WHEN THE POSITIVE SEMIDEFINITE
RELAXATION GUARANTEES A SOLUTION TO
A SYSTEM OF REAL QUADRATIC EQUATIONS

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ABSTRACT. By solving a positive semidefinite program, one can reduce a system of
real quadratic equations to a system of the type \( q_i(x) = \alpha_i \), \( i = 1, \ldots, m \), where
\( q_i : \mathbb{R}^n \rightarrow \mathbb{R} \) are quadratic forms and \( \alpha_i = \text{trace} q_i \). We prove a sufficient condition
for the latter system to have a solution \( x \in \mathbb{R}^n \): assuming that the operator norms
of the \( n \times n \) matrices \( Q_i \) of \( q_i \) do not exceed 1, the smallest eigenvalue the \( m \times m \)
matrix with the \((i,j)\)-th entry equal \( \text{trace}(Q_iQ_j) \) is at least \( \gamma n^{2/3}m^2 \ln n \) for an
absolute constant \( \gamma > 0 \). In particular, this happens when \( n \gg m^6 \) and the forms
\( q_i \) are sufficiently generic. We prove a similar sufficient condition for a homogeneous
system of quadratic equations to have a non-trivial solution.

1. Introduction and main results

(1.1) Systems of real quadratic equations. Let \( q_1, \ldots, q_m : \mathbb{R}^n \rightarrow \mathbb{R} \) be
quadratic forms,
\[
q_i(x) = \langle Q_i x, x \rangle \quad \text{for} \quad i = 1, \ldots, m,
\]
where \( Q_i \) are \( n \times n \) symmetric matrices and
\[
\langle x, y \rangle = \sum_{i=1}^{n} \xi_i \eta_i \quad \text{for} \quad x = (\xi_1, \ldots, \xi_n) \quad \text{and} \quad y = (\eta_1, \ldots, \eta_n)
\]
is the standard scalar product in \( \mathbb{R}^n \).

Let \( \alpha_1, \ldots, \alpha_m \) be real numbers. We want to find out when the system of
equations
\[
q_i(x) = \alpha_i \quad \text{for} \quad i = 1, \ldots, m
\]
is solvable.
has a solution $x \in \mathbb{R}^n$. Such systems of equations appear in various contexts, see, for example, [Bi16], [L+14], [Pa13]. If the number $m$ of equations is fixed in advance, one can decide in polynomial time whether the system has a solution [Ba93], [GP05], [Ba08]. The same is true if the number $n$ of variables is fixed in advance, in which case a polynomial time algorithm to test feasibility exists even if $q_i$ are polynomials of an arbitrary degree, see, for example, [B+06].

If $m$ and $n$ are both allowed to grow, the problem becomes computationally hard. In fact, testing the feasibility of an arbitrary system of real polynomial equations can be easily reduced to testing the feasibility of a system (1.1.1). First, we gradually reduce the degree of polynomials by introducing new variables and equations of the type $\xi_{ij} - \xi_i \xi_j = 0$, and hence reduce a given polynomial system to a system

$$q_i(x) = 0 \quad \text{for} \quad i = 1, \ldots, m,$$

where $q_i$ are quadratic, not necessarily homogeneous, polynomials. Then we introduce yet another variable $\tau$ and replace the above system by a system of homogeneous quadratic equations

$$\tau^2 q_i (\tau^{-1} x) = 0 \quad \text{for} \quad i = 1, \ldots, m$$

with one more quadratic constraint $\tau^2 = 1$.

We are also interested in systems of homogeneous equations

$$(1.1.2) \quad q_i(x) = 0 \quad \text{for} \quad i = 1, \ldots, m,$$

in which case we want to find out whether the system has a non-trivial solution $x \neq 0$.

(1.2) Positive semidefinite relaxation. For an $n \times n$ real symmetric matrix $X$, we write $X \succeq 0$ to say that $X$ is positive semidefinite.

Given (1.1.1), we consider the following system of linear equations

$$(1.2.1) \quad \text{trace}(Q_i X) = \alpha_i \quad \text{for} \quad i = 1, \ldots, m \quad \text{where} \quad X \succeq 0$$

in $n \times n$ positive semidefinite matrices $X$. Unlike (1.1.1), the system (1.2.1) is convex and efficient algorithms are available to test its feasibility, see [Pa13] for a survey. Clearly, if $x = (\xi_1, \ldots, \xi_n)$ is a solution to (1.1.1) then the matrix $X = (x_{ij})$ defined by $x_{ij} = \xi_i \xi_j$ is a positive semidefinite solution to (1.2.1). If $m \leq 2$, then the converse is true: if the system (1.2.1) has a solution then so does (1.1.1), see, for example, Section II.13 of [Ba02]. For $m \geq 3$ the system (1.2.1) may have solutions while (1.1.1) may be infeasible. For example, the system of quadratic equations

$$\xi_1^2 = 1, \quad \xi_2^2 = 1 \quad \text{and} \quad \xi_1 \xi_2 = 0$$

does not have a solution, whereas the $2 \times 2$ identity matrix $I$ is the solution to its positive semidefinite relaxation.
Our goal is to find a simple yet meaningful criterion when a solution to (1.2.1) implies the existence of a solution to (1.1.1).

Let \( X \) be a solution to (1.2.1). Since \( X \succeq 0 \), we can write \( X = TT^* \) for an \( n \times n \) matrix \( T \). Then

\[
\text{trace}(Q_iX) = \text{trace}(Q_iTT^*) = \text{trace}(T^*Q_iT).
\]

Let us define matrices

\[
\tilde{Q}_i = T^*Q_iT \quad \text{for} \quad i = 1, \ldots, m
\]

and the corresponding quadratic forms \( \tilde{q}_i : \mathbb{R}^n \rightarrow \mathbb{R} \),

\[
\tilde{q}_i(x) = \langle \tilde{Q}_i x, x \rangle = q_i(Tx) \quad \text{for} \quad i = 1, \ldots, m.
\]

If \( x \in \mathbb{R}^n \) is a solution to the system

\[
\tilde{q}_i(x) = \alpha_i \quad \text{for} \quad i = 1, \ldots, m
\]

then \( y = Tx \) is a solution to (1.1.1). We note that

\[
\alpha_i = \text{trace} \tilde{Q}_i \quad \text{for} \quad i = 1, \ldots, m.
\]

In other words, a solution \( X \) to (1.2.1) allows us to replace (1.1.1) by a similar system, where the right hand sides \( \alpha_i \) are the traces of the quadratic forms in the left hand side. Ultimately, we are interested in finding out when the system (1.2.4) of quadratic equations with additional conditions (1.2.5) has a solution \( x \in \mathbb{R}^n \).

For an \( n \times n \) real symmetric matrix \( Q \), we denote by \( \|Q\|_{\text{op}} \) the operator norm of \( Q \), that is, the largest absolute value of an eigenvalue of \( Q \).

We prove the following main result.

**Theorem.** Let \( Q_1, \ldots, Q_m \) be \( n \times n \) real symmetric matrices, let \( q_i : \mathbb{R}^n \rightarrow \mathbb{R} \),

\[
q_i(x) = \langle Q_i x, x \rangle \quad \text{for} \quad i = 1, \ldots, m,
\]

be the corresponding quadratic forms, and let

\[
\alpha_i = \text{trace} Q_i \quad \text{for} \quad i = 1, \ldots, m.
\]

We define yet another quadratic form \( \psi : \mathbb{R}^m \rightarrow \mathbb{R} \) by

\[
\psi(t) = \text{trace} \left( \sum_{i=1}^{m} \tau_i Q_i \right)^2 \quad \text{for} \quad t = (\tau_1, \ldots, \tau_m).
\]

Suppose that

\[
\|Q_i\|_{\text{op}} \leq 1 \quad \text{for} \quad i = 1, \ldots, m
\]

and that

\[
\psi(t) \geq \mu \|t\|^2 \quad \text{where} \quad \|t\| = \sqrt{\tau_1^2 + \ldots + \tau_m^2} \quad \text{for} \quad t = (\tau_1, \ldots, \tau_m)
\]

and

\[
\mu \geq 10n^3m^2 \ln(4n) + 2m^2 + m.
\]

Then the system (1.1.1) of equations has a solution \( x \in \mathbb{R}^n \).
(1.4) Discussion. The condition \( \psi(t) \geq \mu \|t\|^2 \) can be stated equivalently as that the eigenvalues of the \( m \times m \) positive semidefinite matrix with the \((i,j)\)-th entry equal to \( \text{trace}(Q_iQ_j) \) are greater than or equal to \( \mu \). This is a standard problem of linear algebra which can be solved in polynomial time.

Under the conditions of Theorem 1.3 we must have \( \mu \leq n \), see Section 3.1, and hence \( n \gg m^6 \). Theorem 1.3 implies roughly that if \( n \) is sufficiently large compared to \( m \) and if the forms \( q_1, \ldots, q_m \) are sufficiently generic, then the system \( q_i(x) = \text{trace} q_i \) of equations has a solution. Moreover, as \( n \) grows compared to \( m \), the condition of being sufficiently generic becomes less and less restrictive: if the \( n \times n \) matrices \( Q_1, \ldots, Q_m \) are sufficiently generic, we expect the smallest eigenvalue \( \mu \) of \( \psi \) to grow roughly linearly in \( n \), while the required threshold for \( \mu \) grows as \( n^{2/3} \ln n \).

Suppose, for example, that the entries of the \( n \times n \) symmetric matrices \( Q_1, \ldots, Q_m \) are sampled independently from a distribution with expectation 0, variance 1 and sub-Gaussian tail, conditioned on the symmetry of the resulting matrices. We assume that \( n \geq m \). As \( n \) grows, with high probability we have (we ignore low-order terms)

\[
\|Q_i\|_{\text{op}} \approx 2\sqrt{n}, \quad \text{trace}(Q_iQ_i) \approx n^2 \quad \text{for} \quad i = 1, \ldots, m
\]

and

\[
\max_{i \neq j} |\text{trace}(Q_iQ_j)| = O(n \ln n),
\]

see Section 2.3 of [Ta12]. Rescaling, we make sure that

\[
\|Q_i\|_{\text{op}} \leq 1, \quad \text{trace}(Q_iQ_i) \approx \frac{n}{4} \quad \text{for} \quad i = 1, \ldots, n
\]

and

\[
\max_{i \neq j} |\text{trace}(Q_iQ_j)| = O(\ln n) \quad \text{for all} \quad i \neq j.
\]

Then we have

\[
\mu \geq \frac{n}{4} - O(m \ln n)
\]

and the condition of Theorem 1.3 is satisfied when \( n \gg m^6 \) up to logarithmic factors, see also [LL16] for the topology of the intersection of a fixed number of random projective quadrics in the homogeneous case.

Let us fix some linearly independent \( n \times n \) real symmetric matrices \( Q_1, \ldots, Q_m \) such that \( \|Q_i\|_{\text{op}} \leq 1 \) for \( i = 1, \ldots, m \) and let \( \psi \) be the corresponding quadratic form of Theorem 1.3. Let \( H \) be a symmetric matrix such that \( \|H\|_{\text{op}} \leq 1 \). We consider the Kronecker products \( Q_1 \otimes H, \ldots, Q_m \otimes H \). Then \( \|Q_i \otimes H\|_{\text{op}} \leq 1 \) for \( i = 1, \ldots, m \) and for the corresponding quadratic form \( \psi_H \) we have

\[
\psi_H = \text{trace}(H^2)\psi.
\]
Hence as soon as \( \text{trace}(H^2) \) is large enough (for which the size of \( H \) should be large enough), the matrices \( Q_1 \otimes H, \ldots, Q_m \otimes H \) satisfy the conditions of Theorem 1.3. We note that if we choose \( H \) to be the \( k \times k \) identity matrix \( I_k \), then for

\[
k = \left\lfloor \frac{\sqrt{8m + 1} - 1}{2} \right\rfloor
\]

the system of quadratic equations (1.1.1) with matrices \( Q_1 \otimes H, \ldots, Q_m \otimes H \) and \( \alpha_i = \text{trace}(Q_i \otimes H) \) will always have a solution, see, for example, Section II.13 of [Ba02].

We prove a similar result for systems of homogeneous equations.

(1.5) Theorem. Let \( Q_1, \ldots, Q_m \) be \( n \times n \) real symmetric matrices such that

\[
\text{trace} Q_i = 0 \quad \text{for} \quad i = 1, \ldots, m
\]

and let \( q_i : \mathbb{R}^n \to \mathbb{R}, \)

\[
q_i(x) = \langle Q_i x, x \rangle \quad \text{for} \quad i = 1, \ldots, m
\]

be the corresponding quadratic forms. We define a quadratic form \( \psi : \mathbb{R}^m \to \mathbb{R} \) by

\[
\psi(t) = \text{trace} \left( \sum_{i=1}^{m} \tau_i Q_i \right)^2 \quad \text{for} \quad t = (\tau_1, \ldots, \tau_m).
\]

Suppose that

\[
\|Q_i\|_{op} \leq 1 \quad \text{for} \quad i = 1, \ldots, m
\]

and that

\[
\psi(t) \geq \mu \|t\|^2 \quad \text{where} \quad \|t\| = \sqrt{\tau_1^2 + \ldots + \tau_m^2} \quad \text{for} \quad t = (\tau_1, \ldots, \tau_m)
\]

and

\[
\mu \geq 10n^2m^2 \ln(4n) + 2m^2 + m.
\]

Then the system (1.1.2) of equations has a solution \( x \neq 0 \).

We prove Theorems 1.3 and 1.5 in Sections 2 and 3. In Section 4, we state some open questions.
2. Enter Gaussian measure

(2.1) Gaussian measure. We consider the standard Gaussian measure in \( \mathbb{R}^n \) with density

\[
\frac{1}{(2\pi)^{n/2}} e^{-\|x\|^2/2} \quad \text{where} \quad \|x\| = \sqrt{\xi_1^2 + \ldots + \xi_n^2} \quad \text{for} \quad x = (\xi_1, \ldots, \xi_n).
\]

Considering a quadratic form \( q(x) = \langle Qx, x \rangle \) as a random variable, we observe that

\[
E q = \text{trace } Q,
\]

so that the equation \( q(x) = \text{trace } Q \) “holds on average”.

In what follows, we denote the imaginary unit by \( \sqrt{-1} \), so as to use \( i \) for indices.

Let \( Q_1, \ldots, Q_m \) be \( n \times n \) real symmetric matrices and let \( I \) be the \( n \times n \) identity matrix. For real \( \tau_1, \ldots, \tau_m \), we consider the matrix

\[
Q(t) = I - \sqrt{-1} \sum_{i=1}^m \tau_i Q_i \quad \text{for} \quad t = (\tau_1, \ldots, \tau_m).
\]

Since the eigenvalues \( \lambda_1(t), \ldots, \lambda_n(t) \) of the linear combination \( \sum_{i=1}^m \tau_i Q_i \) are real, we have

\[
\det Q(t) = \prod_{i=1}^n (1 - \sqrt{-1} \lambda_i(t)) \neq 0 \quad \text{for all} \quad t \in \mathbb{R}^m.
\]

Therefore, we can pick a branch of

\[
-\frac{1}{\frac{1}{2} \det Q(t)},
\]

which we select in such a way so that at \( t = 0 \) we get 1.

It is also more convenient to rescale and define quadratic forms by

\[
q(x) = \frac{1}{2} \langle Qx, x \rangle
\]

for a symmetric matrix \( Q \).

Our proofs of Theorems 1.3 and 1.5 hinge on the following formula.

(2.2) Lemma. Let \( Q_1, \ldots, Q_m \) be \( n \times n \) real symmetric matrices and let

\[
q_i(x) = \frac{1}{2} \langle Q_i x, x \rangle \quad \text{for} \quad i = 1, \ldots, m
\]

be the corresponding quadratic forms. Then for any real \( \alpha_1, \ldots, \alpha_m \) and any real \( \sigma > 0 \), we have

\[
\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp \left\{ -\frac{\sigma^2}{2} \sum_{i=1}^m (q_i(x) - \alpha_i)^2 \right\} e^{-\|x\|^2/2} \, dx
\]

\[
= \frac{1}{\sigma^m (2\pi)^{m/2}} \int_{\mathbb{R}^m} -\frac{1}{2} \left( I - \sqrt{-1} \sum_{i=1}^m \tau_i Q_i \right) \exp \left\{ -\sqrt{-1} \sum_{i=1}^m \alpha_i \tau_i \right\} e^{-\|t\|^2/2\sigma^2} \, dt.
\]
Proof. As is well-known, for a positive definite matrix $Q$ and the corresponding form
\[ q(x) = \frac{1}{2} \langle Qx, x \rangle \]
we have
\[ \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-q(x)} \, dx = -\frac{1}{2} \det Q. \]
Consequently, for \( t \in \mathbb{R}^m, \ t = (\tau_1, \ldots, \tau_m) \), in a sufficiently small neighborhood of 0, we have
\[ \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp \left\{ \sum_{i=1}^{m} \tau_i q_i(x) \right\} e^{-\|x\|^2/2} \, dx = -\frac{1}{2} \det \left( I - \sum_{i=1}^{m} \tau_i Q_i \right). \]
Since both sides of the formula are analytic in \( \tau_1, \ldots, \tau_m \in \mathbb{C} \) for \( \Re \tau_1, \ldots, \Re \tau_m \) in a small neighborhood of 0, we conclude that the above formula holds for all such \( \tau_1, \ldots, \tau_m \) and that, in particular,
\[ \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp \left\{ \sqrt{-1} \sum_{i=1}^{m} \tau_i q_i(x) \right\} e^{-\|x\|^2/2} \, dx = -\frac{1}{2} \det \left( I - \sqrt{-1} \sum_{i=1}^{m} \tau_i Q_i \right) \]
for all real \( \tau_1, \ldots, \tau_m \).

Therefore,
\[ \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp \left\{ \sqrt{-1} \sum_{i=1}^{m} \tau_i (q_i(x) - \alpha_i) \right\} e^{-\|x\|^2/2} \, dx = -\frac{1}{2} \det \left( I - \sqrt{-1} \sum_{i=1}^{m} \tau_i Q_i \right) \exp \left\{ -\sqrt{-1} \sum_{i=1}^{m} \alpha_i \tau_i \right\} \]
for all real \( \tau_1, \ldots, \tau_m \).

Next, we use a well-known formula: for \( \sigma > 0 \) and any real (or complex) \( \alpha \), we have
\[ \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp \left\{ \sqrt{-1} \alpha \tau \right\} \exp \left\{ -\frac{\tau^2}{2\sigma^2} \right\} \, d\tau = \exp \left\{ -\frac{\alpha^2 \sigma^2}{2} \right\}. \]
Integrating both sides of (2.2.1) for \( i = 1, \ldots, m \) over \( \tau_i \in \mathbb{R} \) with density
\[ \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{\tau_i^2}{2\sigma^2} \right\}, \]
we get the desired formula. \( \square \)
(2.3) Corollary. Let \( Q_1, \ldots, Q_m, q_1, \ldots, q_m \) and \( \alpha_1, \ldots, \alpha_m \) be as in Lemma 2.2. Suppose that

\[
\int_{\mathbb{R}^m} \left| \det \left( I - \sqrt{-1} \sum_{i=1}^m \tau_i Q_i \right) \right| \, dt < +\infty
\]

and that

\[
\int_{\mathbb{R}^m} \left| \det \left( I - \sqrt{-1} \sum_{i=1}^m \tau_i Q_i \right) \right| \exp \left\{ -\sqrt{-1} \sum_{i=1}^m \alpha_i \tau_i \right\} \, dt \neq 0.
\]

Then the system (1.1.1) of equations has a solution \( x \in \mathbb{R}^n \).

Proof. By Lemma 2.2, for all \( \sigma > 0 \), we have

\[
\sigma^m \int_{\mathbb{R}^n} \exp \left\{ -\frac{\sigma^2}{2} \sum_{i=1}^m (q_i(x) - \alpha_i)^2 \right\} e^{-\|x\|^2/2} \, dx =
\]

(2.3.3) \[
(2\pi)^{\frac{n-m}{2}} \int_{\mathbb{R}^m} \left| \det \left( I - \sqrt{-1} \sum_{i=1}^m \tau_i Q_i \right) \right| \exp \left\{ -\sqrt{-1} \sum_{i=1}^m \alpha_i \tau_i \right\} e^{-\|x\|^2/2\sigma} \, dt.
\]

As \( \sigma \to +\infty \), the right hand side of (2.3.3) converges to

\[
(2\pi)^{\frac{n-m}{2}} \int_{\mathbb{R}^m} \left| \det \left( I - \sqrt{-1} \sum_{i=1}^m \tau_i Q_i \right) \right| \exp \left\{ -\sqrt{-1} \sum_{i=1}^m \alpha_i \tau_i \right\} \neq 0.
\]

Suppose that the system (1.1.1) has no solutions \( x \in \mathbb{R}^n \). We intend to obtain a contradiction by showing that the left hand side of (2.3.3) converges to 0 as \( \sigma \to +\infty \).

Let

\[
\gamma = (2\pi)^{\frac{n-m}{2}} \int_{\mathbb{R}^m} \left| \det \left( I - \sqrt{-1} \sum_{i=1}^m \tau_i Q_i \right) \right| \, dt.
\]
Let us choose a $\rho > 0$, to be adjusted later. Then

\[
\sigma^m \int_{x \in \mathbb{R}^n : \|x\| > \rho} \exp \left\{ -\frac{\sigma^2}{2} \sum_{i=1}^{m} (q_i(x) - \alpha_i)^2 \right\} e^{-\|x\|^2/2} \, dx
\]

\[
\leq e^{-\rho^2/4} \sigma^m \int_{x \in \mathbb{R}^n : \|x\| > \rho} \exp \left\{ -\frac{\sigma^2}{2} \sum_{i=1}^{m} (q_i(x) - \alpha_i)^2 \right\} e^{-\|x\|^2/4} \, dx
\]

\[
\leq e^{-\rho^2/4} \sigma^m \int_{\mathbb{R}^n} \exp \left\{ -\frac{\sigma^2}{2} \sum_{i=1}^{m} (q_i(x) - \alpha_i)^2 \right\} e^{-\|x\|^2/4} \, dx
\]

\[
= e^{-\rho^2/4} 2^n/2 \sigma^m \int_{\mathbb{R}^n} \exp \left\{ -\frac{(2\sigma)^2}{2} \sum_{i=1}^{m} q_i(x) - \frac{\alpha_i}{2} \right\} e^{-\|x\|^2/2} \, dx
\]

\[
= e^{-\rho^2/4} 2^n/2 \sigma^m \int_{\mathbb{R}^n} \exp \left\{ -\frac{(2\sigma)^2}{2} \sum_{i=1}^{m} q_i(x) - \frac{\alpha_i}{2} \right\} e^{-\|x\|^2/2} \, dx
\]

\[
= e^{-\rho^2/4} 2^n/2 \sigma^m \int_{\mathbb{R}^n} \exp \left\{ -\frac{(2\sigma)^2}{2} \sum_{i=1}^{m} q_i(x) - \frac{\alpha_i}{2} \right\} e^{-\|x\|^2/2} \, dx.
\]

From Lemma 2.2,

\[
(2\sigma)^m \int_{\mathbb{R}^n} \exp \left\{ -\frac{(2\sigma)^2}{2} \sum_{i=1}^{m} q_i(x) - \frac{\alpha_i}{2} \right\} e^{-\|x\|^2/2} \, dx
\]

\[
\leq (2\pi)^{n-m/2} \int_{\mathbb{R}^m} \left| \frac{-1}{2} \det \left( I - \sqrt{-1} \sum_{i=1}^{m} \tau_i Q_i \right) \right| \, dt = \gamma.
\]

Summarizing,

\[
\sigma^m \int_{x \in \mathbb{R}^n : \|x\| > \rho(\epsilon)} \exp \left\{ -\frac{\sigma^2}{2} \sum_{i=1}^{m} (q_i(x) - \alpha_i)^2 \right\} e^{-\|x\|^2/2} \, dx
\]

\[
\leq e^{-\rho^2/4} 2^n/2 \sigma^m \gamma.
\]

Given $\epsilon > 0$, we choose $\rho(\epsilon) > 0$ such that

\[
e^{-\rho^2(\epsilon)/4} 2^n/2 \sigma^m \gamma \leq \frac{\epsilon}{2},
\]

so that for all $\sigma > 0$ we have

\[
(2.3.4) \quad \sigma^m \int_{x \in \mathbb{R}^n : \|x\| > \rho(\epsilon)} \exp \left\{ -\frac{\sigma^2}{2} \sum_{i=1}^{m} (q_i(x) - \alpha_i)^2 \right\} e^{-\|x\|^2/2} \, dx \leq \frac{\epsilon}{2}.
\]
If the system (1.1.1) has no solution then for some $\delta(\epsilon) > 0$, we have
\[ \sum_{i=1}^{m} (q_i(x) - \alpha_i)^2 \geq \delta(\epsilon) \quad \text{provided} \quad \|x\| \leq \rho(\epsilon) \]
and hence
\[ \sigma^m \int_{x \in \mathbb{R}^n: \|x\| \leq \rho(\epsilon)} \exp\left\{-\frac{\sigma^2}{2} \sum_{i=1}^{m} (q_i(x) - \alpha_i)^2\right\} \, dx \]
\[ \leq \sigma^m \rho^n(\epsilon) \nu_n \exp\left\{-\frac{\sigma^2 \delta(\epsilon)}{2}\right\}, \]
where $\nu_n$ is the volume of the unit ball in $\mathbb{R}^n$. Therefore, there is $\sigma_0(\epsilon) > 0$ such that for all $\sigma > \sigma_0(\epsilon)$, we have
\[ \int_{x \in \mathbb{R}^n: \|x\| \leq \rho(\epsilon)} \exp\left\{-\frac{\sigma^2}{2} \sum_{i=1}^{m} (q_i(x) - \alpha_i)^2\right\} \, dx \leq \frac{\epsilon}{2}. \]
Combining (2.3.4) and (2.3.5), we conclude that the limit of the left hand side of (2.3.3) is 0 as $\sigma \to +\infty$, which is the desired contradiction.

Similarly, we obtain a corollary for homogeneous systems.

**(2.4) Corollary.** Let $Q_1, \ldots, Q_m$ and $q_1, \ldots, q_m$ be as in Lemma 2.2 and assume, additionally, that $m < n$. Suppose that
\[ \int_{\mathbb{R}^m} \det^{-\frac{1}{2}}\left( I - \sqrt{-1} \sum_{i=1}^{m} \tau_i Q_i \right) \, dt \neq 0, \]
where the integral converges absolutely. Then the system (1.1.2) of equations has a solution $x \neq 0$.

**Proof.** Seeking a contradiction, suppose that the only solution to the system is $x = 0$. Then for some $\delta > 0$ we have
\[ \sum_{i=1}^{m} q_i^2(x) \geq \delta \quad \text{for all} \quad x \in \mathbb{R}^n \quad \text{such that} \quad \|x\| = 1. \]
From Lemma 2.2, for any $\sigma > 0$, we have
\[ \int_{\mathbb{R}^n} \sigma^m \exp\left\{-\frac{\sigma^2}{2} \sum_{i=1}^{m} q_i^2(x)\right\} e^{-\|x\|^2/2} \, dx \]
\[ = (2\pi)^{\frac{n-m}{2}} \int_{\mathbb{R}^m} \det^{-\frac{1}{2}}\left( I - \sqrt{-1} \sum_{i=1}^{m} \tau_i Q_i \right) e^{-\|x\|^2/2} \, dt. \]
From (2.4.1), the left hand side of (2.4.2) is bounded above (we use polar coordinates) by
\[ \omega_n \int_0^{+\infty} \sigma^m \exp \left\{ -\frac{\delta \sigma^2 \tau^2}{2} \right\} \tau^{n-1} e^{-\tau^2/2} d\tau, \]
where \( \omega_n \) is the surface area of the unit sphere in \( \mathbb{R}^n \). Using the substitution \( \xi = \sigma \tau \), we rewrite the integral as
\[ \omega_n \sigma^{m-n} \int_0^{+\infty} \exp \left\{ -\frac{\delta \xi^2}{2} \right\} \xi^{n-1} e^{-\xi^2/2\sigma^2} d\xi \]
and observe that it converges to 0 as \( \sigma \to +\infty \) (recall that \( m < n \)). On the other hand, the right hand side of (2.4.2) converges to
\[ (2\pi)^{\frac{n-m}{2}} \int_{\mathbb{R}^m} \det \left( I - \sqrt{-1} \sum_{i=1}^m \tau_i Q_i \right) \neq 0, \]
which is the desired contradiction. \( \square \)

3. Proofs of Theorems 1.3 and 1.5

(3.1) Preliminaries. To prove Theorem 1.3, we will use Corollary 2.3. Since the quadratic forms \( q_i \) in Corollary 2.3 are scaled differently than in Theorem 1.3, we need to make some adjustments.

Hence \( Q_1, \ldots, Q_m \) are \( n \times n \) real symmetric matrices, as in Theorem 1.3. We define
\[ q_i(x) = \frac{1}{2} \langle Q_i x, x \rangle \quad \text{for} \quad i = 1, \ldots, m \]
and let
\[ \alpha_i = \frac{1}{2} \text{trace} Q_i \quad \text{for} \quad i = 1, \ldots, m, \]
so we still consider the system of equations
\[ q_i(x) = \alpha_i \quad \text{for} \quad i = 1, \ldots, m. \]
We define the quadratic form \( \psi : \mathbb{R}^m \to \mathbb{R} \) as in Theorem 1.3.

Recall that \( \lambda_1(t), \ldots, \lambda_n(t) \) for \( t = (\tau_1, \ldots, \tau_m) \) denote the eigenvalues of the matrix \( \sum_{i=1}^m \tau_i Q_i \). Since
\[ \left\| \sum_{i=1}^m \tau_i Q_i \right\|_{\text{op}} \leq \sum_{i=1}^m |\tau_i| \leq \sqrt{m} \|t\| \quad \text{for} \quad t = (\tau_1, \ldots, \tau_m), \]
we have
\[ |\lambda_j(t)| \leq \sqrt{m} \|t\| \quad \text{for} \quad j = 1, \ldots, n. \]
In particular,

$$(3.1.2) \quad \psi(t) = \sum_{j=1}^{n} \lambda_j^2(t) \leq nm\|t\|^2.$$  

We choose

$$\rho = \frac{1}{2} n^{-\frac{1}{3}} m^{-\frac{1}{2}}$$

and split each of the integrals (2.3.1) – (2.3.2) into the sum of integrals over two regions in $\mathbb{R}^m$: the inner region where $\|t\| \leq \rho$ and the outer region where $\|t\| > \rho$. We then prove that the integrals over the outer region converge absolutely, which establishes (2.3.1). To establish (2.3.2), we show that the absolute value of the integral over the inner region is bigger than the absolute value of the integral over the outer region.

First, we estimate the integral over the inner region.

**Lemma.** Let

$$\rho = \frac{1}{2} n^{-\frac{1}{3}} m^{-\frac{1}{2}}.$$  

Then

$$\left| \int_{t \in \mathbb{R}^m: \|t\| \leq \rho} \det^{-\frac{1}{2}} \left( I - \sqrt{-1} \sum_{i=1}^{m} \tau_i Q_i \right) \exp \left\{ -\sqrt{-1} \sum_{i=1}^{m} \alpha_i \tau_i \right\} dt \right| \geq \frac{4}{5} (4nm)^{-m/2} \nu_m,$$

where $\nu_m$ is the volume of the unit ball in $\mathbb{R}^m$.

**Proof.** Since $\rho < m^{-1/2}$, by (3.1.1) for all $t \in \mathbb{R}^m$ such that $\|t\| \leq \rho$ we have

$$|\lambda_j(t)| < 1 \quad \text{for} \quad j = 1, \ldots, n$$

and hence

$$\det^{-\frac{1}{2}} \left( I - \sqrt{-1} \sum_{i=1}^{m} \tau_i Q_i \right) \exp \left\{ -\sqrt{-1} \sum_{i=1}^{m} \alpha_i \tau_i \right\}$$

$$= \left( \prod_{j=1}^{n} \left( 1 - \sqrt{-1} \lambda_j(t) \right) \right)^{-1/2} \exp \left\{ -\sqrt{-1} \sum_{i=1}^{m} \alpha_i \tau_i \right\}$$

$$= \exp \left\{ -\frac{1}{2} \sum_{j=1}^{n} \ln \left( 1 - \sqrt{-1} \lambda_j(t) \right) - \sqrt{-1} \sum_{i=1}^{m} \alpha_i \tau_i \right\}$$

$$= \exp \left\{ \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{\infty} \frac{(-1)^k \lambda_j^k(t)}{k} - \sqrt{-1} \sum_{i=1}^{m} \alpha_i \tau_i \right\}.$$
Next, we observe that
\[
\frac{1}{2} \sum_{j=1}^{n} \lambda_j(t) = \frac{1}{2} \text{trace} \left( \sum_{i=1}^{m} \tau_i Q_i \right) = \sum_{i=1}^{m} \alpha_i \tau_i
\]
and that
\[
-\frac{1}{4} \sum_{j=1}^{n} \lambda_j^2(t) = -\frac{1}{4} \psi(t).
\]
Hence the integral in question can be written as
\[
(3.2.1) \quad \int_{t \in \mathbb{R}^m: \|t\| \leq \rho} \exp \left\{ -\frac{1}{4} \psi(t) + \frac{1}{2} \sum_{k=3}^{\infty} \frac{1}{k} \sum_{j=1}^{n} (\sqrt{-1})^k \lambda_j^k(t) \right\} dt.
\]
Next, using (3.1.1), for \( \|t\| \leq \rho \), we bound
\[
\left| \frac{1}{2} \sum_{k=3}^{\infty} \frac{1}{k} \sum_{j=1}^{n} (\sqrt{-1})^k \lambda_j^k(t) \right| \leq \frac{n}{6} \sum_{k=3}^{\infty} (\rho \sqrt{m})^k = \frac{n(\rho \sqrt{m})^3}{6(1 - \rho \sqrt{m})} \leq \frac{1}{24}.
\]
Consequently, the absolute value of the difference of (3.2.1) and the integral
\[
\int_{t \in \mathbb{R}^m: \|t\| \leq \rho} \exp \left\{ -\frac{1}{4} \psi(t) \right\} dt
\]
does not exceed
\[
\left( \exp \left\{ \frac{1}{24} \right\} - 1 \right) \int_{t \in \mathbb{R}^m: \|t\| \leq \rho} \exp \left\{ -\frac{1}{4} \psi(t) \right\} dt
\]
\[
\leq \frac{1}{20} \int_{t \in \mathbb{R}^m: \|t\| \leq \rho} \exp \left\{ -\frac{1}{4} \psi(t) \right\} dt
\]
and hence the absolute value of (3.2.1) is at least
\[
\frac{19}{20} \int_{t \in \mathbb{R}^m: \|t\| \leq \rho} \exp \left\{ -\frac{1}{4} \psi(t) \right\} dt.
\]
On the other hand, by (3.1.2), we have
\[
\int_{t \in \mathbb{R}^m: \|t\| \leq \rho} \exp \left\{ -\frac{1}{4} \psi(t) \right\} dt \geq \int_{t \in \mathbb{R}^m: \|t\| \leq \frac{1}{2} n^{-\frac{1}{2}} m^{-\frac{1}{2}}} \exp \left\{ -\frac{1}{4} n m \|t\|^2 \right\} dt
\]
\[
\geq \exp \left\{ -\frac{1}{16} \right\} 2^{-m} (nm)^{-m/2} \nu_m,
\]
which concludes the proof.

Next, we bound the integral over the outer region.
(3.3) Lemma. Suppose that
\[ \psi(t) \geq \mu \|t\|^2 \] where \( \mu > 2m^2 + m \).

and that
\[ \rho = \frac{1}{2} n^{-\frac{1}{2}} m^{-\frac{1}{2}}. \]

Then
\[
\int \left| \det \left( I - \sqrt{-1} \sum_{i=1}^{m} \tau_i Q_i \right) \right| \, dt
\leq \frac{2m \omega_m}{m^{m/2} (\mu - 2m^2 - m)} \left( 1 + \frac{1}{4n^2/3} \right)^{2m^2 + m - \mu},
\]
where \( \omega_m \) is the surface area of the unit sphere in \( \mathbb{R}^m \).

Proof. For \( t = (\tau_1, \ldots, \tau_m) \), we have
\[
\left| \det \left( I - \sqrt{-1} \sum_{i=1}^{m} \tau_i Q_i \right) \right| = \prod_{j=1}^{n} \left( 1 + \lambda_j^2(t) \right)^{-\frac{1}{2}}.
\]

From (3.1.1), we have
\[ \lambda_j^2(t) \leq m \|t\|^2 \] for \( j = 1, \ldots, n \),
while also
\[ \sum_{j=1}^{n} \lambda_j^2(t) = \psi(t) \geq \mu \|t\|^2. \]

Let us denote
\[ \xi_j = \xi_j(t) = \lambda_j^2(t) \] for \( j = 1, \ldots, n \).

Since the minimum value of the log-concave function
\[ \prod_{j=1}^{n} \left( 1 + \xi_j \right) \] where \( \xi_j \geq 0 \) for \( j = 1, \ldots, n \)

on the polyhedron defined by the inequalities
\[ 0 \leq \xi_j \leq m \|t\|^2 \] for \( j = 1, \ldots, n \) and \( \sum_{j=1}^{n} \xi_j \geq \mu \|t\|^2 \)

14
is attained at a vertex, in which case \( \xi_j \in \{0, m\|t\|^2\} \) for all but possibly one value of \( j \), the value of (3.3.1) on the polyhedron defined by (3.3.2) is at least

\[
(1 + m\|t\|^2)^{\frac{\mu - m}{m}}.
\]

Therefore,

\[
\left| \prod_{j=1}^{n} (1 + \lambda_j^2(t)) \right|^{-\frac{1}{4}} \leq (1 + m\|t\|^2)^{\frac{m - \mu}{4m}}
\]

and the absolute value of the integral in question does not exceed (we write the integral in polar coordinates)

\[
\omega_m \int_{\rho}^{+\infty} (1 + m\tau^2)^{\frac{m - \mu}{4m}} \tau^{m-1} d\tau
\]

\[
\leq \frac{\omega_m}{2m} \int_{\rho}^{+\infty} (1 + m\tau^2)^{\frac{m - \mu}{4m}} \left( 1 + m\tau^2 \right)^{\frac{m - \mu}{m}} (2m\tau) d\tau
\]

\[
= \frac{2m\omega_m}{m^{m/2}(\mu - 2m^2 - m)} \left( 1 + m\rho^2 \right)^{\frac{2m^2 + m - \mu}{4m}}
\]

\[
= \frac{2m\omega_m}{m^{m/2}(\mu - 2m^2 - m)} \left( 1 + 1^{2/3} \right)^{\frac{2m^2 + m - \mu}{4m}}
\]

\[
\square
\]

(3.4) **Proof of Theorem 1.3.** We use Corollary 2.3. From Lemma 3.3 it follows that (2.3.1) holds. To prove (2.3.2), we compare the bounds of Lemma 3.2 and Lemma 3.3. Using that \( m\nu_m = \omega_m \), the bound for \( \mu \) and the estimate

\[
\ln \left( 1 + \frac{1}{4n^{2/3}} \right) > \frac{1}{5n^{2/3}},
\]

we conclude that

\[
\frac{2m\omega_m}{m^{m/2}(\mu - 2m^2 - m)} \left( 1 + \frac{1}{4n^{2/3}} \right)^{\frac{2m^2 + m - \mu}{4m}}
\]

\[
\leq \frac{2\nu_m m^2}{10m^{m/2}2m^{2/3} \ln(4n)} \exp \left\{ \frac{2m^2 + m - \mu}{4m} \ln \left( 1 + \frac{1}{4n^{2/3}} \right) \right\}
\]

\[
< \frac{2\nu_m}{10m^{m/2}2m^{2/3} \ln(4n)} \exp \left\{ -\frac{10n^{2/3}m^2 \ln(4n)}{4m} \frac{1}{5n^{2/3}} \right\}
\]

\[
= \frac{\nu_m}{5m^{m/2}2m^{2/3} \ln(4n)} \exp \left\{ -\frac{1}{2} m \ln(4n) \right\}
\]

\[
= \frac{1}{5} \nu_m (4nm)^{-m/2} \frac{1}{n^{2/3} \ln(4n)} < \frac{4}{5} \nu_m (4nm)^{-m/2}.
\]
Therefore,

\[ \left| \int_{t \in \mathbb{R}^m: \|t\| > \rho} \det \left( I - \sqrt{-1} \sum_{i=1}^{m} \tau_i Q_i \right) \exp \left\{ -\sqrt{-1} \sum_{i=1}^{m} \alpha_i \tau_i \right\} dt \right| \]

\[ < \left| \int_{t \in \mathbb{R}^m: \|t\| \leq \rho} \det \left( I - \sqrt{-1} \sum_{i=1}^{m} \tau_i Q_i \right) \exp \left\{ -\sqrt{-1} \sum_{i=1}^{m} \alpha_i \tau_i \right\} dt \right|, \]

and (2.3.2) holds. We use Corollary 2.3 to complete the proof. \( \square \)

(3.5) **Proof of Theorem 1.5.** We proceed as above, except we use Corollary 2.4 instead of Corollary 2.3. \( \square \)

4. **Open Questions**

(4.1) **Optimal bounds.** It is not clear whether in Theorems 1.3 and 1.5 the lower bound for \( \mu \) is anywhere close to optimal. As we discuss in Section 1.4, if the \( n \times n \) matrices \( Q_1, \ldots, Q_m \) are sufficiently generic, as \( n \) grows relative to \( m \), the threshold for \( \mu \) is achieved when \( n = m^{O(1)} \). It would be interesting to find out if the bound in Theorems 1.3 and 1.5 can be significantly improved, so that we can have \( n \sim m^2 \) or even \( n \sim m \) for generic systems to become feasible.

(4.2) **Finding a solution.** The conditions of Theorem 1.3 and 1.5 can be verified efficiently, in polynomial time. Once the conditions are satisfied, the systems (1.1.1) and (1.1.2) are guaranteed to have solutions. However, neither Theorem 1.3 nor Theorem 1.5 give us an efficient algorithm to construct those solutions. Here the situation appears to be very different from other problems involving positive semi-definite relaxations, where (approximate) solutions can be found via randomized rounding, see [Pa13] for a survey.

**References**


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