Metric spaces: definitions and examples

If \( X \) is a set, then a function \( d : X \times X \to [0, \infty) \) is a metric if

(M1) \( d(x, y) = 0 \) if and only if \( x = y \),
(M2) \( d(x, y) = d(y, x) \) for all \( x, y \in X \), and
(M3) \( d(x, y) + d(y, z) \geq d(x, z) \) for all \( x, y, z \in X \).

\( \mathbb{R}^n \) admits the metrics

\[
d_2(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}
\]

and

\[
d_\infty(\vec{x}, \vec{y}) = \max\{|x_i - y_i| \mid i = 1, \ldots, n\}.
\]

\( d_2 \) is called the Euclidean metric.

If \( X \) is a set, then the discrete metric \( d : X \times X \to [0, \infty) \) is given by \( d(x, y) = 0 \) if \( x = y \) and \( d(x, y) = 1 \) if \( x \neq y \).

If \( (X, d) \) is a metric space and \( A \subset X \), then \( d|_{A \times A} \) is the subspace metric on \( A \).

The space \( C([a, b], \mathbb{R}) \) of all continuous functions \( f : [a, b] \to \mathbb{R} \) admits the metrics \( d_1 \) and \( d_\infty \) where

\[
d_1(f, g) = \int_a^b |f(x) - g(x)| \, dx
\]

and

\[
d_\infty(f, g) = \max\{|f(x) - g(x)| \mid x \in [a, b]\}.
\]

Open sets and continuity

If \( (X, d) \) is a metric space, \( x \in X \) and \( \epsilon > 0 \), then

\[
B(x, \epsilon) = \{ y \in X \mid d(x, y) < \epsilon \}.
\]

A subset \( U \) of \( X \) is open if for all \( x \in U \), there exists \( \epsilon_x > 0 \) such that \( B(x, \epsilon_x) \subset U \).

Lemma: If \( \epsilon > 0 \) and \( x \in X \), then \( B(x, \epsilon) \) is an open subset of \( X \).
Proposition: If $(X, d)$ is a metric space, then
(i) The emptyset $\emptyset$ and $X$ are open.
(ii) If $\{U_\alpha\}_{\alpha \in \Lambda}$ is a collection of open sets, then $\bigcup_{\alpha \in \Lambda} U_\alpha$ is an open set.
(iii) If $\{U_1, \ldots, U_n\}$ is a finite collection of open sets, then $\bigcap_{i=1}^n U_i$ is open.

Proposition: If $(X, d)$ is a metric space and $x$ and $y$ are distinct points in $X$, then there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $y \in V$, i.e. metric spaces are Hausdorff.

A function between metric spaces $f : (X_1, d_1) \rightarrow (X_2, d_2)$ is continuous at $x \in X_1$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $y \in X_1$ and $d_1(x, y) < \delta$, then $d_2(f(x), f(y)) < \epsilon$. $f$ is continuous if it is continuous at every point in $X_1$.

Theorem: A function $f : (X_1, d_1) \rightarrow (X_2, d_2)$ is continuous if and only if whenever $U$ is an open subset of $X_2$, then $f^{-1}(U)$ is open in $X_1$.

Closed sets and continuity

A subset $C$ of a metric space $(X, d)$ is said to be closed in $X$ if its complement $X \setminus C$ is an open set in $X$.

Proposition: If $(X, d)$ is a metric space, then
(i) The emptyset $\emptyset$ and $X$ are closed.
(ii) If $\{C_\alpha\}_{\alpha \in \Lambda}$ is a collection of closed sets, then $\bigcap_{\alpha \in \Lambda} C_\alpha$ is a closed set.
(iii) If $\{C_1, \ldots, C_n\}$ is a finite collection of closed sets, then $\bigcup_{i=1}^n C_i$ is closed.

Lemma: If $(X, d)$ is a metric space, $x \in X$ and $\epsilon > 0$, then the closed disk $D(x, \epsilon) = \{y \in X \mid d(y, x) \leq \epsilon\}$ is a closed subset of $X$.

Theorem: A function $f : (X_1, d_1) \rightarrow (X_2, d_2)$ is continuous if and only if whenever $C$ is a closed subset of $X_2$, then $f^{-1}(C)$ is closed in $X_1$.

Sequences in metric spaces:

A sequence $\{x_n\}$ in a metric space $(X, d)$ is said to be convergent with limit $x \in X$ if for all $\epsilon > 0$, there exists $N$ such that if $n \geq N$, then $d(x_n, x) < \epsilon$. We write $\lim x_n = x$.

Theorem: Suppose that $f : X \rightarrow Y$ is a function between metric spaces. Then $f$ is continuous at $x$ if and only if whenever $\{x_n\}$ is a sequence in $X$ such that $\lim x_n = x$, then $\lim f(x_n) = f(\lim x_n)$.

Lemma: A convergent sequence $\{x_n\}$ in a metric space $X$ is bounded (i.e. there exists $x_0 \in X$ and $R > 0$ so that $x_n \in D(x_0, R)$ for all $n$).
A sequence \( \{x_n\} \) in a metric space \( X \) is said to be a **Cauchy sequence** if for all \( \epsilon > 0 \) there exists \( N \) such that if \( n, m \geq N \), then \( d(x_n, x_m) < \epsilon \).

**Lemma:** Any convergent sequence in a metric space is a Cauchy sequence.

**Closure and interior:**

If \( x \in X \), then an open set \( U \) in \( X \) is an open neighborhood of \( x \) if it contains \( x \). If \( A \) is a subset of a metric space \( X \), then \( x \in \bar{A} \) if and only if whenever \( U \) is an open neighborhood of \( x \), \( U \cap A \) is non-empty. Moreover, \( a \in A^0 \) if and only if there exists an open neighborhood \( U \) of \( a \) which is contained in \( A \). \( \bar{A} \) is called the **closure** of \( A \) and \( A^0 \) is called the **interior** of \( A \).

**Lemma:** If \( A \) is a subset of a metric space \( X \), then

1. \( A^0 \subset A \subset \bar{A} \).
2. \( A^0 \) is open and \( \bar{A} \) is closed in \( X \).
3. If \( C \) is a closed subset of \( X \) and \( A \subset C \), then \( A \subset C \).
4. \( x \in \bar{A} \) if and only there exists a convergent sequence \( \{x_n\} \subset A \) such that \( x = \lim x_n \).
5. \( A \) is closed if and only if \( A = \bar{A} \).
6. \( A \) is open if and only \( A = A^0 \).

**Useful Lemma:** A subset \( V \) of a metric space \((X, d)\) is open in \( X \) if and only if for all \( x \in V \), there exists an open neighborhood \( U_x \) of \( x \) which is contained in \( V \), i.e. \( x \in U_x \subset V \).

**Proposition:** A subset \( C \) of a metric space \((X, d)\) is closed if and only if whenever \( \{x_n\} \) is a convergent sequence such that \( \{x_n\} \subset C \), then \( \lim x_n \in C \).

**Sequential compactness:**

A subset \( C \) of a metric space \( X \) is **sequentially compact** if any sequence \( \{x_n\} \) in \( C \) has a convergent subsequence \( \{x_{j_n}\} \) with limit in \( C \), i.e. \( \lim x_{j_n} \in C \).

A subset \( A \) of a metric space \( X \) is **bounded** if there exists \( x_0 \in X \) and \( R > 0 \) such that \( A \subset D(x_0, R) \).

**Proposition:** A sequentially compact subset of a metric space is closed and bounded.

**Proposition:** If \( C \) is a closed subset of a sequentially compact metric space \( X \), then \( C \) is closed in \( X \).

**Proposition:** If \( f : X \to Y \) is a continuous map between metric spaces and \( C \subset X \) is sequentially compact, then \( f(C) \) is sequentially compact.
**Theorem:** If $f : X \to \mathbb{R}$ is continuous and $C \subset X$ is sequentially compact, then there exists $c \in C$ such that $f(c) = \sup f(C)$, i.e. $f$ achieves its supremum on $C$.

**Theorem:** A subset of $\mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded.

**Lemma:** A Cauchy sequence $\{x_n\}$ in a sequentially compact metric space is convergent.