Let $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ be a polynomial with all roots real and satisfying $x \leq -\delta$ for some $0 < \delta < 1$. We show that for any $0 < \epsilon < 1$, the value of $p(1)$ is determined within relative error $\epsilon$ by the coefficients $a_k$ with $k \leq c \sqrt{\delta} \ln \frac{n}{\epsilon} \sqrt{\delta}$ for some absolute constant $c > 0$. Consequently, if $m_k(G)$ is the number of matchings with $k$ edges in a graph $G$, then for any $0 < \epsilon < 1$, the total number $M(G) = m_0(G) + m_1(G) + \ldots$ of matchings is determined within relative error $\epsilon$ by the numbers $m_k(G)$ with $k \leq c \sqrt{\Delta} \ln(v/\epsilon)$, where $\Delta$ is the largest degree of a vertex, $v$ is the number of vertices of $G$ and $c > 0$ is an absolute constant. We prove a similar result for polynomials with complex roots satisfying $\Re z \leq -\delta$ and apply it to estimate the number of unbranched subgraphs of $G$.

1. Introduction and main results

Our main motivation comes from the observation that in some cases, the total number of combinatorial structures of a particular type is determined with high accuracy by the exact number of the structures of the same type but of a small (sometimes, very small) size. For example, if $G$ is a graph and we are interested in the number $M(G)$ of matchings in $G$, that is, collections of vertex-disjoint edges, then within a given relative error $0 < \epsilon < 1$, the number $M(G)$ is determined by the numbers $m_k(G)$ of matchings with exactly $k$ edges for $k \leq c \sqrt{\Delta(G)} \ln \frac{v(G)}{\epsilon}$, where $\Delta(G)$ is the largest degree of a vertex of the graph, $v(G)$ is the number of vertices of $G$ and $c > 0$ is an absolute constant. We deduce this, and some related results, from some general observation about real-rooted polynomials.

Below we talk about approximating some real and complex values up to “relative error $\epsilon$”. Given a complex number $a \neq 0$, we say that a complex number $b \neq 0$
approximates $a$ up to (or within) relative error $\epsilon > 0$ if we can write $a = e^z$ and $b = e^w$ for some complex numbers $z$ and $w$ such that $|z - w| \leq \epsilon$.

We prove the following main result.

(1.1) Theorem. Suppose that $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ is a polynomial and all roots $x$ of $p$ are real and satisfy $x \leq -\delta$ for some $0 < \delta < 1$. Then, for any $0 < \epsilon < 1$, the value of $p(1)$, up to relative error $\epsilon$, is determined by the coefficients $a_k$ with

$$k \leq \frac{c}{\sqrt{\delta}} \ln \frac{n}{\epsilon \sqrt{\delta}},$$

where $c > 0$ is an absolute constant.

In fact, we present a polynomial time algorithm, which, given $a_0, \ldots, a_k$ with $k$ as in the theorem, computes $p(1)$ within relative error $\epsilon$. We note that we need to know $a_0, \ldots, a_k$ exactly (or, at the very least, with very high precision). Theorem 1.1 is a much improved version of the “personal communication” of the author that was referred to in [PR17].

Using the Heilmann-Lieb Theorem [HL72], see also [GG81], we immediately deduce the result about matchings in graphs. We consider undirected graphs, without loops or multiple edges.

(1.2) Theorem. For a graph $G$, let $m_k(G)$ be the number of matchings that contain exactly $k$ edges and let $M(G) = m_0(G) + m_1(G) + \ldots$ be the total number of matchings. Then, for any $0 < \epsilon < 1$, up to relative error $\epsilon$, the number $M(G)$ is determined by the numbers $m_k(G)$ with

$$k \leq c\sqrt{\Delta(G)} \ln \frac{v(G)}{\epsilon},$$

where $\Delta(G)$ is the largest degree of a vertex of the graph, $v(G)$ is the number of vertices of $G$ and $c > 0$ is an absolute constant.

Again, we have a polynomial time algorithm, which, given $m_k(G)$, with $k$ as in the theorem, produces an estimate of $M(G)$ within relative error $\epsilon$. We note that Patel and Regts [PR17] constructed a polynomial time algorithm for computing $m_k(G)$ with $k = O(\ln v(G))$ provided the largest degree $\Delta(G)$ is fixed in advance (a straightforward enumeration gives only a quasi-polynomial algorithm of $v(G)^{O(\ln v(G))}$ complexity). For general graphs, the complexity of the algorithm roughly matches that of Bayati et al. [B+07], which estimates $M(G)$ using the correlation decay approach. Although our approach and that of [B+07] look completely different, they are both inspired by the concept of “phase transition” coming from statistical physics; more precisely, two related, but different concepts: ours has to do with the Lee-Yang approach via complex zeros of the “partition function” [YL52], [LY52] while that of [B+07] has to do with correlation decay, cf. [DS87] and [KK98]. Hence the fact that the complexity appears to be roughly the same is not entirely accidental.
Using the Chudnovsky-Seymour extension [CS07] of the Heilmann-Lieb Theorem and the Dobrushin-Shearer bound on the roots of the independence polynomial, see, for example, [SS05], we get another combinatorial application. Recall that a subset of vertices of a graph is called an independent set if no two vertices of the subset span an edge of the graph. A graph is called claw-free if it does not contain an induced subgraph consisting of a vertex connected to some other three vertices that are pairwise unconnected. We obtain the following result.

\textbf{(1.3) Theorem.} For a graph \(G\), let \(i_k(G)\) be the number of independent sets with exactly \(k\) vertices and let \(I(G) = i_0(G) + i_1(G) + \ldots\) be the total number of independent sets. Then, for any \(0 < \epsilon < 1\), up to relative error \(\epsilon\), the number \(I(G)\) of a claw-free graph is determined by the numbers \(i_k(G)\) with

\[
    k \leq c \sqrt{\Delta(G)} \ln \frac{v(G)}{\epsilon},
\]

where \(\Delta(G)\) is the largest degree of a vertex of the graph, \(v(G)\) is the number of vertices of \(G\) and \(c > 0\) is an absolute constant.

We have a polynomial time algorithm, which, given \(i_k(G)\), with \(k\) as in the theorem, produces an estimate of \(I(G)\) within relative error \(\epsilon\). Curiously, while the correlation decay approach of [B+07] is essentially harder in the case of independent sets in claw-free graphs than it is in the case of matchings, our approach is the same in both cases (assuming, of course, the hard work done in [HL72] and [CS07]).

Theorems 1.2 and 1.3 vaguely resemble the “approximate inclusion-exclusion” of [LN90] and [K+96]. The methods, however, look completely different. It would be interesting to find out if there is indeed any connection between our Theorem 1.1 and the results of [LN90] and [K+96].

Next, we consider polynomials \(p(z) = a_0 + a_1 z + \ldots + a_n z^n\) with complex roots satisfying \(\Re z \leq -\delta\) for some \(0 < \delta < 1\) (we call such polynomials “stable”). We allow complex coefficients \(a_k\). We obtain the following result.

\textbf{(1.4) Theorem.} Suppose that \(p(z) = a_0 + a_1 z + \ldots + a_n z^n\) is a complex polynomial and all roots \(z\) of \(p\) satisfy \(\Re z \leq -\delta\) for some \(0 < \delta < 1\). Then, for any \(0 < \epsilon < 1\), the value of \(p(1)\), up to relative error \(\epsilon\), is determined by the coefficients \(a_k\) with

\[
    k \leq \frac{c}{\delta} \ln \frac{n}{\epsilon \delta},
\]

where \(c > 0\) is an absolute constant.

We apply Theorem 1.4 to count unbranched subgraphs, that is, collections of edges of the graph such that every vertex of the graph is incident to at most two edges from the collection. From Ruelle’s Theorem [R99a], [R99b], see also [Wa09], we deduce the following result.
Theorem. For a graph $G$, let $u_k(G)$ be the number of unbranched subgraphs with exactly $k$ edges and let $U(G) = u_0(G) + u_1(G) + \ldots$ be the total number of unbranched subgraphs. Then, for any $0 < \epsilon < 1$, up to relative error $\epsilon$, the number $U(G)$ is determined by the numbers $u_k(G)$ with

$$k \leq c (\Delta(G))^3 \ln \frac{v(G)}{\epsilon},$$

where $\Delta(G)$ is the largest degree of a vertex of the graph, $v(G)$ is the number of vertices of $G$ and $c > 0$ is an absolute constant.

One can easily see that if a non-constant polynomial $p$ satisfies the conditions of Theorems 1.1 or 1.4 then so does its derivative $p'$. Therefore, in Theorems 1.2, 1.3 and 1.5, we can not only estimate the number of structures of a given type (matchings, independent sets or unbranched subgraphs) by counting structures up to some small size, but also estimate the average size of a structure, the second moment, etc.

Finally, we mention the following result implicit in Section 2.2 of [Ba16].

Theorem. Let $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ be a polynomial and suppose that for some $0 < \delta < 1$ we have $p(z) \neq 0$ for all $z$ in the $\delta$-neighborhood of the interval $[0,1] \subset \mathbb{C}$ (we measure distances by identifying $\mathbb{C} = \mathbb{R}^2$). Then, for any $0 < \epsilon < 1$, the value of $p(1)$, up to relative error $\epsilon$, is determined by the coefficients $a_k$ with

$$k \leq \exp \left\{ O \left( \frac{1}{\delta} \right) \right\} \ln n \exp \left\{ O \left( \frac{1}{\delta} \right) \right\}.$$

For applications of Theorem 1.6 to computing partition functions, see [Ba16].

We can replace the exponential dependence on $1/\delta$ in Theorem 1.6 by a polynomial dependence if we assume that $p(z) \neq 0$ for $z$ in the $\delta$-neighborhood of the sector $|\arg z| < \alpha$ for some fixed $\alpha > 0$ and some $\delta > 0$. We briefly discuss this in Section 2 and applications to counting subgraphs with prescribed degrees in Section 3.

We prove Theorems 1.1 and 1.4 in Section 2 and Theorems 1.2, 1.3 and 1.5 in Section 3.

2. PROOFS OF THEOREMS 1.1 AND 1.4

We denote the complex plane by $\mathbb{C}$, the Riemann sphere $\mathbb{C} \cup \{\infty\}$ by $\hat{\mathbb{C}}$ and the open unit disc by $\mathbb{D}$, so that

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$ 

By $c$ we denote a positive absolute constant, whose value may change from line to line.

We start with a couple of lemmas.
(2.1) Lemma. Let \(h_1, h_2 : \mathbb{C} \to \mathbb{C}\) be polynomials of degrees \(n_1\) and \(n_2\) respectively and let

\[
g(z) = \frac{h_1(z)}{h_2(z)}; \quad g : \hat{\mathbb{C}} \to \hat{\mathbb{C}},
\]

be a rational function. Let \(\beta > 1\) be a real number and suppose that

\[
h_1(z) \neq 0 \quad \text{and} \quad h_2(z) \neq 0 \quad \text{provided} \quad |z| < \beta,
\]

so \(g\) has neither zeros no poles in the disc \(\beta \mathbb{D} = \{z \in \mathbb{C} : |z| < \beta\}\).

Let us choose a branch of \(f(z) = \ln g(z)\) where \(|z| < \beta\)

and let

\[
T_m(z) = f(0) + \sum_{k=1}^{m} \frac{f^{(k)}(0)}{k!} z^k
\]

be the Taylor polynomial of degree \(m\) of \(f(z)\) computed at \(z = 0\). Then

\[
|f(1) - T_m(1)| \leq \frac{n_1 + n_2}{\beta^m (\beta - 1)(m + 1)}.
\]

Proof. In the case when \(g(z)\) is a polynomial (that is, when \(h_2(z) \equiv 1\)), this is Lemma 2.2.1 of [Ba16]. The proof below in the case of a rational function is very similar.

Let \(\alpha_{11}, \ldots, \alpha_{1n_1}\) be the roots of \(h_1\) and let \(\alpha_{21}, \ldots, \alpha_{2n_2}\) be the roots of \(h_2\), counting multiplicity. Hence

\[
h_1(z) = h_1(0) \prod_{i=1}^{n_1} \left(1 - \frac{z}{\alpha_{1i}}\right) \quad \text{and} \quad h_2(z) = h_2(0) \prod_{j=1}^{n_2} \left(1 - \frac{z}{\alpha_{2j}}\right),
\]

where

\[
|\alpha_{1i}| \geq \beta \quad \text{for} \quad i = 1, \ldots, n_1 \quad \text{and} \quad |\alpha_{2j}| \geq \beta \quad \text{for} \quad j = 1, \ldots, n_2.
\]

Then

\[
f(z) = f(0) + \sum_{i=1}^{n_1} \ln \left(1 - \frac{z}{\alpha_{1i}}\right) - \sum_{j=1}^{n_2} \ln \left(1 - \frac{z}{\alpha_{2j}}\right),
\]

where we choose the branch of the logarithm so that \(\ln 1 = 0\).

Approximating the logarithms by their Taylor polynomials, we obtain

\[
\ln \left(1 - \frac{1}{\alpha_{1i}}\right) = -\sum_{k=1}^{m} \frac{1}{k \alpha_{1i}^k} + \eta_{1i} \quad \text{and} \quad \ln \left(1 - \frac{1}{\alpha_{2j}}\right) = -\sum_{k=1}^{m} \frac{1}{k \alpha_{2j}^k} + \eta_{2j},
\]

where \(\eta_{1i}\) and \(\eta_{2j}\) are the remainders of the Taylor polynomials.

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where
\[ |\eta_{1i}| = \left| \sum_{k=m+1}^{\infty} \frac{1}{k\alpha_{1i}^k} \right| \leq \frac{1}{m+1} \sum_{k=m+1}^{\infty} \frac{1}{\beta^k} = \frac{1}{(m+1)\beta^m(\beta-1)} \]
for \(i = 1, \ldots, n_1\) and, similarly,
\[ |\eta_{2j}| = \left| \sum_{k=m+1}^{\infty} \frac{1}{k\alpha_{2j}^k} \right| \leq \frac{1}{m+1} \sum_{k=m+1}^{\infty} \frac{1}{\beta^k} = \frac{1}{(m+1)\beta^m(\beta-1)} \]
for \(j = 1, \ldots, n_2\).

Since
\[ T_m(1) = -\sum_{i=1}^{n_1} \sum_{k=1}^{m} \frac{1}{k\alpha_{1i}^k} + \sum_{j=1}^{n_2} \sum_{k=1}^{m} \frac{1}{k\alpha_{2j}^k}, \]
the proof follows. \(\square\)

**(2.2) Corollary.** For \(0 < \epsilon < 1\), under the conditions of Lemma 2.1, we have
\[ |f(1) - T_m(1)| \leq \epsilon \]
provided
\[ m \geq \frac{c}{\beta-1} \ln \frac{n_1 + n_2}{\epsilon(\beta-1)} \]
where \(c > 0\) is an absolute constant.

**Proof.** Follows by Lemma 2.1. \(\square\)

To compute the value of \(T_m(1)\) in Lemma 2.1 and Corollary 2.2, we need to compute the derivatives \(f^{(k)}(0)\) for \(k = 0, 1, \ldots, m\). This, in turn, reduces to computing the derivatives \(g^{(k)}(0)\) for \(k = 0, 1, \ldots, m\), as explained in Section 2.2.2 of [Ba16]. For completeness, we describe the procedure here.

**(2.3) Computing \(f^{(k)}(0)\) from \(g^{(k)}(0)\).** We have
\[ f'(z) = \frac{g'(z)}{g(z)} \quad \text{from which} \quad g'(z) = f'(z)g(z) \]
and hence
\[ g^{(k)}(0) = \sum_{j=0}^{k-1} \binom{k-1}{j} f^{(k-j)}(0)g^{(j)}(0) \quad \text{for} \quad k = 1, \ldots, m. \]

(2.3.1)

Now, (2.3.1) is a triangular system of linear equations in the variables \(f^{(k)}(0)\) for \(k = 1, \ldots, m\) with diagonal coefficients \(g^{(0)}(0) = g(0) \neq 0\), so the matrix of the system is invertible. Given the values of \(g(0)\) and \(g^{(k)}(0)\) for \(k = 1, \ldots, m\), one can compute the values of \(f^{(k)}(0)\) for \(k = 1, \ldots, m\) in \(O(m^2)\) time. This is, of course, akin to computing cumulants of a probability distribution from its moments.

Finally, we employ a rational transformation.
(2.4) Lemma. For real $0 < \rho < 1$, let

$$\xi = \xi_\rho = 1 - \sqrt{\frac{\rho}{1 + \rho}}, \quad \beta = \beta_\rho = \xi^{-1} \geq 1 + \sqrt{\rho}.$$ 

and let

$$\psi = \psi_\rho(z) = \frac{\rho}{(1 - \xi z)^2} - \rho, \quad \psi : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}},$$

be a rational function. Then $\psi(0) = 0$, $\psi(1) = 1$ and the image of the disc

$$\beta \mathbb{D} = \{z \in \mathbb{C} : |z| < \beta\}$$

under $\psi$ does not intersect the ray

$$\left\{ z \in \mathbb{C} : \Im z = 0 \text{ and } \Re z \leq -\frac{3\rho}{4} \right\}.$$

Proof. Clearly, $\psi(0) = 0$ and $\psi(1) = 1$. For $z \in \beta \mathbb{D}$, we have $|\xi z| < 1$ and hence

$$\arg \frac{1}{1 - \xi z} < \frac{\pi}{2}.$$

Therefore the image of $\beta \mathbb{D}$ under the map

(2.4.1) $$z \mapsto \frac{\rho}{(1 - \xi z)^2}$$

does not contain the non-positive real ray

$$R_- = \{z \in \mathbb{C} : \Im z = 0, \ \Re z \leq 0\}.$$

The real values of the map (2.4.1) on the disc $\beta \mathbb{D}$ are attained when $z$ is real, and are larger than $\rho/4$, which is attained when $z = -\beta$.

The proof now follows. □

Now we are ready to prove Theorem 1.1.

(2.5) Proof of Theorem 1.1. Let $\rho = 4\delta / 3$ and let $\psi = \psi_\rho : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be the corresponding rational transformation of Lemma 2.4. We consider the composition

$$g(z) = p(\psi(z)).$$

Clearly, $g(0) = p(0)$ and $g(1) = p(1)$. Let

$$\xi = 1 - \sqrt{\frac{\rho}{1 + \rho}} \quad \text{and} \quad \beta = \xi^{-1} \geq 1 + \sqrt{\frac{4\delta}{3}},$$
as in Lemma 2.4. Since the image $\psi(\beta \mathbb{D})$ does not intersect the ray
\[ R = \{ z \in \mathbb{C} : \Re z = 0 \text{ and } \Re z \leq -\delta \}, \]
we conclude that
\[ g(z) \neq 0 \text{ provided } |z| < \beta. \]
For some polynomials $h_1(z)$ and $h_2(z)$, we can write
\[ g(z) = \frac{h_1(z)}{h_2(z)} \text{ where } h_2(z) = (1 - \xi z)^{2n} \text{ and } \deg h_2(z) \leq 2n. \]
Let us choose a branch of
\[ f(z) = \ln g(z) \text{ for } |z| < \beta \]
and let $T_m(z)$ be the Taylor polynomial of $f$ of degree $m$, computed at $z = 0$. From Corollary 2.2, we have
\[ |T_m(1) - f(1)| = |T_m(1) - \ln p(1)| \leq \epsilon, \]
as long as
\[ m \geq \frac{c}{\sqrt{\delta}} \ln \frac{n}{\epsilon \sqrt{\delta}} \]
for some absolute constant $c > 0$.

It remains to show how to compute the values $f^{(k)}(0)$ for $k = 0, \ldots, m$ from the coefficients $a_k$, $k = 0, 1, \ldots, m$, of the polynomial $p$. First, we compute the values $g^{(k)}(0)$ for $k = 0, \ldots, m$. To that effect, let
\[ p_{[m]}(z) = \sum_{k=0}^{m} a_k z^k \]
be the truncation of the polynomial $p$ and let
\[ \psi_{[m]}(z) = \rho \sum_{k=1}^{m} (k + 1) \xi^k z^k \]
be the truncation of the Taylor series expansion of $\psi(z)$ in the disc $\beta \mathbb{D}$. Note that since $\psi(0) = 0$, the constant term of the expansion is 0. We then compute the composition
\[ p_{[m]}(\psi_{[m]}(z)) \]
and discard the terms of degree higher than $m$. A fast way to do it is by Horner’s method, successively computing
\[ \left( \ldots \left( (a_m \psi_{[m]}(z) + a_{m-1}) \psi_{[m]}(z) \right) + \ldots \right) \psi_{[m]}(z) + a_0 \]
and discarding monomials of degree higher than $m$ on each step. This gives us the Taylor polynomial of degree $m$ of $g(z)$, computed at $z = 0$. We then compute the derivatives $f^{(k)}(0)$ as in Section 2.3.

To prove Theorem 1.4, we use a different (simpler) rational transformation. □
(2.6) Lemma. For real $0 < \rho < 1$, let

$$\xi = \xi_\rho = \frac{1}{1 + \rho}, \quad \beta = \beta_\rho = \xi^{-1} = 1 + \rho.$$ 

and let

$$\psi = \psi_\rho(z) = \frac{\rho}{1 - \xi z} - \rho, \quad \psi : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}},$$

be a rational function. Then $\psi(0) = 0$, $\psi(1) = 1$ and the image of the disc

$$\beta \mathbb{D} = \{ z \in \mathbb{C} : |z| < \beta \}$$

under $\psi$ does not intersect the half-plane

$$\{ z \in \mathbb{C} : \Re z \leq -\rho \}.$$ 

Proof. Clearly, $\psi(0) = 0$ and $\psi(1) = 1$. For $z \in \beta \mathbb{D}$, we have $|\xi z| < 1$ and hence

$$\arg \frac{1}{1 - \xi z} < \frac{\pi}{2}.$$ 

Therefore, the image of $\beta \mathbb{D}$ under the map

$$z \mapsto \frac{1}{1 - \xi z}$$

does not intersect the half-plane $\Re z \leq 0$.

The proof now follows. \(\square\)

(2.7) Proof of Theorem 1.4. We define the transformation $\psi = \psi_\delta$ as in Lemma 2.6, consider the composition $g(z) = p(\psi(z))$ and proceed as in the proof on Theorem 1.1 in Section 2.5 with straightforward modifications. \(\square\)

(2.8) Remark: approximating $p'(1)$. It follows from Rolle’s Theorem that if $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ is a non-constant polynomial satisfying the conditions of Theorem 1.1 then $p'(x) = a_1 + 2a_2 x + \ldots + na_n x^{n-1}$ also satisfies the conditions of Theorem 1.1. Similarly, it follows from the Gauss-Lucas Theorem that if a non-constant polynomial $p(z)$ satisfies the conditions of Theorem 1.4, then so does $p'(z)$.

(2.9) Possible ramifications. To prove Theorem 1.6, for $0 < \rho < 1$, we define

$$\xi = \xi_\rho = 1 - e^{-\frac{1}{\rho}}, \quad \beta = \frac{1 - e^{-\frac{1}{\rho}} - \frac{1}{\rho}}{1 - e^{-\frac{1}{\rho}}} > 1.$$
and consider the map
\[ \psi = \psi_\rho(z) = \rho \ln \frac{1}{1 - \xi z} \]
where \(|z| < \beta\).

Then \(\psi(0) = 0, \psi(1) = 1\) and if \(\rho = \rho(\delta) > 0\) is chosen small enough, the image \(\psi(\beta \mathbb{D})\) lies in the \(\delta/2\)-neighborhood of \([0, 1] \subset \mathbb{C}\). However, \(\psi\) is no longer a rational transformation. Instead of \(\psi\), we use its scaled Taylor polynomial approximation \(\tilde{\psi}\) such that we still have \(\tilde{\psi}(0) = 0, \tilde{\psi}(1) = 1\) and \(\tilde{\psi}\) maps the disc \(\beta \mathbb{D}\) inside the \(\delta\)-neighborhood of \([0, 1] \subset \mathbb{C}\), see Section 2.2 of [Ba16] for details. Then the proof proceeds as in Section 2.5.

Suppose now that \(p(z)\) is a polynomial of degree \(n\) such that \(p(z) \neq 0\) whenever \(z\) lies in the \(\delta\)-neighborhood of the sector
\[ S_\alpha = \{ z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \alpha \} \]
for some fixed \(\alpha > 0\) and some \(\delta > 0\). In this case, for \(0 < \rho < 1\) define
\[ \xi = \xi_\rho = 1 - \left( \frac{\rho}{1 + \rho} \right)^{\pi/2\alpha}, \quad \beta = \beta_\rho = \xi^{-1} \left( 1 - \frac{1}{2} \left( \frac{\rho}{1 + \rho} \right)^{\pi/2\alpha} \right) > 1 \]
and consider the (well-defined) map
\[ \psi(z) = \psi_\rho(z) = \rho \left( 1 - \xi z \right)^{-2\alpha/\pi} - \rho \quad \text{for} \quad |z| < \beta. \]

We observe that \(\psi(0) = 0, \psi(1) = 1\) and one can show that by choosing \(\rho\) small enough, we can make sure that the image of the disc \(\beta \mathbb{D}\) lies in a prescribed neighborhood of the sector \(|\arg z| \leq \alpha\). We then can use a sufficiently accurate polynomial approximation \(\tilde{\psi}\) to \(\psi\) to show that the value of \(p(1)\), up to relative error \(\epsilon\), is determined by the lowest
\[ \frac{c}{\delta^{\pi/2\alpha}} \ln \frac{n}{\epsilon \delta^{\pi/2\alpha}} \]
coefficients of \(p\), where \(c > 0\) is an absolute constant.

3. Proofs of Theorems 1.2, 1.3 and 1.5

(3.1) Proof of Theorem 1.2. Given a graph \(G\), we define its matching polynomial by
\[ p(x) = 1 + \sum_{k=1}^{v(G)} m_k(G) x^k. \]

The Heilmann-Lieb Theorem [HL72], see also [GG81], asserts that the roots \(x\) of \(p(x)\) are real and satisfy \(x \leq -\delta\) for
\[ \delta = \frac{1}{4(\Delta - 1)} \quad \text{for} \quad \Delta = \max \{ \Delta(G), 2 \}. \]

The proof now follows from Theorem 1.1. \(\square\)
(3.2) Proof of Theorem 1.3. Given a graph $G$, we define its independence polynomial by

$$p(x) = 1 + \sum_{k=1}^{v(G)} i_k(G)x^k.$$  

Chudnovsky and Seymour proved [CS07] that if $G$ is claw-free, then the roots of $p(x)$ are necessarily non-positive real. On the other hand, the Dobrushin-Shearer bound, cf. [SS05], states that the roots $z$ of the independence polynomial of any graph $G$ satisfy

$$|z| \geq \frac{(\Delta - 1)^{\Delta - 1}}{\Delta^{1/2}} = \frac{1}{\Delta e} \left(1 + O\left(\frac{1}{\Delta}ight)\right) \quad \text{as} \quad \Delta \to \infty,$$

where $\Delta = \max\{2, \Delta(G)\}$. The proof now follows from Theorem 1.1. \hfill \Box

(3.3) Proof of Theorem 1.5. Given a graph $G$, we define its unbranched subgraph polynomial by

$$p(z) = 1 + \sum_{k=1}^{v(G)} u_k(G)z^k.$$  

Ruelle proved [R99a], [R99b], see also [Wa09], that all roots $z$ of $p(z)$ satisfy

$$\Re z \leq -\frac{2}{\Delta^{(\Delta - 1)^2}} \quad \text{for} \quad \Delta = \max\{2, \Delta(G)\}.$$  

The proof now follows from Theorem 1.4. \hfill \Box

(3.4) Estimating averages. We note that the value $\frac{p'(1)}{p(1)}$ is interpreted the average number of edges in a matching in Theorem 1.2, the average number of vertices in an independent set in Theorem 1.3 and the average number of edges in a unbranched subgraph in Theorem 1.5. It follows from Remark 2.8 that we can estimate the averages within relative error $\epsilon > 0$ by inspecting the matchings, independent sets and unbranched subgraphs of pretty much the same size as prescribed by Theorems 1.2, 1.3 and 1.5, though with different absolute constants. Similarly, by computing $\frac{p''(1)}{p(1)}$ we can estimate the second moment, etc.

(3.5) Possible ramifications. For each vertex $w$ of a graph $G$, let us choose a set $A_w$ of allowable degrees of subgraphs. Wagner proved [Wa09] that if $0 \in A_w \subset \{0, 1, 2\}$ for all vertices $w$, then the corresponding subgraph counting polynomial is non-zero in the sector

$$S_{\pi/3} = \left\{ z \in \mathbb{C} : |\arg z| < \frac{\pi}{3} \right\}$$

and is also non-zero in a $\delta$-neighborhood of $z = 0$ for some $\delta = \Omega(1/\Delta(G))$. Using the approach sketched in Section 2.8, one can show that within relative error $\epsilon > 0,$
the total number of such subgraphs is determined by the numbers of subgraphs with
\[ k = \left(\Delta(G)\right)^{O(1)} \ln \frac{v(G)}{\epsilon} \]
edges.

References


