A REMARK ON APPROXIMATING PERMANENTS OF POSITIVE DEFINITE MATRICES

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May 12, 2020

ABSTRACT. Let $A$ be an $n \times n$ positive definite Hermitian matrix with all eigenvalues between 1 and 2. We represent the permanent of $A$ as the integral of some explicit log-concave function on $\mathbb{R}^{2n}$. Consequently, there is a fully polynomial randomized approximation scheme (FPRAS) for per $A$.

1. Introduction and main results

Let $A = (a_{ij})$ be an $n \times n$ complex matrix. The *permanent* of $A$ is defined as

$$
\text{per } A = \sum_{\sigma \in S_n} \prod_{k=1}^{n} a_{k\sigma(k)},
$$

where $S_n$ is the symmetric group of all $n!$ permutations of the set $\{1, \ldots, n\}$. Recently, there was some interest in efficient computing (approximating) per $A$, when $A$ is a positive definite Hermitian matrix (as is known, in that case per $A$ is real and non-negative), see [A+17] and reference therein. In particular, Anari et al. construct in [A+17] a deterministic algorithm approximating the permanent of a positive semidefinite $n \times n$ Hermitian matrix $A$ within a multiplicative factor of $c^n$ for $c = e^{1+\gamma} \approx 4.84$, where $\gamma \approx 0.577$ is the Euler constant.

In this note, we show that there is a fully polynomially randomized approximation scheme (FPRAS) for permanents of positive definite matrices with the eigenvalues between 1 and 2. Namely, we represent per $A$ for such a matrix $A$ as an integral of an explicitly constructed log-concave function $f_A : \mathbb{R}^{2n} \to \mathbb{R}$, so that a Markov Chain Monte Carlo algorithm can be applied to efficiently approximate

$$
\int_{\mathbb{R}^{2n}} f_A(x) \, dx = \text{per } A,
$$

1991 Mathematics Subject Classification. 15A15, 15A57, 68W20, 60J22, 26B25.

Key words and phrases. permanent, positive definite matrices, log-concave measures.

This research was partially supported by NSF Grant DMS 1855428.
see [LV07].

We consider the space $C^n$ with the standard norm

$$\|z\|^2 = |z_1|^2 + \ldots + |z_n|^2, \quad \text{where} \quad z = (z_1, \ldots, z_n).$$

We identify $C^n = \mathbb{R}^{2n}$ by identifying $z = x + iy$ with $(x, y)$. For a complex matrix $L = (l_{jk})$, we denote by $L^* = \left(l^*_{jk}\right)$ its conjugate, so that

$$l^*_{jk} = \overline{l_{kj}} \quad \text{for all} \quad j, k.$$

We prove the following main result.

(1.1) Theorem. Let $A$ be an $n \times n$ positive definite matrix with all eigenvalues between 1 and 2. Let us write $A = I + B$, where $I$ is the $n \times n$ identity matrix and $B$ is an $n \times n$ positive semidefinite Hermitian matrix with eigenvalues between 0 and 1. Further, we write $B = LL^*$, where $L = (l_{jk})$ is an $n \times n$ complex matrix. We define linear functions $\ell_1, \ldots, \ell_n : C^n \rightarrow \mathbb{C}$ by

$$\ell_j(z) = \sum_{k=1}^{n} l_{jk} z_k \quad \text{for} \quad z = (z_1, \ldots, z_n).$$

Let us define $f_A : C^n \rightarrow \mathbb{R}_+$ by

$$f_A(z) = \frac{1}{\pi^n} e^{-\|z\|^2} \prod_{j=1}^{n} \left(1 + |\ell_j(z)|^2\right).$$

(1) Identifying $C^n = \mathbb{R}^{2n}$, we have

$$\text{per } A = \int_{\mathbb{R}^{2n}} f_A(x, y) \, dxdy.$$

(2) The function $f_A : \mathbb{R}^{2n} \rightarrow \mathbb{R}_+$ is log-concave, that is,

$$f_A(\alpha x_1 + (1 - \alpha) x_2) \geq f_A^\alpha(x_1) f_A^{1-\alpha}(x_2)$$

for any $x_1, x_2 \in \mathbb{R}^{2n}$ and any $0 \leq \alpha \leq 1$.

2. Proofs

We start with a known integral representation of the permanent of a positive semidefinite matrix.
(2.1) The integral formula. Let $\mu$ be the Gaussian probability measure in $\mathbb{C}^n$ with density
\[ \frac{1}{\pi^n} e^{-\|z\|^2} \] where $\|z\|^2 = |z_1|^2 + \ldots + |z_n|^2$ for $z = (z_1, \ldots, z_n)$.

Let $\ell_1, \ldots, \ell_n : \mathbb{C}^n \rightarrow \mathbb{C}$ be linear functions and let $B = (b_{jk})$ be the $n \times n$ matrix,
\[ b_{jk} = \mathbf{E} \ell_j \ell_k = \int_{\mathbb{C}^n} \ell_j(z) \ell_k(z) \, d\mu(z) \quad \text{for} \quad j, k = 1, \ldots, n. \]

Hence $B$ is a Hermitian positive semidefinite matrix and the Wick formula (see, for example, Section 3.1.4 of [Ba16]) implies that
\[ \text{per } B = \mathbf{E} \left( |\ell_1|^2 \cdots |\ell_n|^2 \right) = \int_{\mathbb{C}^n} |\ell_1(z)|^2 \cdots |\ell_n(z)|^2 \, d\mu(z). \]

Next, we need a simple lemma.

(2.2) Lemma. Let $q : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a positive semidefinite quadratic form. Then the function
\[ h(x) = \ln(1 + q(x)) - q(x) \]
is concave.

Proof. It suffices to check that the restriction of $h$ onto any affine line $x(\tau) = \tau a + b$ with $a, b \in \mathbb{R}^n$ is concave. Thus we need to check that the univariate function
\[ G(\tau) = \ln(1 + (\alpha \tau + \beta)^2 + \gamma^2) - (\alpha \tau + \beta)^2 - \gamma^2 \quad \text{for} \quad \tau \in \mathbb{R}, \]
where $\alpha \neq 0$, is concave, for which it suffices to check that $G''(\tau) \leq 0$ for all $\tau$. Via the affine substitution $\tau := (\tau - \beta)/\alpha$, it suffices to check that $g''(\tau) \leq 0$, where
\[ g(\tau) = \ln \left( 1 + \tau^2 + \gamma^2 \right) - \left( \tau^2 + \gamma^2 \right). \]
We have
\[ g'(\tau) = \frac{2\tau}{1 + \tau^2 + \gamma^2} - 2 \tau \]
and
\[ g''(\tau) = \frac{2(1 + \tau^2 + \gamma^2) - 4\tau^2}{(1 + \tau^2 + \gamma^2)^2} - 2 = \frac{2(1 + \tau^2 + \gamma^2) - 4\tau^2 - 2(1 + \tau^2 + \gamma^2)^2}{(1 + \tau^2 + \gamma^2)^2} = \frac{2 + 2\tau^2 + 2\gamma^2 - 4\tau^2 - 2 - 2\gamma^4 - 4\tau^4 - 4\gamma^2 - 4\tau^2\gamma^2}{(1 + \tau^2 + \gamma^2)^2} = - \frac{6\tau^2 + 2\gamma^2 + 2\tau^4 + 2\gamma^4 + 4\tau^2\gamma^2}{(1 + \tau^2 + \gamma^2)^2} \leq 0 \]
and the proof follows. \qed
(2.3) Proof of Theorem 1.1. We have

\[ \text{per } A = \text{per}(I + B) = \sum_{J \subset \{1, \ldots, n\}} \text{per } B_J, \]

where \( B_J \) is the principal \(|J| \times |J|\) submatrix of \( B \) with row and column indices in \( J \) and where we agree that \( \text{per } B_\emptyset = 1 \). Let us consider the Gaussian probability measure in \( \mathbb{C}^n \) with density \( \pi^{-n} e^{-\|z\|^2} \). By (2.1.1), we have

\[ \text{per } B_J = E \prod_{j \in J} |\ell_j(z)|^2 \]

and hence

\[ \text{per } A = E \prod_{j=1}^n (1 + |\ell_j(z)|^2) = \int_{\mathbb{R}^{2n}} f_A(x, y) \, dxdy, \]

and the proof of Part (1) follows.

We write

\[ e^{-\|z\|^2} \prod_{j=1}^n (1 + |\ell_j(z)|^2) = e^{-q(z)} \prod_{j=1}^n (1 + |\ell_j(z)|^2) e^{-|\ell_j(z)|^2}, \]

where \( q(z) = \|z\|^2 - \sum_{j=1}^n |\ell_j(z)|^2. \)

By Lemma 2.2 each function \((1 + |\ell_j(z)|^2)e^{-|\ell_j(z)|^2}\) is log-concave on \( \mathbb{R}^{2n} = \mathbb{C}^n \) and hence to complete the proof of Part (2) it suffices to show that \( q \) is a positive semidefinite Hermitian form. To this end, we consider the Hermitian form

\[ p(z) = \sum_{j=1}^n |\ell_j(z)|^2 = \sum_{j=1}^n \left| \sum_{k=1}^n l_{jk} z_k \right|^2 = \sum_{j=1}^n \sum_{1 \leq k_1, k_2 \leq n} l_{jk_1} \overline{l_{jk_2}} z_{k_1} \overline{z_{k_2}}, \]

where

\[ c_{k_1 k_2} = \sum_{j=1}^n l_{jk_1} \overline{l_{jk_2}} \quad \text{for} \quad 1 \leq k_1, k_2 \leq n. \]

Hence for the matrix \( C = (c_{k_1 k_2}) \) of \( p \), we have \( C = LL^* \). We note that \( B = LL^* \) and that the eigenvalues of \( B \) lie between 0 and 1. Therefore, the eigenvalues of \( L^*L \) lie between 0 and 1 (in the generic case, when \( L \) is invertible, the matrices \( LL^* \) and \( L^*L \) are similar). Consequently, the eigenvalues of \( C \) lie between 0 and 1 and hence the Hermitian form \( q(z) \) with matrix \( I - C \) is positive semidefinite, which completes the proof of Part (2). \( \square \)
REFERENCES


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