose that*

$$\tau\delta \ \leq \ \zeta_{kj} \ \leq \ \tau \quad \textit{for all} \quad k, j$$

and some $\tau > 0$. Suppose that $\delta m \leq n$ and $\delta n \leq m$.

Let us define

$$a_k = \sum_{j=1}^{n} \left(\zeta_{kj}^2 + \zeta_{kj}\right) \quad \textit{for} \quad k = 1, \ldots, m \quad \textit{and}$$

$$b_j = \sum_{k=1}^{m} \left(\zeta_{kj}^2 + \zeta_{kj}\right) \quad \textit{for} \quad j = 1, \ldots, n.$$

Let

$$\Delta = \frac{12}{\delta^{15/2}\left(\tau^2 + \tau\right) mn}.$$

Let $L \subset \mathbb{R}^{m+n}$ be a hyperplane not containing the null-space of q. Let us consider the Gaussian probability measure on L with the density proportional to e^{-q}.

Then

$$\left|\mathbf{E}\ (s_{k_1} + t_{j_1})\,(s_{k_2} + t_{j_2})\right| \ \leq \ \Delta \quad \textit{provided} \quad k_1 \neq k_2 \quad \textit{and} \quad j_1 \neq j_2,$$

$$\left|\mathbf{E}\ (s_k + t_{j_1})\,(s_k + t_{j_2}) - \frac{1}{a_k}\right| \ \leq \ \Delta \quad \textit{provided} \quad j_1 \neq j_2,$$

$$\left|\mathbf{E}\ (s_{k_1} + t_j)\,(s_{k_2} + t_j) - \frac{1}{b_j}\right| \ \leq \ \Delta \quad \textit{provided} \quad k_1 \neq k_2 \quad \textit{and}$$

$$\left|\mathbf{E}\ (s_k + t_j)^2 - \frac{1}{a_k} - \frac{1}{b_j}\right| \ \leq \ \Delta \quad \textit{for all} \quad k \quad \textit{and} \quad j.$$

In what follows, we will often deal with the following situation. Let V be Euclidean space, let $\phi : V \longrightarrow \mathbb{R}$ be a positive semidefinite quadratic form and let $L \subset V$ be a subspace such that the restriction of ϕ onto L is strictly positive definite. We consider the Gaussian probability measure on L with the density proportional to $e^{-\phi}$. For a function (random variable) $f : L \longrightarrow \mathbb{R}$ we denote by $\mathbf{E}\,(f;\ \phi|L)$ the expectation of f with respect to that Gaussian measure. Instead of $\mathbf{E}\,(f;\ \phi|V)$ we write simply $\mathbf{E}\,(f;\ \phi)$.

We will use the following standard facts. Suppose that there is a direct sum decomposition $V = L_1 + L_2 + \ldots + L_m$, such that

$$\phi\,(x_1 + \ldots + x_m) = \sum_{i=1}^{m} \phi_i\,(x_i) \quad \text{for all} \quad x_i \in L_i.$$

Then for any two linear functions $\ell_1, \ell_2 : \ V \longrightarrow \mathbb{R}$ we have

$$\mathbf{E}\ (\ell_1 \ell_2;\ \phi) = \sum_{i=1}^m \mathbf{E}\ (\ell_1 \ell_2;\ \phi | L_i)\,.$$

Let us choose an orthonormal basis of V thus identifying $V = \mathbb{R}^d$. Suppose that

$$\phi(x) = \frac{1}{2}\langle x,\ Qx\rangle \quad \text{where} \quad x = (x_1, \ldots, x_d)$$

and Q is a $d \times d$ positive definite matrix. Then the covariance matrix of $x_1, \ldots, x_d$ is Q^{-1}, that is, $\mathbf{E}\ (x_k x_j;\ \phi)$ is the (k,j)-th entry of Q^{-1}.

We deduce Theorem 3.2 from the following statement.

(3.3) Proposition. *Let m and n be positive integers such that*

$$\delta m \ \leq \ n \quad \textit{and} \quad \delta n \ \leq \ m \quad \textit{for some} \quad 0 < \delta < 1.$$

Let ξ_{kj}, $k = 1, \ldots, m$ and $j = 1, \ldots, n$, be numbers such that

$$\alpha \ \leq \ \xi_{kj} \ \leq \ \beta \quad \textit{for all} \quad k, j$$

and some $\beta > \alpha > 0$. Let

$$a_k = \sum_{j=1}^n \xi_{kj} \quad \textit{for} \quad k = 1, \ldots, m \quad \textit{and}$$

$$b_j = \sum_{k=1}^m \xi_{kj} \quad \textit{for} \quad j = 1, \ldots, n.$$

Let us define a quadratic form $\psi : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}$ by

$$\psi(s,t) = \frac{1}{2} \sum_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} \xi_{kj} \left(\frac{s_k}{\sqrt{a_k}} + \frac{t_j}{\sqrt{b_j}} \right)^2.$$

Let $L \subset \mathbb{R}^{m+n}$ be the hyperplane consisting of the points $(s_1, \ldots, s_m;\ t_1, \ldots, t_n)$ such that

$$\sum_{k=1}^m s_k \sqrt{a_k} \ = \ \sum_{j=1}^n t_j \sqrt{b_j}.$$

Then the restriction $\psi | L$ of ψ onto L is strictly positive definite and for

$$\Delta = 3 \left(\frac{\beta}{\alpha} \right)^{7/2} \frac{1}{\sqrt{\delta m n}}$$

we have

$$\begin{aligned}&\left|\mathbf{E}\left(s_k^2;\ \psi|L\right)-1\right|,\ \left|\mathbf{E}\left(t_j^2;\ \psi|L\right)-1\right|\ \leq\ \Delta \quad \textit{for all} \quad k,j,\\ &\left|\mathbf{E}\left(s_{k_1}s_{k_2};\ \psi|L\right)\right|,\ \left|\mathbf{E}\left(t_{j_1}t_{j_2};\ \psi|L\right)\right|\ \leq\ \Delta \quad \textit{for all} \quad k_1\neq k_2 \quad \textit{and} \quad j_1\neq j_2,\\ &\left|\mathbf{E}\left(s_k t_j;\ \psi|L\right)\right|\ \leq\ \Delta \quad \textit{for all} \quad k,j.\end{aligned}$$

Proof. Clearly, the null-space of ψ is one-dimensional and spanned by vector

$$w=\left(\sqrt{a_1},\ldots,\sqrt{a_m};\ -\sqrt{b_1},\ldots,-\sqrt{b_n}\right).$$

We have $L=w^{\perp}$ and hence the restriction of ψ onto L is positive definite.

Next, we observe that

$$v=\left(\sqrt{a_1},\ldots,\sqrt{a_m};\ \sqrt{b_1},\ldots,\sqrt{b_n}\right)$$

is an eigenvector of ψ with eigenvalue 1. Indeed, the gradient of $\psi(x)$ at $x=v$ is equal to $2v$:

$$\begin{aligned}\frac{\partial}{\partial s_k}\psi\Big|_{s_k=\sqrt{a_k},t_j=\sqrt{b_j}} &= \frac{2}{\sqrt{a_k}}\sum_{j=1}^{n}\xi_{kj} = 2\sqrt{a_k} \quad \text{and}\\ \frac{\partial}{\partial t_j}\psi\Big|_{s_k=\sqrt{a_k},t_j=\sqrt{b_j}} &= \frac{2}{\sqrt{b_j}}\sum_{k=1}^{m}\xi_{kj} = 2\sqrt{b_j}.\end{aligned}$$

We write

$$\psi(s,t)=\frac{1}{2}\sum_{k=1}^{m}s_k^2 \ +\ \frac{1}{2}\sum_{j=1}^{n}t_j^2\ +\ \sum_{\substack{k=1,\ldots,m\\ j=1,\ldots,n}}\frac{\xi_{kj}}{\sqrt{a_k}\sqrt{b_j}}s_k t_j. \tag{3.3.1}$$

Let

$$c=a_1+\ldots+a_m=b_1+\ldots+b_n$$

and let us consider another quadratic form $\phi:\mathbb{R}^{m+n}\longrightarrow\mathbb{R}$ defined by

$$\phi(s,t)=\sum_{\substack{1\leq k\leq m\\ 1\leq j\leq n}}\frac{\sqrt{a_k b_j}}{c}t_k s_j. \tag{3.3.2}$$

Clearly, $\phi(s,t)$ is a form of rank 2. Its non-zero eigenvalues are $-1/2$ with the eigenspace spanned by w and $1/2$ with the eigenspace spanned by v.

Let us define a subspace $L_0\subset\mathbb{R}^{m+n}$ of codimension 2 by

$$L_0=(v,w)^{\perp}.$$

In other words, L_0 consists of the points $(s_1, \ldots, s_m; t_1, \ldots, t_n)$ such that

$$\sum_{k=1}^{m} t_k \sqrt{a_k} \ = \ \sum_{j=1}^{n} s_j \sqrt{b_j} = 0.$$

In particular,

$$\phi(s,t) = 0 \quad \text{for all} \quad (s,t) \in L_0.$$

Let us define a quadratic form

$$\tilde{\psi} = \psi - \epsilon^2 \phi \quad \text{for} \quad \epsilon = \frac{\alpha}{\beta}. \tag{3.3.3}$$

We note that $\tilde{\psi}$ is strictly positive definite. Indeed, w and v are eigenvectors of $\tilde{\psi}$ with the eigenvalues $\epsilon^2/2 > 0$ and $1 - \epsilon^2/2 > 0$ respectively and $\tilde{\psi}$ coincides with ψ on the subspace $L_0 = (v, w)^{\perp}$, where ψ is positive definite. Our immediate goal is to bound the covariances

$$\mathbf{E}\left(s_{k_1} s_{k_2};\ \tilde{\psi}\right), \ \mathbf{E}\left(t_{j_1} t_{j_2};\ \tilde{\psi}\right) \quad \text{and} \quad \mathbf{E}\left(s_k t_j;\ \tilde{\psi}\right).$$

We can write

$$\tilde{\psi}(x) = \frac{1}{2}\langle x, (I+P)x \rangle \quad \text{for} \quad x = (s,t),$$

where I is the $(m+n) \times (m+n)$ identity matrix, $P = (p_{il})$ is a symmetric $(m+n) \times (m+n)$ matrix with zero diagonal and $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^{m+n}$. Since

$$\begin{aligned} \alpha n \ &\leq a_k \ \leq \ \beta n \quad \text{for} \quad k = 1, \ldots, m \\ \alpha m \ &\leq b_j \ \leq \ \beta m \quad \text{for} \quad j = 1, \ldots, n \quad \text{and} \\ c \ &\geq \ \alpha m n, \end{aligned} \tag{3.3.4}$$

by (3.3.1) – (3.3.3), for the entries p_{il} of P we have

$$0 \ \leq \ p_{il} \ \leq \ \frac{\beta}{\alpha\sqrt{mn}} = \frac{1}{\epsilon\sqrt{mn}} \quad \text{for all} \quad i, l. \tag{3.3.5}$$

Furthermore, v is the Perron-Frobenius eigenvector of P with the corresponding eigenvalue $1 - \epsilon^2$.

Let us bound the entries of a positive integer power $P^d = \left(p_{il}^{(d)}\right)$ of P. Let

$$\kappa = \frac{\beta}{\alpha^{3/2} \delta^{1/4} (mn)^{3/4}} \quad \text{and let} \quad y = \kappa v, \quad y = (\eta_1, \ldots, \eta_{m+n}).$$

From (3.3.4) we conclude that

$$a_k, \ b_j \ \geq \ \alpha\sqrt{\delta m n} \quad \text{for all} \quad k, j$$

and hence by (3.3.5)

$$p_{il} \ \le \ \eta_i \quad \text{for all} \quad i, l. \tag{3.3.6}$$

Similarly, from (3.3.4), we conclude

$$a_k, \ b_j \ \le \ \beta\sqrt{mn/\delta} \quad \text{for all} \quad k, j$$

and hence

$$\eta_i \ \le \ \frac{\beta^{3/2}}{\alpha^{3/2}\sqrt{\delta mn}} = \frac{1}{\epsilon^{3/2}\sqrt{\delta mn}} \quad \text{for all} \quad i. \tag{3.3.7}$$

Besides, y is an eigenvector of P^d with the eigenvalue $(1-\epsilon^2)^d$. Therefore, for $d \ge 0$ we have

$$\begin{aligned} p_{il}^{(d+1)} =& \sum_{j=1}^{m+n} p_{ij}^{(d)} p_{jl} \\ \le& \sum_{j=1}^{m+n} p_{ij}^{(d)} \eta_j = (1-\epsilon^2)^d \eta_i \\ \le& (1-\epsilon^2)^d \frac{1}{\epsilon^{3/2}\sqrt{\delta mn}}. \end{aligned}$$

Consequently, the series

$$(I+P)^{-1} = I + \sum_{d=1}^{+\infty} (-1)^d P^d$$

converges absolutely and we can bound the entries of $Q = (I+P)^{-1}$, $q = (q_{il})$, by

$$|q_{il}| \ \le \ \frac{1}{\epsilon^2} \frac{1}{\epsilon^{3/2}\sqrt{\delta mn}} = \frac{1}{\epsilon^{7/2}\sqrt{\delta mn}} \quad \text{if} \quad i \ne l$$

and

$$|q_{ii} - 1| \ \le \ \frac{1}{\epsilon^{7/2}\sqrt{\delta mn}}.$$

On the other hand, Q is the matrix of covariances of functions $s_1, \ldots, s_m; t_1, \ldots, t_n$, so we have

$$\begin{gathered} \left|\mathbf{E}\left(s_k^2; \ \tilde{\psi}\right) - 1\right|, \ \left|\mathbf{E}\left(t_j^2; \ \tilde{\psi}\right) - 1\right| \ \le \ \frac{1}{\epsilon^{7/2}\sqrt{\delta mn}} \quad \text{for all} \quad k, j, \\ \left|\mathbf{E}\left(s_{k_1} s_{k_2}; \ \tilde{\psi}\right)\right| \ \le \ \frac{1}{\epsilon^{7/2}\sqrt{\delta mn}} \quad \text{if} \quad k_1 \ne k_2, \\ \left|\mathbf{E}\left(t_{j_1} t_{j_2}; \ \tilde{\psi}\right)\right| \ \le \ \frac{1}{\epsilon^{7/2}\sqrt{\delta mn}} \quad \text{if} \quad j_1 \ne j_2 \\ \text{and} \\ \left|\mathbf{E}\left(s_k t_j; \ \tilde{\psi}\right)\right| \ \le \ \frac{1}{\epsilon^{7/2}\sqrt{\delta mn}} \quad \text{for all} \quad k, j. \end{gathered} \tag{3.3.8}$$

Now we go from $\tilde{\psi}$ back to ψ. Since v and w are eigenvectors of $\tilde{\psi}$ and since $L_0 = (u, v)^{\perp}$, for any linear functions $\ell_1, \ell_2 : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}$ we have

$$\mathbf{E}\left(\ell_1\ell_2;\ \tilde{\psi}|L_0\right) = \mathbf{E}\left(\ell_1\ell_2;\ \tilde{\psi}\right) - \mathbf{E}\left(\ell_1\ell_2;\ \tilde{\psi}|\operatorname{span}(w)\right) - \mathbf{E}\left(\ell_1\ell_2;\ \tilde{\psi}|\operatorname{span}(v)\right).$$

On the other hand, since ψ and $\tilde{\psi}$ coincide on L_0, we have

$$\mathbf{E}\left(\ell_1\ell_2;\ \psi|L_0\right) = \mathbf{E}\left(\ell_1\ell_2;\ \tilde{\psi}|L_0\right).$$

Finally, since v is an eigenvector of ψ and L_0 is the orthogonal complement to v in L, we have

$$\mathbf{E}\left(\ell_1\ell_2;\ \psi|L\right) = \mathbf{E}\left(\ell_1\ell_2;\ \psi|L_0\right) + \mathbf{E}\left(\ell_1\ell_2;\ \psi|\operatorname{span}(v)\right).$$

Therefore,

$$\begin{aligned}\mathbf{E}\left(\ell_1\ell_2;\ \psi|L\right) =& \mathbf{E}\left(\ell_1\ell_2;\ \tilde{\psi}\right) - \mathbf{E}\left(\ell_1\ell_2;\ \tilde{\psi}|\operatorname{span}(w)\right) \\ &- \mathbf{E}\left(\ell_1\ell_2;\ \tilde{\psi}|\operatorname{span}(v)\right) + \mathbf{E}\left(\ell_1\ell_2;\ \psi|\operatorname{span}(v)\right).\end{aligned} \tag{3.3.9}$$

We note that the gradient of function s_k restricted onto $\operatorname{span}(w)$ is $\sqrt{a_k/2c}$. Since w is an eigenvector of $\tilde{\psi}$ with eigenvalue $\epsilon^2/2$, we have

$$\left(\mathbf{E}\, s_{k_1}s_{k_2};\ \tilde{\psi}|\operatorname{span}(w)\right) = \frac{\sqrt{a_{k_1}a_{k_2}}}{2\epsilon^2 c} \leq \frac{1}{2\epsilon^3\sqrt{\delta mn}} \quad \text{for all} \quad k_1, k_2.$$

Similarly,

$$\left(\mathbf{E}\, t_{j_1}t_{j_2};\ \tilde{\psi}|\operatorname{span}(w)\right) = \frac{\sqrt{b_{j_1}b_{j_2}}}{2\epsilon^2 c} \leq \frac{1}{2\epsilon^3\sqrt{\delta mn}} \quad \text{for all} \quad j_1, j_2$$

and

$$\left(\mathbf{E}\, s_k t_j;\ \tilde{\psi}|\operatorname{span}(w)\right) = -\frac{\sqrt{a_k b_j}}{2\epsilon^2 c} \geq -\frac{1}{2\epsilon^3\sqrt{mn}} \quad \text{for all} \quad k, j.$$

Since v is an eigenvector of $\tilde{\psi}$ with eigenvalue $1 - \epsilon^2/2 \geq 1/2$, we obtain

$$\left(\mathbf{E}\, s_{k_1}s_{k_2};\ \tilde{\psi}|\operatorname{span}(v)\right) = \frac{\sqrt{a_{k_1}a_{k_2}}}{4\left(1-\epsilon^2/2\right)c} \leq \frac{1}{2\epsilon\sqrt{\delta mn}} \quad \text{for all} \quad k_1, k_2.$$

Similarly,

$$\left(\mathbf{E}\, t_{j_1}t_{j_2};\ \tilde{\psi}|\operatorname{span}(v)\right) = \frac{\sqrt{b_{j_1}b_{j_2}}}{4\left(1-\epsilon^2/2\right)c} \leq \frac{1}{2\epsilon\sqrt{\delta mn}} \quad \text{for all} \quad j_1, j_2$$

and

$$\Big(\mathbf{E}\, s_k t_j;\ \tilde{\psi} | \operatorname{span}(v)\Big) = \frac{\sqrt{a_k b_j}}{4\left(1-\epsilon^2/2\right)c} \ \leq\ \frac{1}{2\epsilon\sqrt{mn}} \quad \text{for all} \quad k, j.$$

Since v is an eigenvector of ψ with eigenvalue 1, we get

$$(\mathbf{E}\, s_{k_1} s_{k_2};\ \psi | \operatorname{span}(v)) = \frac{\sqrt{a_{k_1} a_{k_2}}}{4c} \ \leq\ \frac{1}{4\epsilon\sqrt{\delta mn}} \quad \text{for all} \quad k_1, k_2.$$

Similarly,

$$(\mathbf{E}\, t_{j_1} t_{j_2};\ \psi | \operatorname{span}(v)) = \frac{\sqrt{b_{j_1} b_{j_2}}}{4c} \ \leq\ \frac{1}{4\epsilon\sqrt{\delta mn}} \quad \text{for all} \quad j_1, j_2$$

and

$$(\mathbf{E}\, s_k t_j;\ \psi | \operatorname{span}(v)) = \frac{\sqrt{a_k b_j}}{4c} \ \leq\ \frac{1}{4\epsilon\sqrt{mn}} \quad \text{for all} \quad k, j.$$

Combining (3.3.8) and (3.3.9), we complete the proof. □

Now we are ready to prove Theorem 3.2.

(3.4) Proof of Theorem 3.2. Let us define

$$\xi_{kj} = \zeta_{kj}^2 + \zeta_{kj} \quad \text{for all} \quad k, j.$$

Hence we have

$$\begin{gathered} \alpha \ \leq\ \xi_{kj} \ \leq\ \beta \quad \text{for all} \quad k, j, \quad \text{where} \\ \alpha = \tau\delta + \tau^2\delta^2 \quad \text{and} \quad \beta = \tau + \tau^2. \end{gathered}$$

We have

$$\frac{\beta}{\alpha} = \frac{\tau + \tau^2}{\tau\delta + \tau^2\delta^2} = \frac{1+\tau}{\delta + \tau\delta^2} \ \leq\ \frac{1}{\delta^2}. \tag{3.4.1}$$

Let

$$a_k = \sum_{j=1}^{n} \xi_{kj} \quad \text{and} \quad b_j = \sum_{k=1}^{m} \xi_{kj}.$$

In particular, we have

$$\begin{aligned} a_k \ &\leq\ \left(\tau + \tau^2\right) n \quad \text{for} \quad k = 1, \ldots, m \quad \text{and} \\ b_j \ &\leq\ \left(\tau + \tau^2\right) m \quad \text{for} \quad j = 1, \ldots, n. \end{aligned} \tag{3.4.2}$$

We apply Proposition 3.3 to the quadratic form

$$\psi = \frac{1}{2} \sum_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} \xi_{kj} \left(\frac{s_k}{\sqrt{a_k}} + \frac{t_j}{\sqrt{b_j}}\right)^2$$

and the hyperplane $L_1 \subset \mathbb{R}^{m+n}$ defined by the equation

$$\sum_{k=1}^{m} s_i \sqrt{a_k} \ = \ \sum_{j=1}^{n} t_j \sqrt{b_j}.$$

Let us consider a linear transformation

$$(s_1, \ldots, s_m;\ t_1, \ldots, t_n) \longmapsto \Big(s_1\sqrt{a_1}, \ldots, s_m\sqrt{a_m};\ t_1\sqrt{b_1}, \ldots, t_n\sqrt{b_n}\Big)$$

and the hyperplane $L_2 \subset \mathbb{R}^{m+n}$ defined by the equation

$$\sum_{k=1}^{m} a_k s_k = \sum_{j=1}^{n} b_j t_j.$$

Then L_2 is mapped onto L_1 and the push-forward of the Gaussian probability measure on L_2 with the density proportional to e^{-q} is the Gaussian probability measure on L_1 with the density proportional to $e^{-\psi}$.

We have

$$\begin{aligned}
&\mathbf{E}\ (s_{k_1}s_{k_2};\ q|L_2) = \frac{1}{\sqrt{a_{k_1}a_{k_2}}}\mathbf{E}\ (s_{k_1}s_{k_2};\ \psi|L_1) \quad \text{for all} \quad k_1 \neq k_2, \\
&\mathbf{E}\ (t_{j_1}t_{j_2};\ q|L_2) = \frac{1}{\sqrt{b_{j_1}b_{j_2}}}\mathbf{E}\ (t_{j_1}t_{j_2};\ \psi|L_1) \quad \text{for all} \quad j_1 \neq j_2, \quad \text{and} \\
&\mathbf{E}\ (s_k t_j;\ q|L_2) = \frac{1}{\sqrt{a_k b_j}}\mathbf{E}\ (s_k t_j;\ \psi|L_1) \quad \text{for all} \quad k, j.
\end{aligned} \tag{3.4.3}$$

We apply Proposition 3.3 to form ψ and hyperplane L_1 with $\beta/\alpha \leq 1/\delta^2$ by (3.4.1). Since by Lemma 3.1, for any hyperplane $L \subset \mathbb{R}^{m+n}$ not containing u we have

$$\mathbf{E}\Big((s_{k_1}+t_{j_1})(s_{k_2}+t_{j_2});\ q|L\Big) = \mathbf{E}\Big((s_{k_1}+t_{j_1})(s_{k_2}+t_{j_2});\ q|L_2\Big),$$

the proof follows by Proposition 3.3 and (3.4.1)–(3.4.3). □

We will need the following result proved in [B97b], see Lemma 2.3 there.

(3.5) Lemma. *Let V be Euclidean space and let $q: V \longrightarrow \mathbb{R}$ be a quadratic form such that* $\operatorname{rank} q = \dim V - 1$. *Let $v \in V$ be the unit eigenvector of q with the eigenvalue* 0 *and let $H = v^{\perp}$ be the orthogonal complement of v. Then for a unit vector $u \in V$ we have*

$$\det q|u^{\perp} = \langle u, v\rangle^2 \det q|H.$$

□

We apply Lemma 3.5 in the following situation. Let $V = \mathbb{R}^{m+n}$ and let q be defined by (1.2.1). Let L be a coordinate hyperplane defined by one of the equations $s_k = 0$ or $t_j = 0$. Then

$$\det q|L = \frac{1}{m+n}\det q|H.$$

In particular, the value of $\det q|L$ does not depend on the choice of the coordinate hyperplane.

Finally, we need the following result.

(3.6) Lemma. *Let $q_0 : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}$ be the quadratic form defined by the formula*

$$q_0(s,t) = \frac{1}{2} \sum_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} (s_k + t_j)^2 .$$

Then the eigenspaces of q_0 are as follows:

The 1-dimensional eigenspace E_1 with the eigenvalue 0 spanned by vector

$$u = \Bigl(\underbrace{1, \ldots, 1}_{m \text{ times}}; \underbrace{-1, \ldots, -1}_{n \text{ times}} \Bigr);$$

The $(n-1)$-dimensional eigenspace E_2 with the eigenvalue $m/2$ consisting of the vectors such that

$$\sum_{j=1}^{n} t_j = 0 \quad \textit{and} \quad s_1 = \ldots = s_m = 0;$$

The $(m-1)$-dimensional eigenspace E_3 with the eigenvalue $n/2$ consisting of the vectors such that

$$\sum_{k=1}^{m} s_k = 0 \quad \textit{and} \quad t_1 = \ldots = t_n = 0$$

and

The 1-dimensional eigenspace E_4 with the eigenvalue $(m+n)/2$ spanned by vector

$$v = \Bigl(\underbrace{n, \ldots, n}_{m \text{ times}}; \underbrace{m, \ldots, m}_{n \text{ times}} \Bigr).$$

Proof. Clearly, E_1 is the eigenspace with the eigenvalue 0. It is then straightforward to check that the gradient of q_0 at $x = (s,t)$ equals mx for $x \in E_2$, equals nx for $x \in E_3$ and equals $(m+n)x$ for $x \in E_4$. □

4. The third degree term

In this section we prove the following main result.

(4.1) Theorem. *Let u_{kj}, $k = 1, \ldots, m$ and $j = 1, \ldots, n$ be Gaussian random variables such that*

$$\mathbf{E}\, u_{kj} = 0 \quad \textit{for all} \quad k, j.$$

Suppose further that for some $\theta > 0$

$$\begin{aligned} \mathbf{E}\, u_{kj}^2 \ &\leq \ \frac{\theta}{m+n} \quad \textit{for all} \quad k, j \quad \textit{and} \\ |\mathbf{E}\, u_{k_1 j_1} u_{k_2 j_2}| \ &\leq \ \frac{\theta}{(m+n)^2} \quad \textit{provided} \quad k_1 \neq k_2 \quad \textit{and} \quad j_1 \neq j_2. \end{aligned}$$

Let

$$U = \sum_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} u_{kj}^3.$$

Then for some absolute constant $\gamma > 0$ and any $0 < \epsilon < 1$ we have

$$\left| \mathbf{E} \exp\{iU\} - \exp\left\{ -\frac{1}{2} \mathbf{E}\, U^2 \right\} \right| \ \leq \ \epsilon$$

provided

$$m + n \ \geq \ \left(\gamma\theta + \gamma \ln \frac{1}{\epsilon} \right)^{\gamma\theta + \gamma \ln \frac{1}{\epsilon}}.$$

Besides,

$$\mathbf{E}\, U^2 \ \leq \ (\gamma\theta)^{\gamma}$$

for some absolute constant $\gamma > 0$. Here $i = \sqrt{-1}$.

We will apply Theorem 4.1 in the following situation. Let $q : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}$ be the quadratic form defined by (1.2.1). Let $L \subset \mathbb{R}^{m+n}$ be a hyperplane not containing the null-space of q. Let us fix the Gaussian probability measure on L with the density proportional to e^{-q}. We define random variables

$$u_{kj} = \sqrt[3]{\frac{\zeta_{kj}\left(\zeta_{kj}+1\right)\left(2\zeta_{kj}+1\right)}{6}} \left(s_k + t_j\right),$$

where $s_1, \ldots, s_m; t_1, \ldots, t_n$ are the coordinates of a point in L. Then we have

$$U = \sum_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} u_{kj}^3 = f(s, t)$$

for f defined by (1.2.2).

(4.2) The expectation of a product of Gaussian random variables. We will use the famous Wick's formula, see, for example, [Zv97]. If $w_1, \ldots, w_l$ are Gaussian random variables such that $\mathbf{E}\, w_1 = \ldots = \mathbf{E}\, w_l = 0$ then

$$\mathbf{E}\, w_1 \cdots w_l = 0 \quad \text{if} \quad l = 2r+1 \quad \text{is odd} \qquad \text{and}$$

$$\mathbf{E}\, w_1 \cdots w_l = \sum \left(\mathbf{E}\, w_{i_1} w_{i_2}\right) \cdots \left(\mathbf{E}\, w_{i_{2r-1}} w_{i_{2r}}\right) \quad \text{if} \quad l = 2r \quad \text{is even}$$

and the sum is taken over all $(2r)!/r!2^r$ unordered partitions of the set of indices $\{1, \ldots, 2r\}$ into r pairwise disjoint unordered pairs $\{i_1, i_2\}, \ldots, \{i_{2r-1}, i_{2r}\}$. Such a partition is called a *matching* of the set $\{1, \ldots, 2r\}$.

In particular, we will use the formula

$$\mathbf{E}\left(w_1^3 w_2^3\right) = 9\left(\mathbf{E}\, w_1^2\right)\left(\mathbf{E}\, w_2^2\right)\left(\mathbf{E}\, w_1 w_2\right) + 6\left(\mathbf{E}\, w_1 w_2\right)^3 \tag{4.2.1}$$

and the formula

$$
\begin{aligned}
\mathbf{cov}\ \left(w_1^4, w_2^4\right) =&\mathbf{E}\ \left(w_1^4 w_2^4\right) - \left(\mathbf{E}\, w_1^4\right)\left(\mathbf{E}\, w_2^4\right) \\
=&9\left(\mathbf{E}\, w_1^2\right)^2\left(\mathbf{E}\, w_2^2\right)^2 + 72\left(\mathbf{E}\, w_1 w_2\right)^2\left(\mathbf{E}\, w_1^2\right)\left(\mathbf{E}\, w_2^2\right) \\
&+ 24\left(\mathbf{E}\, w_1 w_2\right)^4 - 9\left(\mathbf{E}\, w_1^2\right)^2\left(\mathbf{E}\, w_2^2\right)^2 \\
=&72\left(\mathbf{E}\, w_1 w_2\right)^2\left(\mathbf{E}\, w_1^2\right)\left(\mathbf{E}\, w_2^2\right) + 24\left(\mathbf{E}\, w_1 w_2\right)^4
\end{aligned} \tag{4.2.2}
$$

We deduce Theorem 4.1 from the following result.

(4.3) Proposition. *For the Gaussian random variables $\{u_{kj}\}$ of Theorem 4.1, let us define Gaussian random variables $\{v_{kj}\}$, where $k = 1, \ldots, m$ and $j = 1, \ldots, n$ such that*

$$
\begin{aligned}
&\mathbf{E}\, v_{kj} = 0 \quad \textit{for all} \quad k, j \quad \textit{and} \\
&\mathbf{E}\ \left(v_{k_1 j_1} v_{k_2 j_2}\right) = \mathbf{E}\ \left(u^3_{k_1 j_1} u^3_{k_2 j_2}\right) \quad \textit{for all} \quad k_1, k_2 \quad \textit{and} \quad j_1, j_2.
\end{aligned}
$$

Let

$$
V = \sum_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} v_{kj} \quad \textit{and} \quad U = \sum_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} u^3_{kj},
$$

as in Theorem 4.1. Then for a positive integer r we have

$$
\left|\mathbf{E}\, U^{2r} - \mathbf{E}\, V^{2r}\right| \ \leq \ \frac{(\gamma \theta r)^{\gamma r}}{m+n}.
$$

for some constant $\gamma > 0$.

We will use some combinatorics to estimate $\mathbf{E}\, U^{2r}$ and $\mathbf{E}\, V^{2r}$.

(4.4) Weighted graphs and matchings. Let us consider the complete bipartite graph $K_{m,n}$ with $m+n$ vertices and mn edges (k, j) for $k = 1, \ldots, m$ and $j = 1, \ldots, n$. A *weighted graph* G is a multiset of edges (k, j) of $K_{m,n}$ with positive integer weights μ_{ij} on them. The *total weight* of G is the sum of all weights μ_{ij} over the edges of G. Let A be a subset of vertices of G. Then A defines an *induced weighted subgraph* G_A of G consisting of the edges of G that have both vertices in A and such that the weight of an edge in G_A is equal to its weight in G. If every two vertices in A can be connected by a path consisting of edges of G_A and there are no edges of G with one vertex in A and the other not in A then G_A is called a *connected component* of G. We note that if G_A is a connected component of G with the total weight k then A contains at most $k+1$ vertex. Moreover, G_A contains precisely $k+1$ vertex if and only if G_A is a tree with weight 1 on every edge. Clearly, a weighted graph is a vertex-disjoint union of its connected components.

Let x_{kj} be variables indexed by the edges of $K_{m,n}$. Given a weighted graph G, we define the monomial

$$t_G(x) = \prod_{(k,j) \in G} x_{kj}^{\mu_{kj}}.$$

We say that two weighted graphs G_1 and G_2 are *isomorphic* if there are bijections $\phi : \{1, \dots, m\} \longrightarrow \{1, \dots, m\}$ and $\psi : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$ such that (k, j) is an edge of G_1 if and only if $(\phi(k), \psi(j))$ is an edge of G_2 and, moreover, the weight of (k, j) in G_1 and the weight of $(\phi(k), \psi(j))$ in G_2 are equal.

We define a polynomial

$$p_G(x) = \sum_{G'} t_{G'}(x),$$

where the sum is taken over all distinct weighted graphs G' isomorphic to G. We note that

$$\left(\sum_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} x_{kj} \right)^{2r} = \sum_G a_G p_G \left(x_{kj} \right), \tag{4.4.1}$$

where the sum is taken over all pairwise non-isomorphic weighted graphs G of total weight $2r$ and a_G are integers such that

$$1 \ \leq \ a_G \ \leq \ (2r)! \quad \text{for all} \quad G.$$

We need some crude bounds on the number of graphs of various types. First, we consider the number of weighted graphs of the total weight $2r$ on a set of $s_1 + s_2$ vertices. We can obtain such a graph as follows: we consider $2r$ vertex-disjoint edges and glue them together into a graph by identifying one endpoint of every edge with one of the s_1 vertices and the other endpoint of the edge with one of the s_2 vertices. Hence the total number of graphs does not exceed $(s_1 s_2)^{2r}$. Consequently, the number of pairwise non-isomorphic weighted graphs of total weight $2r$ does not exceed $(2r)^{4r+2}$.

If a graph G has $s_1 + s_2$ vertices then the number of monomials in polynomial $p_G \left(x_{kj} \right)$ does not exceed $m^{s_1} n^{s_2} \ \leq \ (m+n)^s$, where $s = s_1 + s_2$ is the total number of vertices.

Wick's formula (see Section 4.2) implies that for Gaussian random variables w_{kj} indexed by the edges $e = (k, j)$ of G and such that $\mathbf{E}\, w_{kj} = 0$ the expectation of the monomial t_G is computed as follows: we replace each edge $e = (k, j)$ of weight μ_{kj} of G by μ_{kj} distinct copies of e and compute

$$\mathbf{E}\, t_G \left(w_{kj} \right) = \sum_M \left(\mathbf{E}\, w_{e_1} w_{e_2} \right) \cdots \left(\mathbf{E}\, w_{e_{2r-1}} w_{2r} \right), \tag{4.4.2}$$

where the sum is taken over all matchings M of the set $\{e_1, \dots, e_{2r}\}$ of the edges.

Recall that γ, possibly with an index, denotes a positive absolute constant whose value may change from line to line.

(4.5) Lemma. *Let u_{kj} and v_{kj} be random variables as in Proposition 4.3. Let G be a weighted graph of weight $2r$.*

(1) *Suppose that each connected component of G is either an isolated edge of weight 1 or a pair of edges of weight 1 each, sharing precisely one common vertex. Suppose further that the number of isolated edges is even. Then*

$$\left|\mathbf{E}\, p_G\left(u_{kj}^3\right)\right|, \ \left|\mathbf{E}\, p_G\left(v_{kj}\right)\right| \ \leq \ (\gamma\theta r)^{\gamma r}.$$

(2) *Suppose that the structure of G differs from that described in Part (1). Then*

$$\left|\mathbf{E}\, p_G\left(u_{ij}^3\right)\right|, \ \left|\mathbf{E}\, p_G\left(v_{ij}\right)\right| \ \leq \ \frac{(\gamma\theta r)^{\gamma r}}{m+n}$$

for some constant $\gamma > 0$.

Proof. Let d be the number of connected components of G that consist of two vertices of G connected by an edge of weight 1. Thus we may write G as a disjoint union $G = G_1 \cup G_2$, where G_1 is a vertex-disjoint union of d edges of weight 1 each and G_2 is a weighted graph of total weight $2r - d$. Moreover, since the weight of each connected component of G_2 is at least 2, graph G_2 contains at most

$$\frac{3}{2}(2r-d) = 3r - \frac{3}{2}d$$

vertices with the equality if and only if d is even and each connected component of G_2 is a pair of edges of weight 1 each sharing precisely one common vertex. Therefore, G has at most

$$2d + 3r - \frac{3}{2}d = 3r + \frac{d}{2}$$

vertices. Therefore, there are at most

$$(m+n)^{3r+d/2}$$

monomials in p_G and if G contains a connected component other than an edge of weight 1 or two edges of weight 1 sharing one common vertex then the number of monomials in p_G does not exceed

$$(m+n)^{3r+d/2-1}.$$

If d is odd then the number of monomials in p_G does not exceed

$$(m+n)^{3r+(d-1)/2}.$$

We handle $p_G(v_{kj})$ first. Applying Wick's formula, see Section 4.2 and (4.2.1) in particular, we obtain

$$\begin{gathered}
|\mathbf{E}\, v_{k_1 j_1} v_{k_2 j_2}| \ \leq \ \frac{15\theta^3}{(m+n)^3} \ \leq \ \frac{(\gamma\theta)^\gamma}{(m+n)^3} \\
\text{for all} \quad k_1, j_1 \quad \text{and all} \quad k_2, j_2 \\
\text{and} \\
|\mathbf{E}\, v_{k_1 j_1} v_{k_2 j_2}| \ \leq \ \frac{9\theta^3}{(m+n)^4} + \frac{6\theta^3}{(m+n)^6} \ \leq \ \frac{(\gamma\theta)^\gamma}{(m+n)^4} \\
\text{provided} \quad k_1 \neq k_2 \quad \text{and} \quad j_1 \neq j_2.
\end{gathered} \tag{4.5.1}$$

Let us replace each edge e of G of weight μ_e by μ_e distinct copies and consider a matching

$$M = \Big\{ \{e_1, e_2\}, \ldots, \{e_{2r-1}, e_{2r}\} \Big\}$$

of the obtained $2r$ edges and the corresponding product of expectations

$$(\mathbf{E}\, v_{e_1} v_{e_2}) \cdots \mathbf{E}\, \big(v_{e_{2r-1}} v_{e_{2r}}\big). \tag{4.5.2}$$

We note that each of the r factors does not exceed $(\gamma\theta)^\gamma/(m+n)^3$ in the absolute value. Moreover, each of the d isolated edges of G has to be paired with an edge from a different connected component (possibly with another isolated edge), from which there are at least $d/2$ factors in (4.5.2) whose value does not exceed $(\gamma\theta)^\gamma/(m+n)^4$. Therefore, the absolute value of (4.5.2) is at most

$$\left(\frac{(\gamma\theta)^\gamma}{(m+n)^4}\right)^{d/2} \left(\frac{(\gamma\theta)^\gamma}{(m+n)^3}\right)^{r-d/2} \ \leq \ \frac{(\gamma\theta)^{\gamma r}}{(m+n)^{3r+d/2}}.$$

Consequently, cf. (4.4.2), for the monomial t_G, we have

$$|\mathbf{E}\, t_G(v_{kj})| \ \leq \ \frac{(2r)!}{r! 2^r} \frac{(\gamma\theta)^{\gamma r}}{(m+n)^{3r+d/2}} \ \leq \ \frac{(\gamma_1 \theta r)^{\gamma_1 r}}{(m+n)^{3r+d/2}}.$$

If d is odd then

$$|\mathbf{E}\, t_G(v_{kj})| \ \leq \ \frac{(\gamma\theta r)^{\gamma r}}{(m+n)^{3r+(d+1)/2}}.$$

Using the bound on the number of monomials in p_G, we complete the estimate for $\mathbf{E}\, p_G(v_{kj})$.

We consider $p_G\left(u_{ij}^3\right)$ next. Let $\hat{G}$ be the weighted graph of the total weight $6r$ which is obtained from G by multiplying the weight of every edge by 3. Let us replace each edge e in $\hat{G}$ of weight μ_e by μ_e distinct copies and let us consider a matching M be of the obtained $6r$ edges and the corresponding product of expectations

$$(\mathbf{E}\, u_{e_1} u_{e_2}) \cdots (\mathbf{E}\, u_{6r-1} u_{6r}). \tag{4.5.3}$$

We note that each of the $3r$ factors does not exceed $\theta/(m+n)$ in the absolute value. Moreover, $\hat{G}$ contains d connected components, each of which is an edge of weight 3. At least one copy of such an edge has to be paired with a copy of an edge from a different connected component, which creates at least $d/2$ factors in (4.5.3) not exceeding $\theta/(m+n)^2$ in the absolute value. Therefore, the absolute value of (4.5.3) is at most

$$\left(\frac{\theta}{(m+n)^2}\right)^{d/2}\left(\frac{\theta}{m+n}\right)^{3r-d/2} = \frac{\theta^{3r}}{(m+n)^{3r+d/2}}$$

with equality possible only if d is even. Consequently, cf. (4.4.2), for every monomial t_G, we have

$$\left|\mathbf{E}\, t_G\left(u_{kj}^3\right)\right| \leq \frac{(6r)!}{(3r)!2^{3r}}\frac{\theta^{3r}}{(m+n)^{3r+d/2}} \leq \frac{(\gamma\theta r)^{\gamma r}}{(m+n)^{3r+d/2}}.$$

If d is odd then

$$\left|\mathbf{E}\, t_G\left(u_{kj}^3\right)\right| \leq \frac{(\gamma\theta r)^{\gamma r}}{(m+n)^{3r+(d+1)/2}}$$

and the proof is concluded as above. □

(4.6) Lemma. *Let u_{kj} and v_{kj} be random variables as in Proposition 4.3. Let G be a weighted graph of weight $2r$ with the connected components as in Part (1) of Lemma 4.5. Then*

$$\left|\mathbf{E}\, p_G\left(v_{kj}\right) - \mathbf{E}\, p_G\left(u_{kj}^3\right)\right| \leq \frac{(\gamma\theta r)^{\gamma r}}{(m+n)^2}$$

for some constant $\gamma > 0$.

Proof. Suppose that the connected components of G are an even number d of vertex disjoint edges of weight 1 each and $r - d/2$ pairs of edges of weight 1 each, each pair sharing precisely one vertex. Let $\mathcal{M}$ be the set of matchings of the edges of G that match each isolated edge with another isolated edge and match each edge in a connected component consisting of two edges with the other edge in that connected component. We define

$$a_G\left(v_{kj}\right) = \sum_{\substack{M\in\mathcal{M} \\ M=\{\{e_1,e_2\},\ldots,\{e_{2r-1},e_{2r}\}\}}} \left(\mathbf{E}\, v_{e_1}v_{e_2}\right)\cdots\left(\mathbf{E}\, v_{e_{2r-1}}v_{e_{2r}}\right).$$

If $M \notin \mathcal{M}$ is a matching of the edges of G, $M = \{\{e_1, e_2\}, \ldots, \{e_{2r-1}, e_{2r}\}\}$, then some $d/2 + 2$ factors match edges from different connected components and hence from (4.5.1) we have

$$\left|\left(\mathbf{E}\, v_{e_1}v_{e_2}\right)\cdots\left(\mathbf{E}\, v_{e_{2r-1}}v_{e_{2r}}\right)\right| \leq \frac{(\gamma\theta)^{\gamma r}}{(m+n)^{3r+d/2+2}}.$$

Therefore, by (4.4.2) for the monomial t_G we have

$$
\begin{aligned}
|\mathbf{E}\, t_G\left(v_{kj}\right) - a_G\left(v_{kj}\right)| \;&\leq\; \frac{(2r)!}{r!2^r}\frac{(\gamma\theta)^{\gamma r}}{(m+n)^{3r+d/2+2}} \\
&\leq \frac{(\gamma_1 \theta r)^{\gamma_1 r}}{(m+n)^{3r+d/2+2}}.
\end{aligned} \tag{4.6.1}
$$

Let us define a weighted graph $\hat{G}$ as in the proof of Lemma 4.5 by multiplying the weight of every edge of G by 3. Hence $\hat{G}$ is represented as a vertex-disjoint union of the connected components

$$
\hat{G} = \left(\bigcup_{i=1}^{d} E_i\right) \bigcup \left(\bigcup_{i=1}^{r-d/2} F_i\right),
$$

where each E_i is an edge of weight 3 and each F_i is a union of two edges of weight 3 each having exactly one common vertex. Let us replace every edge of $\hat{G}$ by 3 distinct copies and let $\hat{\mathcal{M}}$ be the set of matchings of the set of the $6r$ edges such that

for each $1 \leq i \leq d$ there is $1 \leq j \leq d$, $j \neq i$ such that the edges of $E_i \cup E_j$ are matched with edges of $E_i \cup E_j$

and

for each $i = 1, \ldots, r - d/2$, the edges of F_i are matched with edges of F_i.

We define

$$
b_G\left(u_{kj}^3\right) = \sum_{\substack{M \in \hat{\mathcal{M}} \\ M = \{\{e_1, e_2\}, \ldots, \{e_{6r-1}, e_{6r}\}\}}} \left(\mathbf{E}\, u_{e_1} u_{e_2}\right) \cdots \left(\mathbf{E}\, u_{e_{6r-1}} u_{e_{6r}}\right).
$$

If $M \notin \hat{\mathcal{M}}$ is a matching of the edges of $\hat{G}$ then some $d/2+2$ factors match edges in different connected components and hence we have

$$
\left|\left(\mathbf{E}\, u_{e_1} u_{e_2}\right) \cdots \left(\mathbf{E}\, u_{e_{6r-1}} e_{6r}\right)\right| \;\leq\; \frac{\theta^{3r}}{(m+n)^{3r+d/2+2}}
$$

and, consequently,

$$
\begin{aligned}
\left|\mathbf{E}\, t_G\left(u_{kj}^3\right) - b_G\left(u_{kj}^3\right)\right| \;&\leq\; \frac{(6r)!}{(3r)!2^{3r}}\frac{\theta^{3r}}{(m+n)^{3r+d/2+2}} \\
&\leq \frac{(\gamma\theta r)^{\gamma r}}{(m+n)^{3r+d/2+2}}.
\end{aligned} \tag{4.6.2}
$$

On the other hand,

$$a_G(v_{kj}) = b_G\left(u_{kj}^3\right).$$

Therefore, by (4.6.1) and (4.6.2) we conclude that

$$\left|\mathbf{E}\, t_G(v_{kj}) - \mathbf{E}\, t_G\left(u_{kj}^3\right)\right| \ \leq \ \frac{(\gamma\theta r)^{\gamma r}}{(m+n)^{3r+d/2+2}}. \tag{4.6.3}$$

Since graph G contains $3r+d/2$ vertices, polynomial p_G contains not more than $(m+n)^{3r+d/2}$ monomials $t_{G'}$ where G' is isomorphic to G and the proof follows by (4.6.3). □

(4.7) Proof of Proposition 4.3. We use expansion (4.4.1), Lemma 4.6, Part (2) of Lemma 4.5 and the fact that the number of different polynomials $p_G(x)$ in (4.4.1) does not exceed $(2r)^{4r+2}$. □

(4.8) Proof of Theorem 4.1. Let v_{kj} and V be Gaussian random variables of Proposition 4.3. We have

$$\mathbf{E}\, V^2 = \sum_{\substack{1\leq k_1,k_2\leq m\\ 1\leq j_1,j_2\leq n}} \mathbf{E}\, v_{k_1j_1}v_{k_2j_2}.$$

Using (4.5.1), we obtain the estimate

$$\begin{aligned}\mathbf{E}\, V^2 \ &\leq m^2n^2\frac{(\gamma\theta)^\gamma}{(m+n)^4} + mn^2\frac{(\gamma\theta)^\gamma}{(m+n)^3} + nm^2\frac{(\gamma\theta)^\gamma}{(m+n)^3}\\ &\leq (\gamma_1\theta)^{\gamma_1}.\end{aligned}$$

Therefore,

$$\mathbf{E}\, U^2 = \mathbf{E}\, V^2 \ \leq \ (\gamma\theta)^\gamma.$$

By Wick's formula, see Section 4.2, we get

$$\mathbf{E}\, V^{2r} = \frac{(2r)!}{r!2^r}\left(\mathbf{E}\, V^2\right)^r \ \leq \ \frac{(2r)!}{r!2^r}(\gamma\theta)^{\gamma r}.$$

We use the standard estimate

$$\left|e^{ix} - \sum_{s=0}^{2r-1}\frac{(ix)^s}{s!}\right| \ \leq \ \frac{x^{2r}}{(2r)!} \quad \text{for} \quad x\in\mathbb{R}.$$

Thus

$$\left|\mathbf{E}\, e^{iV} - \sum_{s=0}^{2r-1} i^s\frac{\mathbf{E}\, V^s}{s!}\right| \ \leq \ \frac{\mathbf{E}\, V^{2r}}{(2r)!} \ \leq \ \frac{(\gamma\theta)^{\gamma r}}{r!2^r}.$$

Hence we can choose

$$r \leq \gamma \max\left\{\theta,\ \ln \frac{1}{\epsilon}\right\}$$

so that

$$\left| \mathbf{E}\, e^{iV} - \sum_{s=0}^{2r-1} i^s \frac{\mathbf{E}\, V^s}{s!} \right| \leq \frac{\epsilon}{3}.$$

Similarly, we have

$$\left| \mathbf{E}\, e^{iU} - \sum_{s=0}^{2r-1} i^s \frac{\mathbf{E}\, U^s}{s!} \right| \leq \frac{\mathbf{E}\, U^{2r}}{(2r)!}.$$

We note that by symmetry

$$\mathbf{E}\, V^s = \mathbf{E}\, U^s = 0 \quad \text{provided} \quad s \quad \text{is odd.}$$

By Proposition 4.3, as long as

$$m+n \geq \left(\gamma\theta + \gamma \ln \frac{1}{\epsilon}\right)^{\gamma\theta + \gamma \ln \frac{1}{\epsilon}}$$

we have

$$|\mathbf{E}\, V^s - \mathbf{E}\, U^s| \leq \frac{\epsilon}{10} \quad \text{for} \quad s = 0, \ldots, 2r.$$

Since V is Gaussian, we have

$$\mathbf{E}\, e^{iV} = \exp\left\{-\frac{1}{2}\mathbf{E}\, V^2\right\},$$

which completes the proof. □

5. The fourth degree term

We start with the following result.

(5.1) Proposition. *Let u_{kj}, $k = 1, \ldots, m$ and $j = 1, \ldots, n$ be Gaussian random variables as in Theorem 4.1 and let*

$$W = \sum_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} u_{kj}^4.$$

Then for some absolute constant $\gamma > 0$ we have

(1)

$$\mathbf{E}\, W \leq \gamma \theta^2 \quad \textit{and}$$

(2)

$$\mathbf{var}\, W \leq \frac{\gamma \theta^4}{m+n}.$$

Proof. Using Wick's formula, see Section 4.2, we get

$$\mathbf{E}\, u_{kj}^4 = 3\left(\mathbf{E}\, u_{kj}^2\right)^2 \ \le\ \frac{3\theta^2}{(m+n)^2}$$

and hence

$$\mathbf{E}\, W = \sum_{\substack{1\le k\le m\\ 1\le j\le n}} \mathbf{E}\, u_{kj}^4 \ \le\ (mn)\frac{3\theta^2}{(m+n)^2} \ \le\ 3\theta^2,$$

which proves Part (1).

To prove Part (2), we note that

$$\mathbf{var}\, W = \sum_{\substack{1\le k_1,k_2\le m\\ 1\le j_1,j_2\le n}} \mathbf{cov}\ \left(u_{k_1j_1}^4, u_{k_2j_2}^4\right).$$

Using (4.2.2), we get

$$\mathbf{cov}\ \left(u_{k_1j_1}^4, u_{k_2j_2}^4\right) = 72\left(\mathbf{E}\, u_{k_1j_1}u_{k_2j_2}\right)^2 \left(\mathbf{E}\, u_{k_1j_1}^2\right)\left(\mathbf{E}\, u_{k_2j_2}^2\right) + 24\left(\mathbf{E}\, u_{k_1j_1}u_{k_2j_2}\right)^4.$$

Hence

$$\mathbf{cov}\ \left(u_{k_1j_1}^4, u_{k_2j_2}^4\right) \ \le\ \frac{96\theta^4}{(m+n)^4} \quad \text{for all} \quad k_1, k_2 \quad \text{and} \quad j_1, j_2.$$

Additionally,

$$\begin{aligned}\mathbf{cov}\ \left(u_{k_1j_1}^4, u_{k_2j_2}^4\right) \ &\le\ \frac{72\theta^4}{(m+n)^6} + \frac{24\theta^4}{(m+n)^8}\\ &\le\ \frac{96\theta^4}{(m+n)^6} \quad \text{provided} \quad k_1\ne k_2 \quad \text{and} \quad j_1\ne j_2.\end{aligned}$$

Summarizing,

$$\mathbf{var}\, W \ \le\ m^2n^2\frac{96\theta^4}{(m+n)^6} + \left(mn^2+nm^2\right)\frac{96\theta^4}{(m+n)^4} \ \le\ \frac{\gamma\theta^4}{m+n}.$$

□

We will apply Proposition 5.1 in the following situation. Let $q:\mathbb{R}^n \longrightarrow \mathbb{R}$ be the quadratic form defined by (1.2.1). Let $L\subset \mathbb{R}^{m+n}$ be a hyperplane not containing the null-space of q. Let us fix the Gaussian probability measure in L with the density proportional to e^{-q}. We define random variables

$$u_{kj} = \sqrt[4]{\frac{\zeta_{kj}\left(\zeta_{kj}+1\right)\left(6\zeta_{kj}^2+6\zeta_{kj}+1\right)}{24}}\left(s_k+t_j\right),$$

where $s_1,\ldots,s_m;t_1,\ldots,t_n$ are the coordinates of a point in L. Then we have

$$W = \sum_{\substack{1\le k\le m\\ 1\le j\le n}} u_{kj}^4 = h(s,t)$$

for h defined by (1.2.2).

Next, we want to show that W has an exponential tail. For that, we use the following result of Duoandikoetxea [Du87], see Corollary 5 there.

(5.2) Lemma. *Let p be a polynomial of degree d in random Gaussian variables $w_1, \ldots, w_l$.*

Then for $r > 2$ we have

$$\left(\mathbf{E}\,|p|^r\right)^{1/r} \ \leq \ r^{d/2}\left(\mathbf{E}\,p^2\right)^{1/2}.$$

□

We obtain the following corollary.

(5.3) Lemma. *For some absolute constant γ we have*

$$\mathbf{P}\left\{W \ > \ \gamma\theta^2 + 1\right\} \ \leq \ \exp\left\{-(m+n)^{1/5}\right\}$$
$$\text{provided} \quad m+n \ \geq \ \gamma\theta^{\gamma}.$$

Proof. Applying Lemma 5.2 with $d = 4$ to $p(W) = W - \mathbf{E}\,W$, from Part (2) of Proposition 5.1, we get

$$\mathbf{E}\,|W - \mathbf{E}\,W|^r \ \leq \ r^{2r}\left(\mathbf{var}\,W\right)^{r/2} \ \leq \ r^{2r}\left(\frac{\gamma\theta^4}{m+n}\right)^{r/2}.$$

Let us choose

$$r = (m+n)^{1/5}.$$

Then

$$\begin{aligned} r^{2r}\left(\frac{\gamma\theta^4}{m+n}\right)^{r/2} &= \exp\left\{2r\ln r + \frac{r}{2}\ln\left(\gamma\theta^4\right) - \frac{r}{2}\ln(m+n)\right\} \\ &= \exp\left\{-\frac{1}{10}(m+n)^{1/5}\ln(m+n) + \frac{\ln\left(\gamma\theta^4\right)}{2}(m+n)^{1/5}\right\} \\ &\leq \exp\left\{-\frac{(m+n)^{1/5}}{10}\right\} \end{aligned}$$

provided $\ln(m+n) > 5\ln\left(\gamma\theta^4\right) + 10$.

By Part (1) of Proposition 5.1, we have $\mathbf{E}\,W \leq \gamma\theta^2$ and the proof follows by Markov's inequality. □

6. Computing the integral over a neighborhood of the origin

We consider the integral

$$\int_{\Pi_0} F(s,t)\ dsdt$$

of Corollary 2.3.

Recall that the quadratic form $q : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}$ is defined by

$$q(s,t) = \frac{1}{2}\sum_{\substack{1\leq k\leq m \\ 1\leq j\leq n}}\left(\zeta_{kj}^2 + \zeta_{kj}\right)\left(s_k + t_j\right)^2,$$

cf. (1.2.1). In this section, we prove the following main result.

(6.1) Theorem. *Let us fix a number* $0 < \delta < 1$. *Suppose that* $m \geq \delta n$, $n \geq \delta m$ *and that*

$$\delta\tau \ \leq \ \zeta_{kj} \ \leq \ \tau \quad \textit{for all} \quad k, j$$

and some $\tau > \delta$.

Let $f(s,t)$ *and* $h(s,t)$ *be the polynomials defined by (1.2.2). Let us define a neighborhood* $U \subset \Pi_0$ *by*

$$U = \left\{ (s,t) \in \Pi_0 : \quad |s_k|, |t_j| \ \leq \ \frac{\ln(m+n)}{\tau\sqrt{m+n}} \quad \textit{for all} \quad k, j \right\}.$$

Let us identify the hyperplane $t_n = 0$ *containing* Π_0 *with* $\mathbb{R}^{m+n-1}$, *let*

$$\Xi = \int_{\mathbb{R}^{m+n-1}} e^{-q} \ dsdt$$

and let us consider the Gaussian probability measure in $\mathbb{R}^{m+n-1}$ *with the density* $\Xi^{-1}e^{-q}$. *Let*

$$\mu = \mathbf{E}\, f^2 \quad \textit{and} \quad \nu = \mathbf{E}\, h.$$

Then

(1)

$$\mu, \ \nu \ \leq \gamma(\delta)$$

for some constant $\gamma(\delta) > 0$.

(2)

$$\Xi \ \geq \ \frac{(2\pi)^{(m+n-1)/2}}{m^{(n-1)/2} n^{(m-1)/2} (\tau + \tau^2)^{(m+n-1)/2}}.$$

(3)

$$\left| \int_U F(s,t) \ dsdt - \exp\left\{ -\frac{\mu}{2} + \nu \right\} \Xi \right| \ \leq \ \epsilon \exp\left\{ -\frac{\mu}{2} + \nu \right\} \Xi$$

provided

$$m + n \ \geq \ \left(\gamma(\delta) + \ln \frac{1}{\epsilon} \right)^{\gamma(\delta) + \ln \frac{1}{\epsilon}}$$

for some constant $\gamma(\delta) > 0$.

(4)

$$\left| \int_U |F(s,t)| \ dsdt - \exp\{\nu\}\Xi \right| \ \leq \ \epsilon \exp\{\nu\}\, \Xi$$

provided

$$m + n \ \geq \ \left(\gamma(\delta) + \ln \frac{1}{\epsilon} \right)^{\gamma(\delta) + \ln \frac{1}{\epsilon}}$$

for some constant $\gamma(\delta) > 0$.

Proof. From the Taylor series expansion, we write

$$\begin{aligned} F(s,t) &= \exp\left\{-q(s,t) - if(s,t) + h(s,t) + \rho(s,t)\right\} \quad \text{and} \\ |F(s,t)| &= \exp\left\{-q(s,t) + h(s,t) + \rho_1(s,t)\right\} \quad \text{where} \\ &\qquad |\rho(s,t)|,\ |\rho_1(s,t)| \ \leq\ \gamma(1+\tau)^5 \sum_{\substack{1\leq k\leq m\\ 1\leq j\leq n}} |s_k+t_j|^5 \quad \text{for} \quad (s,t)\in\Pi. \end{aligned}$$

In particular,

$$|\rho(s,t)|,\ |\rho_1(s,t)| \ \leq\ \gamma\frac{\ln^5(m+n)}{\sqrt{m+n}} \quad \text{for} \quad (s,t)\in U$$

for some absolute constant $\gamma>0$. Therefore, for any $0<\epsilon<1/2$ we have

$$\begin{aligned} &\left|\int_U F\ dsdt - \int_U \exp\left\{-q-if+h\right\}\ dsdt\right| \ \leq\ \epsilon\int_U \exp\{-q+h\}\ dsdt \\ &\qquad\qquad \text{and} \\ &\left|\int_U |F|\ dsdt - \int_U \exp\left\{-q+h\right\}\ dsdt\right| \ \leq\ \epsilon\int_U \exp\{-q+h\}\ dsdt \\ &\qquad\qquad \text{provided} \quad m+n>\epsilon^{-\gamma} \end{aligned} \tag{6.1.1}$$

for some absolute constant $\gamma>0$.

We think of the coordinate functions s_k and t_j as of random variables with respect to the Gaussian probability measure on $\mathbb{R}^{m+n-1}$ with the density $\Xi^{-1}e^{-q}$. By Theorem 3.2, we have

$$\begin{aligned} &\mathbf{E}\ (s_k+t_j)^2 \ \leq\ \frac{\gamma(\delta)}{\tau^2(m+n)} \quad \text{for all} \quad k,j \qquad \text{and} \\ &|\mathbf{E}\ (s_{k_1}+t_{j_1})\,(s_{k_2}+t_{j_2})| \ \leq\ \frac{\gamma(\delta)}{\tau^2(m+n)^2} \quad \text{if} \quad k_1\neq k_2 \quad \text{and} \quad j_1\neq j_2. \end{aligned} \tag{6.1.2}$$

Letting

$$u_{kj} = \sqrt[4]{\frac{\zeta_{kj}\,(\zeta_{kj}+1)\left(6\zeta_{kj}^2+6\zeta_{kj}+1\right)}{24}}\,(s_k+t_j)$$

We obtain Gaussian random variables u_{kj} such that

$$\begin{aligned} &\mathbf{E}\,u_{kj}^2 \ \leq\ \frac{\gamma(\delta)}{m+n} \quad \text{for all} \quad k,j \qquad \text{and} \\ &|\mathbf{E}\,u_{k_1j_1}u_{k_2j_2}| \ \leq\ \frac{\gamma(\delta)}{(m+n)^2} \quad \text{provided} \quad k_1\neq k_2 \quad \text{and} \quad j_1\neq j_2. \end{aligned}$$

Besides,

$$h = \sum_{\substack{1 \le k \le m \\ 1 \le j \le n}} u_{kj}^4.$$

By Part (1) of Proposition 5.1, we have

$$\nu = \mathbf{E}\, h \ \le\ \gamma(\delta), \tag{6.1.3}$$

which proves one of the inequalities in Part (1).

By Part (2) of Proposition 5.1, we have

$$\mathbf{var}\, h = \mathbf{E}\,(h-\nu)^2 \ \le\ \frac{\gamma(\delta)}{(m+n)}.$$

In particular, by Chebyshev's inequality

$$\mathbf{P}\left\{|h-\nu| > \epsilon\right\} \ \le\ \frac{\gamma(\delta)}{(m+n)\epsilon^2}. \tag{6.1.4}$$

Besides, by Lemma 5.3, we have

$$\mathbf{P}\left\{h > \gamma(\delta)\right\} \ \le\ \exp\left\{-(m+n)^{1/5}\right\}$$

provided $m+n \ \ge\ \gamma_1(\delta)$. Since

$$h(s,t) \le \gamma(\delta)\ln^4(m+n) \quad \text{for} \quad (s,t) \in U,$$

we conclude that $h(s,t)$ is essentially bounded in U by some constant $\gamma(\delta)$:

$$\int\limits_{\substack{(s,t)\in U: \\ h(s,t)\ge\gamma(\delta)}} \exp\{-q+h\}\ dsdt \ \le\ \epsilon\Xi$$

provided $m+n > \gamma_1(\delta)\ln(1/\epsilon)$. From this and (6.1.4) we have that for any $0 < \epsilon < 1/2$

$$\left|\int_U \exp\{-q-if+h\}\ dsdt - e^{\nu}\int_U \exp\{-q-if\}\ ds\ dt\right| \ \le\ \epsilon\Xi \quad \text{provided} \quad m+n \ge \epsilon^{-\gamma(\delta)} \tag{6.1.5}$$

and similarly,

$$\left|\int_U \exp\{-q+h\}\ dsdt - e^{\nu}\int_U \exp\{-q\}\ ds\ dt\right| \ \le\ \epsilon\Xi \quad \text{provided} \quad m+n \ge \epsilon^{-\gamma(\delta)}. \tag{6.1.6}$$

Since s_k+t_j is a Gaussian random variable, the first inequality of (6.1.2) implies that

$$\mathbf{P}\left\{|s_k+t_j| \ \geq \ \gamma(\delta)\frac{\ln(m+n)}{\tau\sqrt{m+n}}\right\} \ \leq \ \exp\left\{-\gamma_1(\delta)\ln^2(m+n)\right\}$$

for some constants $\gamma(\delta), \gamma_1(\delta)>0$. Since these inequalities hold for $t_n=0$ as well, we obtain similar bounds for $|s_k|$ and then for $|t_j|$ for all k and j. Therefore,

$$\int_{\mathbb{R}^{m+n-1}\setminus U} e^{-q} \ dsdt \ \leq \ \exp\left\{-\gamma(\delta)\ln^2(m+n)\right\}\Xi,$$

which together with (6.1.6) and (6.1.1) proves Part (4).

Moreover, for any $0<\epsilon<1/2$

$$(6.1.7)\qquad \left|\int_U \exp\{-q+if\} \ dsdt - \int_{\mathbb{R}^{m+n-1}} \exp\{-q+if\} \ dsdt\right| \ \leq \ \epsilon\Xi \quad \text{provided} \quad m+n\geq \epsilon^{-\gamma(\delta)}.$$

Let us now redefine

$$u_{kj} = \sqrt[3]{\frac{\zeta_{kj}\left(\zeta_{kj}+1\right)\left(2\zeta_{kj}+1\right)}{6}}\left(s_k+t_j\right) \quad \text{for all} \quad k,j.$$

Again, by Theorem 3.2 we obtain Gaussian random variables such that

$$\begin{aligned} \mathbf{E}\, u_{kj}^2 \ &\leq \ \frac{\gamma(\delta)}{m+n} \quad \text{for all} \quad k,j \qquad \text{and} \\ |\mathbf{E}\, u_{k_1j_1}u_{k_2j_2}| \ &\leq \ \frac{\gamma(\delta)}{(m+n)^2} \quad \text{provided} \quad k_1\neq k_2 \quad \text{and} \quad j_1\neq j_2. \end{aligned}$$

Besides,

$$f=\sum_{\substack{1\leq k\leq m\\ 1\leq j\leq n}} u_{kj}^3.$$

By Theorem 4.1, we have

$$(6.1.8)\qquad \left|\int_{\mathbb{R}^{m+n-1}} \exp\{-q+if\} \ dsdt - e^{-\mu/2}\right| \ \leq \ \epsilon\Xi \quad \text{provided} \quad m+n>\left(\gamma+\ln(1/\epsilon)\right)^{\gamma+\ln(1/\epsilon)}$$

and that

$$(6.1.9)\qquad \mu=\mathbf{E}\, f^2 \ \leq \ \gamma(\delta).$$

Inequalities (6.1.3) and (6.1.9) complete the proof of Part (1). Combining Part (1), (6.1.1), (6.1.5) and (6.1.8) we complete the proof of Part (3).

It remains to prove Part (2). Let q_0 be the quadratic form defined in Lemma 3.6. Then

$$q(s,t) \ \leq \ (\tau+\tau^2)q_0(s,t)$$

and, therefore,

$$\int_{\mathbb{R}^{m+n-1}} \exp\{-q(s,t)\} \ dsdt \ \geq \ \int_{\mathbb{R}^{m+n-1}} \exp\left\{-(\tau+\tau^2)q_0(s,t)\right\} \ dsdt.$$

We compute the integral in the right hand side.

Let us consider the hyperplane $H \subset \mathbb{R}^{m+n}$ that is the orthogonal complement to the null-space of q_0. By Lemma 3.6, the eigenvalues of the restriction $q_0|H$ are $m/2$ with multiplicity $(n-1)$, $n/2$ with multiplicity $m-1$ and $(m+n)/2$ with multiplicity 1. Therefore,

$$\begin{aligned}\int_H \exp\left\{-(\tau+\tau^2)q_0(s,t)\right\} \ dsdt = & \frac{(2\pi)^{(m+n-1)/2}}{(\tau^2+\tau)^{(m+n-1)/2}\sqrt{\det 2q_0|H}} \\ = & \frac{(2\pi)^{(m+n-1)/2}}{m^{(n-1)/2}n^{(m-1)/2}(\tau+\tau^2)^{(m+n-1)/2}(m+n)^{1/2}}.\end{aligned}$$

The proof now follows by Lemma 3.5. $\square$

7. Bounding the integral outside of a neighborhood of the origin

Again, we consider the integral representation of Corollary 2.3. In this section we prove that the integral of $F(s,t)$ outside of the neighborhood U of the origin is asymptotically negligible. We prove the following main result.

(7.1) Theorem. *Let us fix a number $0 < \delta < 1$. Suppose that $m \geq \delta n$, $n \geq \delta m$ and that*

$$\delta\tau \ \leq \ \zeta_{kj} \ \leq \ \tau \quad \text{for all} \quad k, j$$

and some $\delta \leq \tau \leq (m+n)^{1/\delta}$. Let

$$U = \left\{(s,t) \in \Pi_0 : \quad |s_k|, |t_j| \ \leq \ \frac{\ln(m+n)}{\tau\sqrt{m+n}} \quad \text{for all} \quad k, j\right\}.$$

Then for any $\kappa > 0$

$$\int_{\Pi_0 \setminus U} |F(s,t)| \ dsdt \ \leq \ (m+n)^{-\kappa}\left|\int_U F(s,t) \ dsdt\right|$$

provided $m+n > \gamma(\delta, \kappa)$.

The general plan to prove Theorem 7.1 is as follows. First, we notice that by Parts (1)–(4) of Theorem 6.1, the integrals

$$\int_U |F(s,t)| \, dsdt \quad \text{and} \quad \left| \int_U F(s,t) \, dsdt \right| \tag{7.1.1}$$

have the same order, so we are going to prove that the integral

$$\int_{\Pi_0 \setminus U} |F(s,t)| \, dsdt$$

is negligible compared to the first integral of (7.1.1). We prove it in two steps. First, by a string of combinatorial arguments we show that the integral

$$\int_{\Pi_0 \setminus I} |F(s,t)| \, dsdt$$

is negligible compared to

$$\int_I |F(s,t)| \, dsdt, \tag{7.1.2}$$

where $I \subset \Pi_0$ is a larger neighborhood of the origin,

$$I = \Big\{ (s,t) \in \Pi_0 : \quad |s_k|, |t_j| \ \leq \ \epsilon/\tau \quad \text{for all} \quad k, j \Big\},$$

where $\epsilon > 0$ is any fixed number. This is the only place where we use that τ is bounded above by a polynomial in $m+n$. Then we notice that for a sufficiently small $\epsilon = \epsilon(\delta)$, the function $|F(s,t)|$ is strictly log-concave on I and we use a concentration inequality for strictly log-concave measures to deduce that the integral

$$\int_{I \setminus U} |F(s,t)| \, dsdt$$

is negligible compared to (7.1.2).

(7.2) The absolute value of $F(s,t)$. Let

$$\alpha_{kj} = 2\zeta_{kj} \left(1 + \zeta_{kj}\right) \quad \text{for all} \quad k, j.$$

Then for some positive constants $\gamma_1(\delta)$ and $\gamma_2(\delta)$ we have

$$\gamma_1(\delta)\tau^2 \ \leq \ \alpha_{kj} \ \leq \ \gamma_2(\delta)\tau^2 \quad \text{for all} \quad k, j \tag{7.2.1}$$

For $(s,t) \in \Pi_0$ we write

$$|F(s,t)| = |F_1(s,t)| \cdot |F_2(s,t)|, \quad \text{where}$$

$$|F_1(s,t)| = \left(\prod_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n-1}} \frac{1}{1 + \alpha_{kj} - \alpha_{kj} \cos\left(s_k + t_j\right)} \right)^{1/2} \quad \text{and} \tag{7.2.2}$$

$$|F_2(s,t)| = \left(\prod_{k=1}^{m} \frac{1}{1 + \alpha_{kn} - \alpha_{kn} \cos s_k} \right)^{1/2}$$

(7.3) Metric ρ. Let us introduce the following function

$$\rho : \mathbb{R} \longrightarrow [0, \pi], \quad \rho(x) = \min_{k \in \mathbb{Z}} |x - 2\pi k|.$$

In words: $\rho(x)$ is the distance from x to the closest integer multiple of 2π. Clearly,

$$\rho(-x) = \rho(x) \quad \text{and} \quad \rho(x+y) \ \leq \ \rho(x) + \rho(y)$$

for all $x, y \in \mathbb{R}$. We will use that

$$\cos x \ \leq \ 1 - \gamma \rho^2(x) \tag{7.3.1}$$

and that

$$\cos x - \cos y \ \geq \ \gamma \left(\rho^2(y) - \rho^2(x) \right) \tag{7.3.2}$$

for some positive absolute constant γ.

(7.4) Lemma. *For $\epsilon > 0$ and a point $(s, t) \in \Pi_0$ let us define the following two sets:*

Let $K = K(s, t; \epsilon) \subset \{1, \ldots, m\}$ be the set of all indices k such that

$$\rho\left(s_k + t_j\right) \ \leq \ \epsilon/\tau$$

for more than $(n-1)/2$ distinct indices $j = 1, \ldots, n-1$

and

Let $J = J(s, t; \epsilon) \subset \{1, \ldots, n-1\}$ be the set of all indices j for which there exist $k \in K$ such that

$$\rho\left(s_k + t_j\right) \ \leq \ \epsilon/\tau.$$

Let $\overline{K} = \{1, \ldots, m\} \setminus K$ and let $\overline{J} = \{1, \ldots, n-1\} \setminus J$.

Then

(1)

$$|F(s, t)| \ \leq \ (1 + \gamma(\epsilon, \delta))^{-|\overline{K}|(n-1)/4}$$

for some constant $\gamma(\epsilon, \delta) > 0$;

(2)

$$|F(s, t)| \ \leq \ (1 + \gamma(\epsilon, \delta))^{-|\overline{J}||K|/2}$$

for some $\gamma(\epsilon, \delta) > 0$;

(3)

$$\rho\left(s_{k_1} - s_{k_2}\right) \ \leq \ 2\epsilon/\tau$$

for all $k_1, k_2 \in K$;

(4)

$$\rho\left(s_k + t_j\right) \ \leq \ 3\epsilon/\tau$$

for all $k \in K$ and all $j \in J$;

(5)

$$\rho\left(t_{j_1} - t_{j_2}\right) \ \leq \ 4\epsilon/\tau$$

for all $j_1, j_2 \in J$.

Proof. For every $k \in \overline{K}$ there are at least $(n-1)/2$ distinct j for which $\rho\,(s_k + t_j) > \epsilon/\tau$ and hence so $\cos\,(s_k + t_j) \leq 1 - \gamma(\epsilon)/\tau^2$ by (7.3.1). Part (1) follows by (7.2.2) and (7.2.1).

For every $j \in \overline{J}$ and all $k \in K$ we have $\rho\,(s_k + t_j) > \epsilon/\tau$ and Part (2) follows by (7.2.2) and (7.2.1) as above.

For every $k_1, k_2 \in K$ there is at least one common $j \in J$ such that $\rho\,(s_{k_1} + t_j) \leq \epsilon/\tau$ and $\rho\,(s_{k_2} + t_j) \leq \epsilon/\tau$. Then

$$\rho\,(s_{k_1} - s_{k_2}) = \rho\,(s_{k_1} + t_j - s_{k_2} - t_j) \;\leq\; \rho\,(s_{k_1} + t_j) + \rho\,(s_{k_2} + t_j) \;\leq\; 2\epsilon/\tau,$$

and Part (3) follows.

For every $j \in J$ there is a $k_1 \in K$ such that $\rho\,(s_{k_1} + t_j) \leq \epsilon/\tau$. Hence, by Part (3),

$$\rho\,(s_k + t_j) = \rho\,(s_k + t_j - s_{k_1} + s_{k_1}) \;\leq\; \rho\,(s_{k_1} + t_j) + \rho\,(s_k - s_{k_1}) \;\leq\; 3\epsilon/\tau$$

and Part (4) follows.

Given $j_1, j_2 \in J$ we have $\rho\,(s_{k_1} + t_{j_1}),\ \rho\,(s_{k_2} + t_{j_2}) \leq \epsilon/\tau$ for some $k_1, k_2 \in K$ and by Part (3) we have

$$\begin{aligned}\rho\,(t_{j_1} - t_{j_2}) =& \rho\,(s_{k_1} + t_{j_1} - s_{k_2} - t_{j_2} + s_{k_2} - s_{k_1}) \\ \leq& \rho\,(s_{k_1} + t_{j_1}) + \rho\,(s_{k_2} + t_{j_2}) + \rho\,(s_{k_2} - s_{k_1}) \;\leq\; 4\epsilon/\tau.\end{aligned}$$

$\square$

(7.5) Corollary. *For an $\epsilon > 0$ and $\kappa > 0$ let us define a set $V = V(\epsilon, \kappa)$, $V \subset \Pi_0$ consisting of the points (s, t) such that*

$$|\overline{K}| \;\leq\; \kappa \ln m \quad \textit{and} \quad |\overline{J}| \;\leq\; \kappa \ln n,$$

where $\overline{K}$ and $\overline{J}$ are defined in Lemma 7.4. Then

$$\int_{\Pi_0 \setminus V} |F(s,t)|\; ds dt \;\leq\; (m+n)^{-(m+n)} \int_{\Pi_0} |F(s,t)|\; ds dt$$

provided

$$\kappa \;\geq\; \gamma(\epsilon, \delta) \quad \textit{and} \quad m + n \;\geq\; \gamma_1(\epsilon, \delta)$$

for some positive constants $\gamma(\epsilon, \delta)$ and $\gamma_1(\epsilon, \delta)$.

Proof. Follows by Parts (1) and (2) of Lemma 7.4 and by Parts (1), (2) and (4) of Theorem 6.1. Here we use that τ is bounded by a polynomial in $m + n$. $\square$

For the next several statements we will be using various constants depending on two parameters, ϵ and δ. For brevity, we say that $m + n$ is *large enough* provided $m + n \geq \gamma(\epsilon, \delta)$ for some positive constant $\gamma(\epsilon, \delta)$.

(7.6) Proposition. *For a positive $\epsilon > 0$ let us define a set $X = X(\epsilon) \subset \Pi_0$ by*

$$X = X(\epsilon) = \Big\{ (s,t) \in \Pi_0 : \quad \rho\left(s_k + t_j\right) \leq \epsilon/\tau \qquad \text{for all} \quad k = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n-1 \Big\}.$$

Then

$$\int_{\Pi_0 \setminus X(\epsilon)} |F(s,t)| \, ds dt \ \leq \ e^{-\gamma \cdot (m+n)} \int_{\Pi_0} |F(s,t)| \, ds dt,$$

$\gamma = \gamma(\epsilon, \delta) > 0$ is some constant and $m+n$ is large enough.

Proof. Let us choose a $\kappa > 0$ and consider a set $V = V(\epsilon/60, \kappa) \subset \Pi_0$ as in Corollary 7.5 so that the conclusion of Corollary 7.5 holds.

For subsets $A \subset \{1, \ldots, m\}$, $B \subset \{1, \ldots, n-1\}$ such that

$$|\overline{A}| \ \leq \ \kappa \ln m \quad \text{and} \quad |\overline{B}| \ \leq \ \kappa \ln n \tag{7.6.1}$$

let us define a set $P_{A,B} \subset \Pi_0$ (we call it a *piece*) by

$$P_{A,B} = \Big\{ (s,t) \in \Pi_0 : \quad \rho\left(s_k + t_j\right) \leq \epsilon/20\tau \quad \text{for all} \quad k \in A \quad \text{and all} \quad j \in B \Big\}.$$

From Part (4) of Lemma 7.4 we conclude that $(s,t) \in P_{A,B}$ if $A = K(s,t;\epsilon/60)$ and $B = J(s,t;\epsilon/60)$. Therefore,

$$V \ \subset \ \bigcup_{A,B} P_{A,B}, \tag{7.6.2}$$

where the union is taken over all A, B satisfying (7.6.1). In particular, from Corollary 7.5, we have

$$\int_{\Pi_0 \setminus \bigcup_{A,B} P_{A,B}} |F(s,t)| \, ds dt \ \leq \ (m+n)^{-(m+n)} \int_{\Pi_0} |F(s,t)| \, ds dt \tag{7.6.3}$$

provided $m+n$ is large enough.

As in the proof of Lemma 7.4, we conclude that for any $(s,t) \in P_{A,B}$ we have

$$\begin{aligned} &\rho\left(s_{k_1} - s_{k_2}\right) \ \leq \ \epsilon/10\tau \quad \text{for all} \quad k_1, k_2 \in A \qquad \text{and} \\ &\rho\left(t_{j_1} - t_{j_2}\right) \ \leq \ \epsilon/10\tau \quad \text{for all} \quad j_1, j_2 \in B. \end{aligned} \tag{7.6.4}$$

Let us choose a point $(s,t) \in P_{A,B} \setminus X(\epsilon)$. Hence we have $\rho\left(s_{k_0} + t_{j_0}\right) > \epsilon/\tau$ for some k_0 and j_0. Let us pick $k_1 \in A$ and $j_1 \in B$. Then

$$\begin{aligned} \rho\left(s_{k_0} + t_{j_0}\right) =& \rho\left(s_{k_0} + t_{j_0} + s_{k_1} + t_{j_1} - s_{k_1} - t_{j_1}\right) \\ & \leq \rho\left(s_{k_0} + t_{j_1}\right) + \rho\left(s_{k_1} + t_{j_0}\right) + \rho\left(s_{k_1} + t_{j_1}\right). \end{aligned}$$

Since $\rho\left(s_{k_1}+t_{j_1}\right) \leq \epsilon/20\tau$, we must have either $\rho\left(s_{k_0}+t_{j_1}\right) > 19\epsilon/40\tau > 0.4\epsilon/\tau$, in which case $k_0 \in \overline{A}$ necessarily, or $\rho\left(s_{k_1}+t_{j_0}\right) > 19\epsilon/40\tau > 0.4\epsilon/\tau$, in which case $j_0 \in \overline{B}$ necessarily. In the first case (7.6.4) implies that $\rho\left(s_{k_0}+t_j\right) > 0.3\epsilon/\tau$ for all $j \in B$ and in the second case (7.6.4) implies that $\rho\left(s_k+t_{j_0}\right) > 0.3\epsilon/\tau$ for all $k \in A$.

For $k \in \overline{A}$ we define

$$Q_{A,B;k} = \Big\{(s,t) \in P_{A,B}: \quad \rho\left(s_k+t_j\right) > 0.3\epsilon/\tau \quad \text{for all} \quad j \in B\Big\} \tag{7.6.5}$$

and for $j \in \overline{B}$ we define

$$R_{A,B;j} = \Big\{(s,t) \in P_{A,B}: \quad \rho\left(s_k+t_j\right) > 0.3\epsilon/\tau \quad \text{for all} \quad k \in A\Big\}. \tag{7.6.6}$$

Then

$$P_{A,B} \setminus X(\epsilon) \quad \subset \quad \left(\bigcup_{k \in \overline{A}} Q_{A,B;k} \quad \bigcup \quad \bigcup_{j \in \overline{B}} R_{A,B;j}\right). \tag{7.6.7}$$

Let us compare the integrals

$$\int_{Q_{A,B;k}} |F(s,t)| \, dsdt \quad \text{and} \quad \int_{P_{A,B}} |F(s,t)| \, dsdt.$$

Given a point $(s,t) \in P_{A,B}$ we obtain another point in $P_{A,B}$ if we arbitrarily choose coordinates $s_k \in [-\pi,\pi]$ for $k \in \overline{A}$ and $t_j \in [-\pi,\pi]$ for $j \in \overline{B}$. Let us pick a particular non-empty set $Q_{A,B;k}$ for some $k \in \overline{A}$. We obtain a *fiber* $E \subset P_{A,B}$ if we let the coordinate s_k vary arbitrarily between $-\pi$ and π while fixing all other coordinates of some point $(s,t) \in P_{A,B}$. Geometrically, each fiber E is an interval of length 2π. We construct a set $I \subset E$ as follows: we choose an arbitrary coordinate $j_1 \in B$ and let s_k vary in such a way that $\rho\left(s_k+t_{j_1}\right) \leq 0.1\epsilon/\tau$. Geometrically, I is an interval of length $0.2\epsilon/\tau$ or a union of two non-overlapping intervals of the total length $0.2\epsilon/\tau$. Moreover, by (7.6.4), we have

$$\rho\left(s_k+t_j\right) \;\leq\; 0.2\epsilon/\tau \quad \text{for all} \quad j \in B \quad \text{and all} \quad (s,t) \in I. \tag{7.6.8}$$

Comparing (7.6.5) and (7.6.8) and using (7.2.1), (7.2.2), (7.3.2) and (7.6.1) we conclude that for any $(s,t) \in Q_{A,B;k} \cap E$ and any $\left(\tilde{s},\tilde{t}\right) \in I$, we have

$$|F(s,t)| \;\leq\; e^{-\gamma\cdot(m+n)} \left|F\left(\tilde{s},\tilde{t}\right)\right|$$

for some constant $\gamma=\gamma(\epsilon,\delta)>0$, provided $m+n$ is large enough (here we use that τ is bounded by a polynomial in $m+n$). Therefore,

$$\int_{E \cap Q_{A,B;k}} |F(s,t)| \, ds_k \;\leq\; e^{-\gamma\cdot(m+n)} \int_E |F(s,t)| \, ds_k$$

for some $\gamma = \gamma(\epsilon, \delta) > 0$ provided $m+n$ is large enough (again, we use that τ is bounded by a polynomial in $m+n$). Integrating over all fibers $E \subset P_{A,B}$, we prove that

$$\int_{Q_{A,B;k}} |F(s,t)| \, dsdt \;\leq\; e^{-\gamma \cdot (m+n)} \int_{P_{A,B}} |F(s,t)| \, dsdt$$

provided $m+n$ is large enough. Similarly, we prove that for sets $R_{A,B;j}$ defined by (7.6.6) we have

$$\int_{R_{A,B;j}} |F(s,t)| \, dsdt \;\leq\; e^{-\gamma \cdot (m+n)} \int_{P_{A,B}} |F(s,t)| \, dsdt$$

for some constant $\gamma(\epsilon, \delta) > 0$ provided $m+n$ is large enough. Then from (7.6.7) we deduce that

$$\int_{P_{A,B} \setminus X(\epsilon)} |F(s,t)| \, dsdt \;\leq\; e^{-\gamma \cdot (m+n)} \int_{P_{A,B}} |F(s,t)| \, dsdt$$

provided $m+n$ is large enough. Finally, because of (7.6.1) the number of pieces $P_{A,B}$ does not exceed $\exp\left\{\gamma_1(\epsilon,\delta) \ln^2(mn)\right\}$ and the proof follows by (7.6.3). $\square$

Next, we look at the structure of set $X(\epsilon)$ of Proposition 7.6.

(7.7) Lemma. *For $\epsilon > 0$ let us define sets $Y(\epsilon), Z(\epsilon) \subset \Pi_0$ by*

$$\begin{aligned} Y(\epsilon) = \Big\{ (s,t) \in \Pi_0 : \quad & |s_k + t_j| \leq \epsilon/\tau \\ & \text{for all} \quad k = 1, \ldots, m \quad \text{and all} \quad j = 1, \ldots, n-1 \Big\} \qquad \text{and} \\ Z(\epsilon) = \Big\{ (s,t) \in \Pi_0 : \quad & s_k, t_j \in [-\pi, \ -\pi + \epsilon/\tau] \cup [\pi - \epsilon/\tau, \ \pi] \\ & \text{for all} \quad k = 1, \ldots, m \quad \text{and all} \quad j = 1, \ldots, n-1 \Big\}. \end{aligned}$$

Then for the set $X(\epsilon)$ of Proposition 7.6, we have

$$X(\epsilon) \;\subset\; Y(\epsilon) \;\cup\; Z(2\epsilon).$$

Proof. We note that if $\rho(x) \leq \epsilon/\tau$ for $-2\pi \leq x \leq 2\pi$ if and only if $|x| \leq \epsilon/\tau$ or $x \geq 2\pi - \epsilon/\tau$ or $x \leq -2\pi + \epsilon/\tau$.

Let us pick an arbitrary $(s,t) \in X(\epsilon)$. Suppose that $-\pi + 2\epsilon/\tau < s_{k_0} < \pi - 2\epsilon/\tau$ for some k_0. Since $-\pi \leq t_j \leq \pi$, we have $-2\pi + 2\epsilon/\tau < s_{k_0} + t_j < 2\pi - 2\epsilon/\tau$ and hence $|s_{k_0} + t_j| \leq \epsilon/\tau$ for $j = 1, \ldots, n-1$. In particular, $-\pi + \epsilon/\tau < t_j < \pi - \epsilon/\tau$ for all $j = 1, \ldots, n-1$. Arguing as before, we conclude that $|s_k + t_j| \leq \epsilon/\tau$ for all $k = 1, \ldots, m$ and $j = 1, \ldots, n-1$.

Similarly, if $-\pi + 2\epsilon/\tau < t_{j_0} < \pi - 2\epsilon/\tau$ for some $j_0 = 1, \ldots, n-1$ we argue that $|s_k + t_j| \leq \epsilon/\tau$ for all k and j. Thus the only remaining possibility is that $(s,t) \in Z(2\epsilon)$. $\square$

(7.8) Proposition. *For $0 < \epsilon < \tau/2$ let us define a cube*

$$I = I(\epsilon) = \Big\{(s,t) \in \Pi_0 : \quad |s_k|, |t_j| \leq \epsilon/\tau \quad \text{for all} \quad k, j\Big\}.$$

Then

$$\int_{\Pi_0 \setminus I(\epsilon)} |F(s,t)| \; dsdt \; \leq \; e^{-\gamma \cdot (m+n)} \int_{\Pi_0} |F(s,t)| \; dsdt$$

for some $\gamma = \gamma(\epsilon, \delta) > 0$ and $m+n$ large enough.

Proof. We use Proposition 7.6 and Lemma 7.7. By Proposition 7.6,

$$\int_{\Pi_0 \setminus X(\epsilon/9)} |F(s,t)| \; dsdt \; \leq \; e^{-\gamma(m+n)} \int_{\Pi_0} |F(s,t)| \; dsdt \tag{7.8.1}$$

for some constant $\gamma = \gamma(\epsilon, \delta) > 0$ provided $m+n$ is large enough. By Lemma 7.7, we have

$$X(\epsilon/9) \quad \subset \quad Y(\epsilon/9) \; \cup \; Z(2\epsilon/9). \tag{7.8.2}$$

Next, we show that the integral over $Z(2\epsilon/9)$ is negligible.

The set $Z(2\epsilon/9)$ is a union of pairwise disjoint 2^{m+n-1} *corners*, where each corner is determined by a choice of the interval $[-\pi, \; -\pi + 2\epsilon/9\tau]$ or $[\pi - 2\epsilon/9\tau, \; \pi]$ for each coordinate s_k and t_j. The transformation

$$s_k \longmapsto \begin{cases} s_k + \pi & \text{if } s_k \in [-\pi, \; -\pi + 2\epsilon/9\tau] \\ s_k - \pi & \text{if } s_k \in [\pi - 2\epsilon/9\tau, \; \pi] \end{cases}$$

and

$$t_j \longmapsto \begin{cases} t_j + \pi & \text{if } s_k \in [-\pi, \; -\pi + 2\epsilon/9\tau] \\ t_j - \pi & \text{if } t_j \in [\pi - 2\epsilon/9\tau, \; \pi] \end{cases}$$

maps the corners one-to-one on the corresponding 2^{m+n-1} *corners* of the cube $I(2\epsilon/9)$, where each corner is determined by a choice of the interval $[0, \; 2\epsilon/9\tau]$ or $[-2\epsilon/9\tau, \; 0]$ for each coordinate s_k and t_j. This transformation does not change factor $|F_1(s,t)|$ of but makes factor $|F_2(s,t)|$ of (7.2.2) exponentially in m bigger. Therefore,

$$\int_{Z(2\epsilon/9)} |F(s,t)| \; dsdt \; \leq \; e^{-\gamma m} \int_{I(2\epsilon/9)} |F(s,t)| \; dsdt \tag{7.8.3}$$

for some $\gamma = \gamma(\epsilon, \delta) > 0$ and $m+n$ large enough.

Finally, we estimate the integral over $Y(\epsilon/9)$.

Let us choose a point $(s,t) \in Y(\epsilon/9)$. We have

$$l\epsilon/9\tau \; \leq \; s_1 \; \leq \; (l+1)\epsilon/9\tau$$

for some integer l. Therefore

$$(-l-2)\epsilon/9\tau \ \leq \ t_j \ \leq (-l+1)\epsilon/9\tau \quad \text{for} \quad j=1, \ldots, n-1$$

and

$$(l-2)\epsilon/9\tau \ \leq \ s_k \ \leq \ (l+3)\epsilon/9\tau \quad \text{for} \quad k=1, \ldots, m.$$

Let us denote

$$w = \left(\underbrace{\epsilon/9\tau, \ldots, \epsilon/9\tau}_{m \text{ times}}; \ \underbrace{-\epsilon/9\tau, \ldots, -\epsilon/9\tau}_{n-1 \text{ times}} \right).$$

Hence we conclude that $Y(\epsilon/9)$ is covered by polynomially many $(m+n)^{\gamma(\epsilon,\delta)}$ of translations of cubes

$$I(\epsilon/3) + lw, \quad \text{where} \quad l \quad \text{is an integer}$$

(here we use again that τ is bounded by a polynomial in $(m+n)$). The translation by a multiple of w does not change factor $|F_1(s,t)|$ but possibly changes factor $|F_2(s,t)|$ of (7.2.2). For $(s,t) \in I(\epsilon/3)$ we have $|s_k| \leq \epsilon/3\tau$ for all k. For $(s,t) \in I(\epsilon/3) + lw$ with $|l| \geq 7$, we have $|s_k| \geq 4\epsilon/9\tau$ for all k. Hence the integral of $|F(s,t)|$ of the translations $I(\epsilon/3)+lw$ with $|l| \geq 7$ is exponentially in m smaller than the integral of $|F(s,t)|$ over $I(\epsilon/3)$, see (7.3.2). Furthermore, for $|l| \leq 6$ we have $I(\epsilon/3)+lw \subset I(\epsilon)$. Thus we proved that

$$(7.8.4) \qquad \int_{Y(2\epsilon/9)\setminus I(\epsilon)} |F(s,t)| \ dsdt \ \leq \ e^{-\gamma m} \int_{I(\epsilon)} |F(s,t)| \ dsdt,$$

for some $\gamma = \gamma(\epsilon, \delta) > 0$ and $m+n$ large enough. Combining (7.8.1)–(7.8.4), we complete the proof. □

We need a concentration result for strictly log-concave probability measures, see, for example, Section 2.2 of [Le01] or Theorem 8.1 and its proof in [Ba97], which, although stated for the Gaussian measure is adapted in a straightforward way to our situation.

(7.9) Theorem. *Let V be Euclidean space with the norm $\| \cdot \|$, let $B \subset V$ be a convex body, let us consider a probability measure supported on B with the density e^{-u}, where $u: B \longrightarrow \mathbb{R}$ is a function satisfying*

$$u(x) + u(y) - 2u\left(\frac{x+y}{2}\right) \ \geq \ c\|x-y\|^2 \quad \textit{for all} \quad x, y \in B$$

and some constant $c > 0$. For a point $x \in V$ and a closed subset $A \subset V$ we define the distance

$$\operatorname{dist}(x, A) = \min_{y \in A} \|x - y\|.$$

Let $A \subset B$ be a closed set such that $\mathbf{P}(A) \geq 1/2$. Then, for any $r \geq 0$ we have

$$\mathbf{P}\left\{x \in B: \quad \operatorname{dist}(x, A) \geq r\right\} \ \leq \ 2e^{-cr^2}.$$

□

Here is how we apply this theorem.

(7.10) Lemma. *Let us fix* $0 < \delta < 1$ *and let* α_{kj}, $k = 1, \ldots, m$, $j = 1, \ldots, n$ *be numbers satisfying (7.2.1) with some* $\delta \leq \tau \leq (m+n)^{1/\delta}$. *Suppose that* $m \geq \delta n$ *and* $n \geq \delta m$. *Then there exists an* $\epsilon = \epsilon(\delta) > 0$ *such that the following holds.*

In the space $\mathbb{R}^{m+n}$ *of points* $(s_1, \ldots, s_m; t_1, \ldots, t_n)$ *let us consider the hyperplane* H *defined by the equation*

$$s_1 + \ldots + s_m = t_1 + \ldots + t_n.$$

Let $B \subset H$ *be a convex body centrally symmetric about the origin:* $(s, t) \in B$ *if and only if* $(-s, -t) \in B$, *and such that for all* $(s, t) \in B$ *we have*

$$\begin{aligned} |s_k| \leq \epsilon/\tau \quad &\textit{for} \quad k = 1, \ldots, m \\ |t_j| \leq \epsilon/\tau \quad &\textit{for} \quad j = 1, \ldots, n. \end{aligned}$$

Let us consider the probability measure on B *with the density proportional to* $|F(s,t)|$. *Then, for any* $\kappa > 0$ *we have*

$$\mathbf{P}\left\{(s,t) \in B : \quad |s_k|, |t_j| \ \leq \ \frac{\ln(m+n)}{2\tau\sqrt{m+n}} \quad \textit{for all} \quad k, j\right\} \ \geq \ 1 - (m+n)^{-\kappa},$$

provided $m + n \geq \gamma(\delta, \kappa)$ *for some constant* $\gamma(\delta, \kappa) > 0$.

Proof. Let us consider a univariate function

$$f(x) = \ln(1 + \alpha - \alpha \cos x).$$

Then

$$f''(x) = \frac{\alpha(1+\alpha)\cos x - \alpha^2}{(1 + \alpha - \alpha\cos x)^2}.$$

If

$$\gamma_1(\delta)\tau^2 \ \leq \ \alpha \ \leq \ \gamma_2(\delta)\tau^2$$

for some constants $\gamma_1(\delta), \gamma_2(\delta) > 0$ and some $\tau \geq \delta$ then

$$f''(x) \ \geq \ \gamma(\delta)\tau^2 \quad \text{provided} \quad |x| \ \leq \ 2\epsilon/\tau$$

for some constants $\gamma(\delta) > 0$ and $\epsilon = \epsilon(\delta) > 0$.

Therefore,

$$f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) \ \geq \ \gamma(\delta)\tau^2(x-y)^2 \quad \text{provided} \quad |x|, |y| \ \leq \ 2\epsilon/\tau$$

for some $\gamma(\delta) > 0$.

Let us consider the map

$$M : (s_1, \ldots, s_m; t_1, \ldots, t_n) \longmapsto (\ldots s_k + t_j \ldots)$$

as a map $M : H \longmapsto \mathbb{R}^{mn}$. From Lemma 3.6

$$\|Mx\|^2 \ \geq \ \min\{m,n\}\|x\|^2 \quad \text{for all} \quad x \in H,$$

where $\|\cdot\|$ is the Euclidean norm in the corresponding space.

Since

$$-\ln|F(s,t)| = \frac{1}{2}\sum_{\substack{1\leq k\leq m\\ 1\leq j\leq n}} \ln\left(1+\alpha_{kj}-\alpha_{kj}\cos\left(s_k+t_j\right)\right),$$

we conclude that for any constant a and for

$$u(x) = -\ln|F(s,t)| + a \quad \text{where} \quad x=(s,t) \tag{7.10.1}$$

we have

$$u(x)+u(y)-2u\left(\frac{x+y}{2}\right) \ \geq \ \gamma(\delta)\tau^2(m+n)\|x-y\|^2$$

provided x and y belong to the cube $I(\epsilon)$ defined by the inequalities $|s_k|, |t_j| \leq \epsilon/\tau$. We choose constant a in (7.10.1) so that e^{-u} is a probability density on B.

Now we apply Theorem 7.9 with

$$c=\gamma(\delta)\tau^2(m+n).$$

For $k=1,\ldots,m$, let S_k^+ be the set consisting of the points $(s,t)\in B$ with $s_k \geq 0$, let S_k^- be the set consisting of the points $(s,t)\in B$ with $s_k\leq 0$, let T_j^+ be the set consisting of the points $(s,t)\in B$ with $t_j \geq 0$ and let T_j^- be the set consisting of the points $(s,t)\in B$ with $t_j \leq 0$. Since both B and the probability measure are invariant under the symmetry

$$(s,t)\longmapsto(-s,-t),$$

we have

$$\mathbf{P}\left(S_k^+\right), \mathbf{P}\left(S_k^-\right), \mathbf{P}\left(T_j^+\right), \mathbf{P}\left(T_j^-\right) = \frac{1}{2}.$$

Therefore, by Theorem 7.9, all but a $(m+n)^{-\kappa}$ fraction of all points in B lie within a distance

$$r=\frac{\ln(m+n)}{2\tau\sqrt{m+n}}$$

from each set $S_k^+, S_k^-, T_j^+, T_j^-$ provided $m+n \geq \gamma(\delta)$ for some constant $\gamma(\delta,\kappa)>0$. $\square$

Now we are ready to prove Theorem 7.1.

(7.11) Proof of Theorem 7.1. Let $\epsilon = \epsilon(\delta) > 0$ be the number constructed in Lemma 7.10 and let $H \subset \mathbb{R}^{m+n}$ be the hyperplane defined in Lemma 7.10. We consider a linear transformation $T: H \longrightarrow \mathbb{R}^{m+n-1} \subset \mathbb{R}^{m+n}$,

$$(s_1, \ldots, s_m;\ t_1, \ldots, t_n) \longmapsto (s_1 + t_n, \ldots, s_m + t_n;\ t_1 - t_n, \ldots, t_{n-1} - t_n, 0).$$

The inverse linear transformation

$$T^{-1}: \quad \left(s_1', \ldots, s_m';\ t_1', \ldots, t_{n-1}', 0\right) \longmapsto (s_1, \ldots, s_m;\ t_1, \ldots, t_n)$$

is computed as follows:

$$t_n = \frac{s_1' + \ldots + s_m' - t_1' - \ldots - t_{n-1}'}{m+n}, \quad s_k = s_k' - t_n, \quad t_j = t_j' + t_n.$$

Let us consider the cube $I = I(\epsilon/2) \subset \mathbb{R}^{m+n-1}$ defined by the inequalities

$$|s_k|,\ |t_j|\ \leq\ \epsilon/2\tau \quad \text{for} \quad k = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n-1.$$

By Proposition 7.8 we have

$$\int_{\Pi_0 \setminus I} |F(s,t)|\, dsdt\ \leq\ e^{-\gamma \cdot (m+n)} \int_{\Pi_0} |F(s,t)|\, dsdt \tag{7.11.1}$$

for some constant $\gamma = \gamma(\delta) > 0$ and $m + n > \gamma_1(\delta)$ for some other constant $\gamma_1(\delta) > 0$.

Let $B = T^{-1}(I) \subset H$. Then B is centrally symmetric and convex and for all $(s,t) \in B$ we have $|s_k|, |t_j| \leq \epsilon/\tau$ for all k and j. Let

$$\begin{aligned} Y = \Big\{(s,t) \in B: \quad |s_k| \leq \frac{\ln(m+n)}{2\tau\sqrt{m+n}} \quad &\text{for} \quad k = 1, \ldots, m \qquad \text{and} \\ |t_j| \leq \frac{\ln(m+n)}{2\tau\sqrt{m+n}} \quad &\text{for} \quad j = 1, \ldots, n\Big\}. \end{aligned}$$

By Lemma 7.10,

$$\int_{B \setminus Y} |F(s,t)|\, dsdt\ \leq\ (m+n)^{-\kappa} \int_B |F(s,t)|\, dsdt$$

provided $m + n \geq \gamma(\delta, \kappa)$. Now, the push-forward of the probability measure on B with the density proportional to $|F(s,t)|$ is the probability measure on I with the density proportional to $|F(s,t)|$. Moreover, the image $T(Y)$ lies in the cube U defined by the inequalities

$$|s_k|\ |t_j|\ \leq\ \frac{\ln(m+n)}{\tau\sqrt{m+n}} \quad \text{for} \quad k = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n-1.$$

Therefore,

$$\int_{I \setminus U} |F(s,t)\ dsdt\ \leq\ (m+n)^{-\kappa} \int_I |F(s,t)|\, dsdt. \tag{7.11.2}$$

provided $m+n \geq \gamma(\delta, \kappa)$. The proof now follows by (7.11.1), (7.11.2) and Theorem 6.1. $\square$

8. Concluding remarks

First, we formally finish the proof of Theorem 1.3.

(8.1) Proof of Theorem 1.3. The proof follows by the integral representation of Corollary 2.3, Theorem 7.1 bounding away the integral outside of the neighborhood U of the origin, Part (3) of Theorem 6.1 computing the asymptotic of the integral over U, the observation that

$$\Xi = \int_{\mathbb{R}^{m+n-1}} e^{-q} \ dsdt = \frac{\pi^{(m+n-1)/2}}{\sqrt{\det q|\mathbb{R}^{m+n-1}}},$$

Lemma 3.5 implying that

$$\det q|\mathbb{R}^{m+n-1} = \frac{1}{m+n} \det q|H,$$

where H is the orthogonal complement to the null-space of q and Lemma 3.1 implying that the expectations

$$\mu = \mathbf{E}\, f^2 \quad \text{and} \quad \nu = \mathbf{E}\, h$$

can be computed with respect to the Gaussian probability measure with the density proportional to e^{-q} in an arbitrary hyperplane $L \subset \mathbb{R}^{m+n}$ not containing the null-space of q. $\square$

It appears quite plausible that a similar approach can be used to obtain asymptotic formulas for other combinatorial quantities of interest, such as the number of graphs on n vertices with prescribed degrees at the vertices.

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