

# COUNTING, EQUIDISTRIBUTION AND ENTROPY GAPS AT INFINITY WITH APPLICATIONS TO CUSPED HITCHIN REPRESENTATIONS

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ABSTRACT. We show that if an eventually positive, non-arithmetic, locally Hölder continuous potential for a topologically mixing countable Markov shift with (BIP) has an entropy gap at infinity, then one may apply the renewal theorem of Kesseböhmer and Kombrink to obtain counting and equidistribution results. We apply these general results to obtain counting and equidistribution results for cusped Hitchin representations, and more generally for cusped Anosov representations of geometrically finite Fuchsian groups.

## 1. INTRODUCTION

In this paper, we use the Renewal Theorem of Kesseböhmer and Kombrink [31] to establish counting and equidistribution results for well-behaved potentials on topologically mixing countable Markov shifts with (BIP) in the spirit of Lalley's work [34] on finite Markov shifts. Inspired by work of Schapira-Tapie [62, 63], Dal'bo-Otal-Peigné [17], Iommi-Riquelme-Veloza [25] and Veloza [67] in the setting of geodesic flows on negatively curved Riemannian manifolds, we define notions of entropy gap at infinity for our potentials. Our results require that the potentials are non-arithmetic, eventually positive and have an entropy gap at infinity.

Our main motivation for this general analysis was provided by cusped Hitchin representations of a geometrically finite Fuchsian group into  $\mathrm{SL}(d, \mathbb{R})$ . Given a linear functional  $\phi$  on the Cartan algebra  $\mathfrak{a}$  of  $\mathrm{SL}(d, \mathbb{R})$  which is a positive linear combination of simple roots, we can define the  $\phi$ -translation length  $\ell^\phi(A) = \phi(\ell(A))$  (where  $\ell$  is the Jordan projection) for  $A \in \mathrm{SL}(d, \mathbb{R})$ . The first consequence of the general theory we develop is that if  $\rho$  is cusped Hitchin, then

$$\#\{[\gamma] \in [\Gamma] \mid 0 < \ell^\phi(\rho(\gamma)) \leq t\} \sim \frac{e^{t\delta}}{t\delta}$$

where  $\delta = \delta_\phi(\rho)$  is the  $\phi$ -entropy of  $\rho$  (and  $[\Gamma]$  is the collection of conjugacy classes of elements of  $\Gamma$ .) We also obtain a Manhattan curve theorem and equidistribution results in this context. In later work, we plan to use these results to construct pressure metrics on cusped Hitchin components. A longer term goal is the development of a geometric theory of the augmented Hitchin component which parallels the study of the augmented Teichmüller space as the metric completion of Teichmüller space with the Weil-Petersson metric (see Masur [42]).

**General Thermodynamical results:** We now give more precise statements of our general results. We assume throughout that  $(\Sigma^+, \sigma)$  is a topologically mixing, one-sided, countable Markov shift with alphabet  $\mathcal{A}$  which has (BIP). Moreover, all of our functions will be assumed to be locally Hölder continuous (see Section 2 for precise definitions).

We now introduce the crucial assumptions we will make in our work. Given a locally Hölder continuous function  $f : \Sigma^+ \rightarrow \mathbb{R}$  and  $a \in \mathcal{A}$ , we let

$$I(f, a) = \inf \{f(x) \mid x \in \Sigma^+, x_1 = a\} \quad \text{and} \quad S(f, a) = \sup \{f(x) \mid x \in \Sigma^+, x_1 = a\}.$$

Note that  $I(f, a)$  and  $S(f, a)$  are finite since  $f$  is locally Hölder continuous.

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We say that  $f$  has a *strong entropy gap at infinity* if the series

$$Z_1(f, s) = \sum_{a \in \mathcal{A}} e^{-sS(f, a)}$$

has a finite critical exponent  $d(f) > 0$  and diverges when  $s = d(f)$ .

We say that  $f$  has a *weak entropy gap at infinity* if  $Z_1(f, s)$  has a finite critical exponent  $d(f) > 0$  and there exists  $\delta = \delta(f) > d(f) > 0$  so that  $P(-\delta f) = 0$  where  $P$  is the Gurevich pressure function associated to  $(\Sigma^+, \sigma)$  (defined in Section 2). We will see later (in Section 3), that a strong entropy gap at infinity implies a weak entropy gap at infinity.

We say that  $f$  is *strictly positive* if  $c(f) = \inf\{f(x) \mid x \in \Sigma^+\} > 0$ . We say that  $f$  is *eventually positive* if there exist  $N \in \mathbb{N}$  and  $B > 0$  so that

$$S_n f(x) = f(x) + f(\sigma(x)) + \cdots + f(\sigma^{n-1}(x)) > B$$

for all  $n \geq N$  and  $x \in \Sigma^+$ . Recall that  $f$  is *arithmetic* if the subgroup of  $\mathbb{R}$  generated by  $\{S_n f(x) \mid x \in \text{Fix}^n, n \in \mathbb{N}\}$  is cyclic, where  $x \in \text{Fix}^n$  if  $\sigma^n(x) = x$ .

We begin by stating our general counting results. For all  $n \in \mathbb{N}$ , let

$$\mathcal{M}_f(n, t) = \{x \in \Sigma^+ : x \in \text{Fix}^n \text{ and } S_n f(x) \leq t\} \quad \text{and let} \quad M_f(t) = \sum_{n=1}^{\infty} \frac{1}{n} \#\mathcal{M}_f(n, t).$$

**Theorem A** (Growth rate of closed orbits). *Suppose that  $(\Sigma^+, \sigma)$  is a topologically mixing, one-sided, countable Markov shift which has (BIP). If  $f : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous, non-arithmetic, eventually positive and has a weak entropy gap at infinity, and  $P(-\delta f) = 0$ , then*

$$\lim_{t \rightarrow \infty} M_f(t) \frac{t\delta}{e^{t\delta}} = 1.$$

Similarly, for all  $k \in \mathbb{N}$ , let

$$\mathcal{R}_f(k, t) = \{x \in \mathcal{M}_f(k, t) \mid x \notin \mathcal{M}_f(n, t) \text{ if } n < k\} \quad \text{and let} \quad R_f(t) = \sum_{k=1}^{\infty} \frac{1}{k} \#\mathcal{R}_f(k, t).$$

If  $x \in \mathcal{M}_f(n, t) - \mathcal{R}_f(n, t)$ , then there exists  $j \geq 2$  so that  $x \in \mathcal{M}_f(\frac{n}{j}, \frac{t}{j})$ , so

$$M_f(t) - M_f\left(\frac{t}{2}\right) \leq R_f(t) \leq M_f(t).$$

Therefore, the following result is an immediate corollary of Theorem A.

**Corollary 1.1** (Growth rate of closed prime orbits). *Suppose that  $(\Sigma^+, \sigma)$  is a topologically mixing, one-sided, countable Markov shift which has (BIP). If  $f : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous, non-arithmetic, eventually positive and has a weak entropy gap at infinity, and  $P(-\delta f) = 0$ , then*

$$\lim_{t \rightarrow \infty} R_f(t) \frac{t\delta}{e^{t\delta}} = 1.$$

Another immediate corollary of our counting result is Bowen's formula for the critical exponent. Let  $\Sigma_f$  be the suspension flow of  $f$  and let  $\mathcal{O}_f$  be the collection of closed orbits of  $\Sigma_f$

$$\mathcal{O}_f(t) = \{\lambda \mid \ell_f(\lambda) \leq t\}$$

where  $\ell_f(\lambda)$  is the period of  $\lambda$ . Notice that  $\#\mathcal{O}_f(t) = M_f(t)$ , since if  $\lambda \in \mathcal{O}_f(t)$ , then there exists  $x \in \text{Fix}^n$  for some  $n$ , so that  $S_n f(x) = \ell_f(\lambda)$  and  $x$  is well-defined up to cyclic permutation.

**Corollary 1.2** (Bowen's formula). *Suppose that  $(\Sigma^+, \sigma)$  is a topologically mixing, one-sided, countable Markov shift which has (BIP). If  $f : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous, non-arithmetic, eventually positive, has a weak entropy gap at infinity and  $P(-\delta f) = 0$ , then*

$$\delta = \lim_{t \rightarrow \infty} \frac{1}{t} \log \#\mathcal{O}_f(t).$$

If  $f : \Sigma^+ \rightarrow \mathbb{R}$  and  $g : \Sigma^+ \rightarrow \mathbb{R}$  are two eventually positive locally Hölder continuous functions, then there is a natural identification of the set  $\mathcal{O}_f$  of closed orbits of  $\Sigma_f$  and the set  $\mathcal{O}_g$  of closed orbits of  $\Sigma_g$ . If  $f$  is strictly positive and has a weak entropy gap at infinity so that  $P(-\delta f) = 0$ , then the equilibrium state for  $-\delta f$  induces a measure of maximal entropy on the suspension flow on  $\Sigma_f$ . We obtain an equidistribution result for this equilibrium state which roughly says that it behaves like a Patterson-Sullivan measure.

In the following theorem, if  $\phi$  and  $\psi$  are real-valued functions, we say that

$$\phi \sim \psi \quad \text{if} \quad \lim_{t \rightarrow \infty} \frac{\phi(t)}{\psi(t)} = 1.$$

**Theorem B** (Equidistribution). *Suppose that  $(\Sigma^+, \sigma)$  is a topologically mixing, one-sided, countable Markov shift which has (BIP) and  $f : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous, non-arithmetic, eventually positive, has a weak entropy gap at infinity,  $P(-\delta f) = 0$  and  $\mu_{-\delta f}$  is the equilibrium state for  $-\delta f$ . If  $g : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous, eventually positive, and there exists  $C > 0$  such that*

$$|f(x) - g(x)| < C$$

for all  $x \in \Sigma^+$ , then

$$\sum_{\gamma \in \mathcal{O}_f(t)} \frac{l_g(\gamma)}{l_f(\gamma)} \sim \left( \frac{\int g \, d\mu_{-\delta f}}{\int f \, d\mu_{-\delta f}} \right) \cdot \frac{e^{t\delta}}{t\delta}$$

as  $t \rightarrow \infty$ . Equivalently,

$$\sum_{k=1}^{\infty} \frac{1}{k} \sum_{x \in \mathcal{M}_f(k,t)} \frac{S_k g(x)}{S_k f(x)} \sim \left( \frac{\int g \, d\mu_{-\delta f}}{\int f \, d\mu_{-\delta f}} \right) \cdot \frac{e^{t\delta}}{t\delta}$$

as  $t \rightarrow \infty$ .

We can obtain a completely analogous statement if we instead consider the set  $\mathcal{P}_f$  of primitive closed orbits of the suspension flow  $\Sigma_f$ .

Suppose that  $f : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous, eventually positive, and has a strong entropy gap at infinity and that  $g : \Sigma^+ \rightarrow \mathbb{R}$  is also eventually positive and locally Hölder continuous, and that there exists  $C > 0$  so that  $|f(x) - g(x)| < C$  for all  $x \in \Sigma^+$ . (Notice that this implies that  $d(f) = d(g)$ .) Inspired by Burger [12], we define, the *Manhattan curve*

$$\mathcal{C}(f, g) = \{(a, b) \in \mathbb{R}^2 \mid P(-af - bg) = 0 \quad a \geq 0, b \geq 0, a + b > 0\}.$$

The Manhattan curve has the following properties.

**Theorem C** (Manhattan curve). *Suppose that  $(\Sigma^+, \sigma)$  is a topologically mixing, one-sided countable Markov shift with (BIP),  $f : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous, eventually positive and has a strong entropy gap at infinity and that  $g : \Sigma^+ \rightarrow \mathbb{R}$  is also eventually positive and locally Hölder continuous. If there exists  $C > 0$  so that  $|f(x) - g(x)| < C$  for all  $x \in \Sigma^+$ , then*

- (1)  $(\delta(f), 0), (0, \delta(g)) \in \mathcal{C}(f, g)$ .
- (2) If  $a \geq 0, b \geq 0$ , and  $a + b > 0$ , then there exists a unique  $t > \frac{d(f)}{a+b}$  so that  $(ta, tb) \in \mathcal{C}(f, g)$ .
- (3)  $\mathcal{C}(f, g)$  is a closed subsegment of an analytic curve.
- (4)  $\mathcal{C}(f, g)$  is strictly convex, unless

$$S_n f(x) = \frac{\delta(g)}{\delta(f)} S_n g(x)$$

for all  $x \in \text{Fix}^n$  and  $n \in \mathbb{N}$ .

Moreover, the tangent line to  $\mathcal{C}(f, g)$  at  $(a, b) \in \mathcal{C}(f, g)$  has slope

$$s(a, b) = -\frac{\int_{\Sigma^+} g \, d\mu_{-af-bg}}{\int_{\Sigma^+} f \, d\mu_{-af-bg}}$$

where  $\mu_{-af-bg}$  is the equilibrium state of the function  $-af - bg$ .

**Applications to cusped Hitchin representations:** Let  $S = \mathbb{H}^2/\Gamma$  be a geometrically finite, hyperbolic surface, and let  $\Lambda(\Gamma) \subset \partial\mathbb{H}^2$  be the limit set of  $\Gamma \subset \mathbb{H}^2$ . Following Fock and Goncharov [20], a *cusped Hitchin representation* is a representation  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  such that if  $\gamma \in \Gamma$  is parabolic, then  $\rho(\gamma)$  is a unipotent element with a single Jordan block and there exists a  $\rho$ -equivariant positive map  $\xi_\rho : \Lambda(\Gamma) \rightarrow \mathcal{F}_d$ . If  $S$  is compact, cusped Hitchin representations are just the traditional Hitchin representations introduced by Hitchin [24] and further studied by Labourie [32], while if  $\Gamma$  is convex cocompact, they are the Hitchin representations studied by Labourie-McShane [33]. As these are covered by the traditional theory of Anosov representations, we will focus on the case where  $\Gamma$  is not convex cocompact. If  $d = 3$  and  $S$  has finite area, then a cusped Hitchin representation is simply the holonomy map of a finite area strictly convex projective structure on  $S$  (see Marquis [40]). More generally, if  $\rho : \Gamma \rightarrow \mathrm{SL}(3, \mathbb{R})$  acts geometrically finitely, in the sense of Crampon-Marquis [16, Def. 5.14], on a strictly convex domain with  $C^1$  boundary, then  $\rho$  is cusped Hitchin by [20, 1.3. Thm.].

Let

$$\mathfrak{a} = \{\vec{a} \in \mathbb{R}^d \mid a_1 + \cdots + a_d = 0\}$$

be the standard Cartan algebra for the Lie algebra  $\mathfrak{sl}(d, \mathbb{R})$  of  $\mathrm{SL}(d, \mathbb{R})$ . If  $T \in \mathrm{SL}(d, \mathbb{R})$ , let

$$\lambda_1(T) \geq \cdots \geq \lambda_d(T)$$

be the (ordered) moduli of (generalized) eigenvalues of  $T$  (with multiplicity). The Jordan (or Lyapunov) projection

$$\ell : \mathrm{SL}(d, \mathbb{R}) \rightarrow \mathfrak{a} \text{ is given by } \ell(T) = (\log \lambda_1(T), \dots, \log \lambda_d(T)).$$

For each  $k = 1, \dots, d-1$ , let  $\alpha_k : \mathfrak{a} \rightarrow \mathbb{R}$  be given by  $\alpha_k(\vec{a}) = a_k - a_{k+1}$  and let

$$\Delta = \left\{ \sum_{k=1}^{d-1} t_k \alpha_k \mid t_k \geq 0 \forall k \text{ and } t_k > 0 \text{ for some } k \right\} \subset \mathfrak{a}^*.$$

For example, if  $\alpha_H$  is the Hilbert length functional given by  $\alpha_H(\vec{a}) = a_1 - a_d$ , then  $\alpha_H = \sum_{k=1}^{d-1} \alpha_k \in \Delta$ . Similarly, if  $\omega_1(\vec{a}) = a_1$ , then  $\omega_1 = \sum_{k=1}^{d-1} \frac{d-k}{d} \alpha_k \in \Delta$ . Given non-trivial  $\phi \in \Delta$  and  $T \in \mathrm{SL}(d, \mathbb{R})$ , we define the  $\phi$ -translation length

$$\ell^\phi(T) = \phi(\ell(T)).$$

Let  $(\Sigma^+, \sigma)$  be the Stadlbauer-Ledrappier-Sarig coding [36, 64] (if  $S$  has finite area) or Dal'bo-Peigné coding [19] (if not) of the recurrent portion of the geodesic flow on  $T^1S$ . It is topologically mixing and has (BIP). Moreover, it comes equipped with a map

$$G : \mathcal{A} \rightarrow \Gamma$$

so that if  $\gamma \in \Gamma$  is hyperbolic, then there exists  $x = \overline{x_1 \cdots x_n} \in \Sigma^+$  so that  $G(x_1) \cdots G(x_n)$  is conjugate to  $\gamma$ . Moreover,  $x$  is unique up to powers of  $\sigma$ . Given a cusped Hitchin representation  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  we will define a vector-valued roof function  $\tau_\rho : \Sigma^+ \rightarrow \mathfrak{a}$  with the property that if  $x = \overline{x_1 \cdots x_n}$  is a periodic element of  $\Sigma^+$ , then

$$S_n \tau_\rho(x) = \tau_\rho(x) + \tau_\rho(\sigma(x)) + \cdots + \tau_\rho(\sigma^{n-1}(x)) = \ell(\rho(G(x_1) \cdots G(x_n)))$$

so  $\tau_\rho$  encodes all the spectral data of  $\rho(\Gamma)$ .

The following result allows us to use the general thermodynamical machinery we developed to study cusped Hitchin representations.

**Theorem D** (Roof functions). *Suppose that  $\Gamma$  is a torsion-free, geometrically finite Fuchsian group which is not convex cocompact,  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  is a cusped Hitchin representation and  $\phi \in \Delta$ . Then there exists a locally Hölder continuous function  $\tau_\rho^\phi = \phi \circ \tau_\rho : \Sigma^+ \rightarrow \mathbb{R}$  such that*

- (1)  $\tau_\rho^\phi$  is eventually positive and non-arithmetic.
- (2) If  $x = \overline{x_1 \cdots x_n}$  is a periodic element of  $\Sigma^+$ , then

$$S_n \tau_\rho^\phi(x) = \ell^\phi(\rho(G(x_1) \cdots G(x_n))).$$

- (3)  $\tau_\rho^\phi$  has a strong entropy gap at infinity. Moreover, if  $\phi = a_1 \alpha_1 + \cdots + a_{d-1} \alpha_{d-1}$ , then

$$d(\tau_\rho^\phi) = \frac{1}{2(a_1 + \cdots + a_{d-1})}.$$

- (4) If  $\eta : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  is another cusped Hitchin representation, then there exists  $C > 0$  so that

$$|\tau_\rho^\phi(x) - \tau_\eta^\phi(x)| \leq C$$

for all  $x \in \Sigma^+$ .

We obtain a counting result for cusped Hitchin representations as an immediate consequence of Theorem A.

**Corollary 1.3.** *If  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  is a cusped Hitchin representation and  $\phi \in \Delta$ , then there exists a unique  $\delta = \delta_\phi(\rho)$  so that  $P(-\delta \tau_\rho^\phi) = 0$ , and*

$$\#\{[\gamma] \in [\Gamma] \mid 0 < \ell^\phi(\rho(\gamma)) \leq t\} \sim \frac{e^{t\delta}}{t\delta}$$

as  $t \rightarrow \infty$ .

We will refer to  $\delta_\phi(\rho)$  as the  $\phi$ -topological entropy of  $\rho$ .

If  $\rho, \eta : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  are cusped Hitchin representations and  $\phi \in \Delta$ , we define the *Manhattan curve*

$$\mathcal{C}^\phi(\rho, \eta) = \{(a, b) \in \mathbb{R}^2 \mid P(-a\tau_\rho^\phi - b\tau_\eta^\phi) = 0, a \geq 0, b \geq 0, a + b > 0\}.$$

Theorem C immediately gives the following information about  $\mathcal{C}^\phi(\rho, \eta)$ .

**Corollary 1.4.** *If  $\rho, \eta : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  are cusped Hitchin representations and  $\phi \in \Delta$ , then*

- (1)  $\mathcal{C}^\phi(\rho, \eta)$  is a closed subsegment of an analytic curve,
- (2) the points  $(\delta_\phi(\rho), 0)$  and  $(0, \delta_\phi(\eta))$  lie on  $\mathcal{C}^\phi(\rho, \eta)$ ,
- (3) and  $\mathcal{C}^\phi(\rho, \eta)$  is strictly convex, unless

$$\ell^\phi(\rho(\gamma)) = \frac{\delta_\phi(\eta)}{\delta_\phi(\rho)} \ell^\phi(\eta(\gamma))$$

for all  $\gamma \in \Gamma$ .

Moreover, the tangent line to  $\mathcal{C}^\phi(\rho, \eta)$  at  $(h^\phi(\rho), 0)$  has slope

$$s^\phi(\rho, \eta) = -\frac{\int \tau_\eta^\phi d\mu_{-\delta_\phi(\rho)\tau_\rho^\phi}}{\int \tau_\rho^\phi d\mu_{-\delta_\phi(\rho)\tau_\rho^\phi}}$$

We call  $I^\phi(\rho, \eta) = -s^\phi(\rho, \eta)$  the  $\phi$ -pressure intersection. We also define the *renormalized  $\phi$ -pressure intersection* by

$$J^\phi(\rho, \eta) = \frac{\delta_\phi(\eta)}{\delta_\phi(\rho)} I^\phi(\rho, \eta).$$

As a further corollary of Theorem C we obtain the following rigidity result for renormalized pressure intersection. This corollary will later play a key role in our forthcoming construction of pressure metrics on the space of cusped Hitchin representations.

**Corollary 1.5.** *If  $\rho, \eta : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  are cusped Hitchin representations and  $\phi \in \Delta$ , then*

$$J^\phi(\rho, \eta) \geq 1$$

*with equality if and only if*

$$\ell^\phi(\rho(\gamma)) = \frac{\delta_\phi(\eta)}{\delta_\phi(\rho)} \ell^\phi(\eta(\gamma))$$

*for all  $\gamma \in \Gamma$ .*

As a corollary of Theorem B we obtain the following geometric interpretation of the pressure intersection. Let

$$R_T^\phi(\rho) = \{[\gamma] \in [\Gamma] \mid 0 < \ell^\phi(\rho(\gamma)) \leq T\}.$$

**Corollary 1.6.** *If  $\rho, \eta : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  are cusped Hitchin representations and  $\phi \in \Delta$  then*

$$I^\phi(\rho, \eta) = \lim_{T \rightarrow \infty} \frac{1}{\#(R_T^\phi(\rho))} \sum_{[\gamma] \in R_T^\phi(\rho)} \frac{\ell^\phi(\eta(\gamma))}{\ell^\phi(\rho(\gamma))}.$$

In a companion paper, Canary, Zhang and Zimmer [13] study the geometry of cusped Hitchin representation showing that they are “relatively” Borel Anosov in a sense which generalizes work of Labourie [32]. They also show that cusped Hitchin representations are stable with respect to type-preserving deformation in  $\mathrm{SL}(d, \mathbb{C})$ . As a consequence, they see that limit maps are Hölder and vary analytically. In a sequel paper, we will combine the work in this paper and in [13] to construct pressure metrics on cusped Hitchin components.

This project is motivated by the hope that there is an augmented Hitchin component which generalizes the notion of augmented Teichmüller space. Masur [42] proved that the augmented Teichmüller space is the metric completion of Teichmüller space with the Weil-Petersson metric. The strata at infinity of augmented Teichmüller space consists of Teichmüller spaces of cusped hyperbolic surfaces. These strata naturally inherit a Weil-Petersson metric from the completion. The potential analogy is clearest when  $d = 3$ , where Hitchin components are spaces of convex projective structures on closed surfaces. Work of Loftin [37] and Loftin-Zhang [38] explores the analytic structure and topology of this bordification. We hope that our work on pressure metrics will aid in showing that there is an augmented Hitchin component which arises as the metric completion of the Hitchin component with the pressure metric.

**Other applications:** These results have immediate generalizations for  $P_k$ -Anosov representations of geometrically finite Fuchsian groups.

We also recover (mild generalizations of) many of Sambarino’s results on counting and equidistribution for uncusped Anosov representations in our framework (see [53, 54, 55]).

**Historical remarks:** Counting and equidistribution results have long been a central theme of the Thermodynamical Formalism (see, for example, the seminal work of Bowen, Parry, Pollicott and Ruelle [5, 6, 44, 51]). Lalley’s innovation [34] was the introduction of renewal theory and the development of a Renewal Theorem which allowed him to obtain precise counting and equidistribution results. Our work harnesses Kesseböhmer and Kombrink’s extension [31] of Lalley’s Renewal Theorem to the setting of countable Markov shifts to obtain similar results in our setting.

Bishop and Steger [3] proved a rigidity theorem in the setting of finite area hyperbolic surfaces which is the precursor to the study of Manhattan curves. Lalley [35] extended Bishop and Steger’s rigidity theorem to the setting of closed negatively curved surfaces. The formulation in terms of a Manhattan curve is due to Burger [12] who worked in the setting of convex cocompact representations into rank one Lie groups. Kao [26] established a Manhattan curve theorem for geometrically finite Fuchsian groups and Bray-Canary-Kao [9] extended his result to the setting of geometrically finite quasifuchsian representations.

Dal'bo and Peigné [19] used renewal theorems in their work obtaining counting and mixing results on geometrically finite negatively curved surfaces. They also applied renewal techniques to study counting results for the modular surface [18]. Thirion [65] used related techniques to obtain asymptotic results for orbital counting functions for ping pong groups. Thirion's ping pong groups overlap with the class of (images of) cusped  $P_1$ -Anosov representations.

Corollary 1.3 generalizes results of Sambarino [53, 54, 55] from the Anosov setting, while Corollaries 1.5 and 1.6 generalize results of Bridgeman-Canary-Labourie-Sambarino [10].

In the case of cusped Hitchin representations,  $d(\tau_\rho^\phi)$  is simply the maximum critical exponent of the  $\phi$ -length Poincaré series associated to any unipotent subgroup of  $\rho(\Gamma)$ . Thus, having a strong entropy gap at infinity is analogous to the critical exponent gap used in the work of Dal'bo-Peigné [19] and Dal'bo-Otal-Peigné [17]. Schapira and Tapie [62, Prop. 7.16] showed that for a geometrically finite negatively curved manifold then there is a critical exponent gap if and only if the geodesic flow has an entropy gap at infinity. Our definition is inspired by their work. In turn, Schapira and Tapie were motivated, in part, by work on strongly positive recurrent potentials for countable Markov shifts due to Gurevich and Savchenko [23, 61], Sarig [57, 58], Ruelle [52], and Boyle-Buzzi-Gómez [8]. Other relevant precursors to our results include the work of Iommi-Riquelme-Velozo [25], Riquelme-Velozo [50], and Velozo [67].

In recent work, Pollicott and Urbanski [47] use related techniques to obtain fine counting results for conformal dynamical systems. Their main technical tools come from the study of complexified Ruelle-Perron-Frobenius operators, generalizing early work of Parry-Pollicott [44] in the setting of finite Markov shifts. (The proof of Kesseböhmer and Kombrink's Renewal Theorem [31] also relies on the study of complexified Ruelle-Perron-Frobenius operators.) Pollicott and Urbanski give extensive applications to the study of circle packings, rational functions, continued fractions, Fuchsian groups and Schottky groups and other topics.

Feng Zhu [69] obtained closely related counting and equidistribution results for the Hilbert length functional on geometrically finite strictly convex projective manifolds. When  $d = 3$ , cusped Hitchin representations are holonomy maps of strictly convex projective surfaces, so our results overlap with his in this case.

**Outline of paper:** In Section 2, we recall the relevant background material from the theory of countable Markov shifts. In Section 3, we use this theory to explore the consequences of entropy gaps at infinity. In Section 4, we recall the Renewal Theorem of Kesseböhmer and Kombrink [31] and show that we can apply it in our context. Section 5 contains the crucial technical material needed in the proof of Theorems A. Sections 6, 7 and 8 contain the proof of Theorems A, B and C (respectively). In Section 9, we develop the background material needed for our applications. Section 10 contains the proof of (a generalization of) Theorem D and Section 11 derives its consequences.

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## 2. BACKGROUND FROM THE THERMODYNAMIC FORMALISM

In this section, we recall the background results we will need from the Thermodynamic Formalism for countable Markov shifts as developed by Gurevich-Savchenko [23], Mauldin-Urbanski [43] and Sarig [57].

Given a countable alphabet  $\mathcal{A}$  and a transition matrix  $\mathbb{T} = (t_{ab}) \in \{0, 1\}^{\mathcal{A} \times \mathcal{A}}$  a one-sided Markov shift is

$$\Sigma^+ = \{x = (x_i) \in \mathcal{A}^{\mathbb{N}} \mid t_{x_i x_{i+1}} = 1 \text{ for all } i \in \mathbb{N}\}$$

equipped with a shift map  $\sigma : \Sigma^+ \rightarrow \Sigma^+$  which takes  $(x_i)_{i \in \mathbb{N}}$  to  $(x_{i+1})_{i \in \mathbb{N}}$ .

We will work in the setting of topologically mixing Markov shifts with (BIP), where many of the classical results of Thermodynamic formalism generalize. The shift  $(\Sigma^+, \sigma)$  is *topologically mixing* if for all  $a, b \in \mathcal{A}$ , there exists  $N = N(a, b)$  so that if  $n \geq N$ , then there exists  $x \in \Sigma$  so that  $x_1 = a$  and  $x_n = b$ . It has the big images and pre-images property (BIP) if there exists a finite subset  $\mathcal{B} \subset \mathcal{A}$  so that if  $a \in \mathcal{A}$ , then there exist  $b_0, b_1 \in \mathcal{B}$  so that  $t_{b_0 a} = 1 = t_{a b_1}$ .

The theory works best for locally Hölder continuous potentials. We say that  $g : \Sigma^+ \rightarrow \mathbb{R}$  is *locally Hölder continuous* if there exist  $A > 0$  and  $\alpha > 0$  so that

$$|g(x) - g(y)| \leq A e^{-\alpha n}$$

whenever  $x_i = y_i$  for all  $i \leq n$ . When we want to record the constants we will say that  $g$  is locally  $\alpha$ -Hölder continuous with constant  $A$ . The *Gurevich pressure* of  $g$  is given by

$$P(g) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\{x \in \text{Fix}^n \mid x_1 = a\}} e^{S_n g(x)}$$

for some (any)  $a \in \mathcal{A}$  where

$$S_n g(x) = \sum_{i=1}^n g(\sigma^{i-1}(x))$$

is the *ergodic sum* and  $\text{Fix}^n = \{x \in \Sigma^+ \mid \sigma^n(x) = x\}$ .

We say that two locally Hölder continuous functions  $f$  and  $g$  are *cohomologous* if there exists a locally Hölder continuous function  $h$  so that

$$f - g = h - h \circ \sigma.$$

The analogue of Livsic's theorem holds in this setting.

**Theorem 2.1.** (Sarig [60, Thm 1.1]) *Suppose that  $\Sigma^+$  is a topologically mixing, one-sided countable Markov shift with (BIP). If  $f : \Sigma^+ \rightarrow \mathbb{R}$  and  $g : \Sigma^+ \rightarrow \mathbb{R}$  are both locally Hölder continuous, then  $f$  is cohomologous to  $g$  if and only if  $S_n f(x) = S_n g(x)$  for all  $n \in \mathbb{N}$  and  $x \in \text{Fix}^n$ . In particular, if  $f$  and  $g$  are cohomologous, then  $P(-tf) = P(-tg)$  whenever  $P(-tf)$  is finite.*

A  $\sigma$ -invariant Borel probability measure  $\mu$  on  $\Sigma^+$  is an *equilibrium state* for a locally Hölder continuous function  $g : \Sigma^+ \rightarrow \mathbb{R}$  if

$$P(g) = h_\sigma(\mu) + \int_{\Sigma^+} g d\mu$$

where  $h_\sigma(\mu)$  is the measure-theoretic entropy of  $\sigma$  with respect to the measure  $\mu$ .

A Borel probability measure  $\mu$  on  $\Sigma^+$  is a *Gibbs state* for a locally Hölder continuous function  $g : \Sigma^+ \rightarrow \mathbb{R}$  if there exists  $B > 1$  so that

$$\frac{1}{B} \leq \frac{\mu([a_1, \dots, a_n])}{e^{S_n g(x) - nP(g)}} \leq B$$

for all  $x \in [a_1, \dots, a_n]$ , where  $[a_1, \dots, a_n]$  is the *cylinder* consisting of all  $x \in \Sigma^+$  so that  $x_i = a_i$  for all  $1 \leq i \leq n$ .

**Theorem 2.2.** (Mauldin-Urbanski [43, Thm 2.2.9], Sarig [60, Thm 4.9]) *If  $\Sigma^+$  is a topologically mixing, one-sided countable Markov shift with (BIP),  $g : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous, it admits a shift invariant Gibbs state  $\mu_g$ , and  $-\int g d\mu_g < +\infty$ , then  $\mu_g$  is the unique equilibrium state for  $g$ .*

Recall from the introduction that for  $g : \Sigma^+ \rightarrow \mathbb{R}$  a locally Hölder continuous function we define

$$I(g, a) = \inf \{g(x) \mid x \in \Sigma^+, x_1 = a\} \quad \text{and} \quad S(g, a) = \sup \{g(x) \mid x \in \Sigma^+, x_1 = a\}.$$

We will make crucial use of the following criterion for a potential to admit an equilibrium state.



**Theorem 2.3.** (Mauldin-Urbanski [43, Thm 2.2.4 and 2.2.9, Lemma 2.2.8], Sarig [60, Thm 4.9]) *If  $\Sigma^+$  is a topologically mixing, one-sided countable Markov shift with (BIP),  $g : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous, and*

$$\sum_{a \in \mathcal{A}} I(g, a) e^{-S(g, a)}$$

*converges, then  $-g$  admits a unique equilibrium state  $\mu_{-g}$ . Moreover,*

$$\int_{\Sigma^+} g \, d\mu_{-g} < +\infty.$$

We will need to be able to take the derivatives of the pressure function and to be able to apply the Implicit Function Theorem. We say that  $\{g_u : \Sigma^+ \rightarrow \mathbb{R}\}_{u \in M}$  is a *real analytic family* if  $M$  is a real analytic manifold and for all  $x \in \Sigma^+$ ,  $u \rightarrow g_u(x)$  is a real analytic function on  $M$ . Mauldin and Urbanski [43, Thm. 2.6.12, Prop. 2.6.13] (see also Sarig [59, Cor. 4]), prove real analyticity properties of the pressure function and evaluate its derivative.

**Theorem 2.4.** (Mauldin-Urbanski, Sarig) *Suppose that  $\Sigma^+$  is a topologically mixing, one-sided countable Markov shift with (BIP). If  $\{g_u : \Sigma^+ \rightarrow \mathbb{R}\}_{u \in M}$  is a real analytic family of locally Hölder continuous functions such that  $P(g_u) < \infty$  for all  $u$ , then  $u \rightarrow P(g_u)$  is real analytic.*

*Moreover, if  $v \in T_{u_0}M$  and there exists a neighborhood  $U$  of  $u_0$  in  $M$  so that if  $u \in U$  and  $-\int_{\Sigma^+} g_u d\mu_{g_{u_0}} < \infty$ , then*

$$D_v P(g_u) = \int_{\Sigma^+} D_v(g_u(x)) \, d\mu_{g_{u_0}}.$$

Recall that if  $f : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous the *transfer operator* is defined by

$$\mathcal{L}_f \phi(x) := \sum_{y \in \sigma^{-1}(x)} e^{f(y)} \phi(y)$$

where  $\phi : \Sigma^+ \rightarrow \mathbb{R}$  is a *bounded* locally Hölder continuous function. The transfer operator, in particular, gives us crucial information about equilibrium states.

**Theorem 2.5.** (Mauldin-Urbanski [43, Cor. 2.7.5], Sarig [60, Thm. 4.9]) *Suppose that  $\Sigma^+$  is a topologically mixing, one-sided countable Markov shift with (BIP). If  $g : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous,  $P(g) < +\infty$ , and  $\sup g < +\infty$  then there exist unique probability measures  $\mu_g$  and  $\nu_g$  on  $\Sigma^+$  and a positive function  $h_g : \Sigma^+ \rightarrow \mathbb{R}$  so that*

$$\mu_g = h_g \nu_g, \quad \mathcal{L}_g h_g = e^{P(g)} h_g, \quad \text{and} \quad \mathcal{L}_g^* \nu_g = e^{P(g)} \nu_g.$$

*Moreover,  $h_g$  is bounded away from both 0 and  $+\infty$  and  $\mu_g$  is an equilibrium state for  $g$ .*

We will also use the following estimate on the behavior of powers of the transfer operator.

**Theorem 2.6.** (Mauldin-Urbanski [43, Theorem 2.4.6]) *Suppose that  $\Sigma^+$  is a topologically mixing, one-sided countable Markov shift with (BIP). If  $g : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous,  $P(g) < +\infty$ , and  $\sup g < +\infty$ , then there exist  $R > 0$  and  $\eta \in (0, 1)$  so that if  $n \in \mathbb{N}$  and  $\phi : \Sigma^+ \rightarrow \mathbb{R}$  is bounded and locally  $\eta$ -Hölder continuous with constant  $A$ , then*

$$(2.1) \quad \left\| e^{-nP(g)} \mathcal{L}_g^n \phi - h_g(x) \int \phi \, d\nu_g \right\| \leq R \eta^n \left( \sup_{x \in \Sigma^+} |\phi(x)| + A \right).$$

### 3. ENTROPY GAPS AT INFINITY

In this section, we show that a strong entropy gap at infinity implies a weak entropy gap at infinity and explore the thermodynamical consequences of entropy gaps at infinity.

Recall that  $d(f)$  is the critical exponent of the series

$$Z_1(f, s) = \sum_{a \in \mathcal{A}} e^{-sI(f, a)}.$$

Notice that if  $f$  is locally Hölder continuous, there exists  $C > 0$  so that  $S(f, a) - I(f, a) \leq C$  for all  $a \in \mathcal{A}$ . So the series

$$\sum_{a \in \mathcal{A}} e^{-sI(f, a)}$$

has critical exponent  $d(f)$  and diverges at  $d(f)$  if and only if  $f$  has a strong entropy gap at infinity.

We first observe a bound on the number of letters with  $I(f, a) \leq t$ .

**Lemma 3.1.** *Suppose that  $\Sigma^+$  is a topologically mixing, one-sided countable Markov shift with (BIP). If  $f : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous,  $d(f)$  is finite and  $b > d(f)$ , then there exists  $D = D(f, b) > 0$  so that*

$$B_1(f, t) = \#\{a \in \mathcal{A} \mid I(f, a) \leq t\} \leq De^{bt}$$

for all  $t > 0$ , and

$$\sum_{y \in \sigma^{-1}(x)} \mathbf{1}_{\{f(y) \leq t\}}(y) \leq De^{bt}$$

for all  $x \in \Sigma^+$  and  $t > 0$ .

*Proof.* Fix  $b > d(f)$ . If there does not exist  $D$  so that  $B_1(f, t) \leq De^{bt}$  for all  $t > 0$ , then there exists a sequence  $t_n \rightarrow \infty$  so that

$$B_1(f, t_n) \geq ne^{bt_n}.$$

But then

$$\sum_{a \in \mathcal{A}} e^{-bI(f, a)} \geq \sum_{\{a \mid I(f, a) \leq t_n\}} e^{-bI(f, a)} \geq ne^{bt_n} e^{-bt_n} = n$$

for all  $n \in \mathbb{N}$ , which contradicts our assumption that  $b > d(f)$ .

Finally, notice that if  $x \in \Sigma^+$ , then

$$\sum_{y \in \sigma^{-1}(x)} \mathbf{1}_{\{f(y) \leq t\}}(y) \leq B_1(f, t) \leq De^{bt}$$

for all  $t > 0$ . □

It will often be convenient to work with a strictly positive potential. We observe that an eventually positive potential is always cohomologous to a strictly positive potential with the same entropy gaps.

**Lemma 3.2.** *Suppose that  $\Sigma^+$  is a topologically mixing, one-sided countable Markov shift with (BIP) and that  $f : \Sigma^+ \rightarrow \mathbb{R}$  is eventually positive, locally Hölder continuous and  $d(f)$  is finite. Then  $f$  is cohomologous to a strictly positive, locally Hölder continuous function  $g$  so that*

- (1) *there exists  $C$  so that  $|f(x) - g(x)| \leq C$  for all  $x \in \Sigma^+$ ,*
- (2)  *$d(f) = d(g)$ ,*
- (3)  *$f$  has a weak entropy gap at infinity if and only if  $g$  has a weak entropy gap at infinity, and*
- (4)  *$f$  has a strong entropy gap at infinity if and only if  $g$  has a strong entropy gap at infinity.*

*Proof.* Notice that (1) implies that  $|I(f, a) - I(g, a)| \leq C$ . Moreover, if  $f$  is cohomologous to  $g$ , and both are locally Hölder continuous, then  $P(-tf) = P(-tg)$  for all  $t > d(f)$ , see Theorem 2.1. Therefore, (2)–(4) follow immediately once we construct a strictly positive, locally Hölder continuous function  $g$  that is cohomologous to  $f$  so that (1) holds.

Let

$$R = \left| \inf_{x \in \Sigma^+} f(x) \right|.$$

Note that  $R = |\inf_{a \in \mathcal{A}} I(f, a)|$  is finite since there exists  $s > d(f) > 0$  so that  $\sum_{a \in \mathcal{A}} e^{-sI(f, a)}$  is finite. Since  $f$  is eventually positive, there exists  $N \in \mathbb{N}$  and  $B > 0$  so that if  $n \geq N$  and  $x \in \Sigma^+$ , then

$$S_n f(x) \geq B.$$

Let

$$\mathcal{F} = \{a \in \mathcal{A} \mid I(f, a) \leq RN + B\}.$$

Since  $d(f)$  is finite,  $\mathcal{F}$  must be finite. To see this, observe that for  $s > d(f) > 0$

$$\infty > \sum_{a \in \mathcal{A}} e^{-sI(f, a)} \geq \sum_{a \in \mathcal{F}} e^{-sI(f, a)} \geq \sum_{a \in \mathcal{F}} e^{-s(RN+B)}.$$

For all  $n \in \mathbb{N}$ , define

$$\begin{aligned} C_n f(x) &= \sum_{i=1}^n \left( f(\sigma^{i-1}(x)) \mathbf{1}_{\{x_i \in \mathcal{F}\}}(x) + (RN + B) \mathbf{1}_{\{x_i \notin \mathcal{F}\}}(x) \right) \\ &= S_n f(x) - \sum_{i=1}^n \left( f(\sigma^{i-1}(x)) - (RN + B) \right) \mathbf{1}_{\{x_i \notin \mathcal{F}\}}(x). \end{aligned}$$

By construction,

$$RN^2 + NB + TN \geq C_N f(x) \geq B$$

for all  $x \in \Sigma^+$ , where

$$T = \sup\{f(x) \mid x_1 \in \mathcal{F}\}.$$

(The lower bound holds, since  $C_N f(x) = S_N f(x) \geq B$  if  $x_i \in \mathcal{F}$  for all  $i \leq N$ , and otherwise one of the summands of  $C_N f(x)$  is  $RN + B$  and each of the remaining terms are bounded below by  $-R$ .)

We then define  $g : \Sigma^+ \rightarrow \mathbb{R}$  by

$$g(x) = \frac{1}{N} C_N f(x) + (f(x) - (RN + B)) \mathbf{1}_{\{x_1 \notin \mathcal{F}\}}(x).$$

By construction,  $g$  is continuous and

$$g(x) \geq \frac{B}{N} > 0$$

for all  $x \in \Sigma^+$ , so  $g$  is strictly positive.

Moreover, if  $x_1 \in \mathcal{F}$ , then  $|g(x) - f(x)| \leq RN + B + 2T$ , and if  $x_1 \notin \mathcal{F}$ , then  $|g(x) - f(x)| \leq RN + B + \frac{1}{N} C_N f(x) \leq 2(RN + B)$ . It follows that

$$|g(x) - f(x)| \leq 2(RN + B + T) =: C$$

for all  $x \in \Sigma^+$ .

To show  $g$  is locally Hölder continuous, consider  $x, y \in \Sigma^+$  for which  $x_i = y_i$  for all  $i = 1, \dots, n$ , and note that it suffices to consider  $n \geq N$ . Then

$$|g(x) - g(y)| = \left| \frac{1}{N} \left( \sum_{i=1}^N (f(\sigma^{i-1}(x)) - f(\sigma^{i-1}(y))) \mathbf{1}_{\{x_i \in \mathcal{F}\}}(x) \right) + (f(x) - f(y)) \mathbf{1}_{\{x_1 \notin \mathcal{F}\}}(x) \right|.$$

Since  $n \geq N$ , applying local Hölder continuity of  $f$  gives the desired conclusion.

Finally, if  $x = \overline{x_1 \dots x_r} \in \text{Fix}^r$ , then one may check that  $S_r f(x) = S_r g(x)$ . To see this, observe that

$$\begin{aligned} S_r g(x) &= S_r \left( \frac{1}{N} C_N f(x) \right) + S_r \left( (f(x) - (RN + B)) \mathbf{1}_{\{x_1 \notin \mathcal{F}\}}(x) \right) \\ &= \frac{1}{N} S_r C_N f(x) + \sum_{j=1}^r (f(\sigma^{j-1}(x)) - (RN + B)) \mathbf{1}_{\{x_j \notin \mathcal{F}\}}(x) \end{aligned}$$

and since  $\sigma^r(x) = x$ ,

$$\begin{aligned} S_r C_N f(x) &= S_r S_N f(x) - \sum_{j=1}^r \sum_{i=1}^N (f(\sigma^{i-1}(x)) - (RN + B)) \mathbf{1}_{\{x_i \notin \mathcal{F}\}}(x) \\ &= N S_r f(x) - N \sum_{j=1}^r (f(\sigma^{j-1}(x)) - (RN + B)) \mathbf{1}_{\{x_j \notin \mathcal{F}\}}(x). \end{aligned}$$

Theorem 2.1 then implies that  $f$  and  $g$  are cohomologous.  $\square$

We next study the behavior of  $P(-tf)$  for  $t > d(f)$ , showing among other things that a strong entropy gap at infinity implies a weak entropy gap at infinity.

**Lemma 3.3.** *Suppose that  $\Sigma^+$  is a topologically mixing, one-sided countable Markov shift with (BIP) and  $f : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous and eventually positive.*

- (1) *If  $d(f)$  is finite, then  $P(-tf)$  is finite if  $t > d(f)$  and infinite if  $t < d(f)$ , and the function  $t \rightarrow P(-tf)$  is monotone decreasing and analytic on  $(d(f), \infty)$ .*
- (2) *There exists at most one  $\delta \in (d(f), \infty)$  so that  $P(-\delta f) = 0$ .*
- (3) *If  $f$  has a strong entropy gap at infinity, then  $t \rightarrow P(-tf)$  is proper on  $(d(f), \infty)$ . In particular,  $f$  has a weak entropy gap at infinity.*

*Proof.* Mauldin and Urbanski [43, Theorem 2.1.9] proved that if  $\Sigma^+$  is topologically mixing and has (BIP), then  $P(-sf)$  is finite if and only if

$$Z_1(-f, s) = \sum_{a \in \mathcal{A}} e^{\sup\{-sf(x) \mid x_1=a\}}$$

converges. Therefore,  $P(-tf)$  is finite if  $t > d(f)$  and infinite if  $t < d(f)$ . Notice that  $t \rightarrow P(-tf)$  is monotone decreasing by definition and analytic by Theorem 2.4, so (1) follows. (2) is an immediate consequence of (1).

It remains to show (3). The fact that  $\lim_{t \rightarrow d(f)} P(-tf) = +\infty$  is essentially contained in Mauldin and Urbanski's proof of [43, Theorem 2.1.9], but we elaborate here for completeness. They show that there exist constants  $q, s, M, m > 0$  so that for any locally Hölder continuous function  $g$ ,

$$\sum_{i=n}^{n+s(n-1)} Z_i(g, 1) \geq \frac{e^{-M+(M-m)n}}{q^{n-1}} Z_1(g, 1)^n.$$

where

$$Z_n(g, 1) = \sum_{p \in \Lambda_k} e^{\sup_{x \in p} S_n g(x)},$$

and  $\Lambda_k$  is the set of  $k$ -cylinders of  $\Sigma^+$ . They observe [43, Equation (2.1)] that  $\lim \frac{1}{n} \log Z_n(g, 1) = P(g)$ . Thus there exists  $A > 0$  such that for all  $n$ , there exists  $\hat{n} \in [n, n + s(n-1)]$  so that  $Z_{\hat{n}}(g, 1) \geq A^n Z_1(g, 1)^n$ , so  $P(g) \geq \frac{1}{1+s} \log AZ_1(g, 1)$ . Therefore, if  $f$  has a strong entropy gap at infinity, then  $\lim_{t \rightarrow d(f)} Z_1(-tf, 1) = +\infty$  and hence

$$\lim_{t \rightarrow d(f)} P(-tf) \geq \lim_{t \rightarrow d(f)} \frac{1}{1+s} \log AZ_1(-tf, 1) = +\infty.$$

We now show that  $\lim_{t \rightarrow \infty} P(-tf) = -\infty$ . Notice that since there exists  $N > 0$  such that  $S_n f(x) > B > 0$  for all  $n \geq N$  and  $x \in \Sigma^+$ , we have  $S_{kN} f(x) > kB$  for every  $k \geq 1$ . Then,

$$\sum_{\{x \in \text{Fix}^{kN} \mid x_1=a\}} e^{-2td(f)S_{kN} f(x)} \leq \sum_{\{x \in \text{Fix}^{kN} \mid x_1=a\}} e^{-2(t-1)d(f)kB - 2d(f)S_{kN} f(x)}$$

which implies

$$\begin{aligned} P(-2td(f)f) &\leq \lim_{k \rightarrow \infty} \frac{1}{kN} \log \sum_{\{x \in \text{Fix}^{kN} \mid x_1 = a\}} e^{-2(t-1)d(f)kB - 2d(f)S_{kN}f(x)} \\ &= \frac{-2(t-1)d(f)B}{N} + P(-2d(f)f) \end{aligned}$$

and so  $\lim_{t \rightarrow \infty} P(-tf) = -\infty$ .

Since  $t \rightarrow P(-tf)$  is proper and monotone decreasing on  $(d(f), \infty)$ , it follows that there exists  $\delta > d(f)$  so that  $P(-\delta f) = 0$ . Therefore,  $f$  has a weak entropy gap at infinity and we have established (3).  $\square$

We next observe that  $-tf$  admits an equilibrium state if  $t > d(f)$ .

**Lemma 3.4.** *Suppose that  $\Sigma^+$  is a topologically mixing, one-sided countable Markov shift with (BIP). If  $f : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous and eventually positive and  $t > d(f)$ , then there exists a unique equilibrium state  $\mu_{-tf}$  for  $-tf$ . Moreover,*

$$0 < \int_{\Sigma^+} f d\mu_{-tf} < +\infty.$$

*Proof.* Theorem 2.3 implies that there exists a unique equilibrium state for  $-tf$  if and only if

$$\sum_{a \in \mathcal{A}} tI(f, a)e^{-tS(f, a)} < +\infty.$$

Indeed, this series converges since

$$\sum_{a \in \mathcal{A}} e^{-sS(f, a)} < +\infty$$

for all  $s > d(f)$ . Theorem 2.3 also ensures that  $\int_{\Sigma^+} f d\mu_{-tf} < +\infty$ . Since  $f$  is eventually positive, it is cohomologous to a strictly positive function  $g$ . Then  $-tf$  and  $-tg$  are cohomologous and hence have the same integral with respect to any shift-invariant measure, and also share the same shift-invariant equilibrium state, i.e.  $\mu_{-tf} = \mu_{-tg}$  (see [43, Theorem 2.2.7] and Theorem 2.3). Hence,

$$\int_{\Sigma^+} f d\mu_{-tf} = \int_{\Sigma^+} g d\mu_{-tg} > 0.$$

$\square$

Theorem 2.5 and Lemma 3.3 have the following corollary which we will use repeatedly.

**Corollary 3.5.** *Suppose that  $\Sigma^+$  is a topologically mixing, one-sided countable Markov shift with (BIP). If  $f : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous, eventually positive, and has a weak entropy gap at infinity and  $t > d(f)$ , then there exist unique probability measures  $\mu_{-tf}$  and  $\nu_{-tf}$  on  $\Sigma^+$  and a positive function  $h_{-tf} : \Sigma^+ \rightarrow \mathbb{R}$  so that*

$$\mu_{-tf} = h_{-tf}\nu_{-tf}, \quad \mathcal{L}_{-tf}h_{-tf} = e^{P(-tf)}h_{-tf}, \quad \text{and} \quad \mathcal{L}_{-tf}^*\nu_{-tf} = e^{P(-tf)}\nu_{-tf}$$

and  $h_{-tf}$  is bounded away from both 0 and  $+\infty$ . Moreover,  $\mu_{-tf}$  is the equilibrium state of  $-tf$ .

We will need analogues of these results for functions of the form  $-zg - \delta f$  where  $g$  is comparable to  $f$  and  $z$  is close to 0.

**Proposition 3.6.** *Suppose that  $\Sigma^+$  is a topologically mixing, one-sided countable Markov shift with (BIP),  $f : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous, eventually positive and has a weak entropy gap at infinity and  $P(-\delta f) = 0$  for  $\delta = \delta(f) > d(f) > 0$ . If  $g : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous, eventually positive, and there exists  $C$  so that  $|f(x) - g(x)| \leq C$  for all  $x \in \Sigma^+$ , then*

- (1) if  $z > d(f) - \delta$ , then  $P(-zg - \delta f)$  is finite,  $z \rightarrow P(-zg - \delta f)$  is monotone decreasing and analytic on  $(d(f) - \delta, \infty)$  and  $\sup_{x \in \Sigma^+} (-zg - \delta f) < +\infty$ .
- (2) if  $z > d(f) - \delta$ , then there exist unique probability measures  $\mu_{-zg-\delta f}$  and  $\nu_{-zg-\delta f}$  on  $\Sigma^+$  and a positive function  $h_{-zg-\delta f} : \Sigma^+ \rightarrow \mathbb{R}$  so that

$$\begin{aligned} \mu_{-zg-\delta f} &= h_{-zg-\delta f} \nu_{-zg-\delta f}, & \mathcal{L}_{-zg-\delta f} h_{-zg-\delta f} &= e^{P(-zg-\delta f)} h_{-zg-\delta f}, \\ & & \text{and } \mathcal{L}_{-zg-\delta f}^* \nu_{-zg-\delta f} &= e^{P(-zg-\delta f)} \nu_{-zg-\delta f}. \end{aligned}$$

Moreover,  $h_{-zg-\delta f}$  is bounded away from both 0 and  $+\infty$  and  $\mu_{-zg-\delta f}$  is the unique equilibrium state of  $-zg - \delta f$ .

*Proof.* Notice that

$$\sum_{\{x \in \text{Fix}^n \mid x_1 = a\}} e^{S_n(-zg-\delta f)} \leq \sum_{\{x \in \text{Fix}^n \mid x_1 = a\}} e^{nzC} e^{S_n((-z-\delta)f)}$$

so  $P(-zg - \delta f)$  is finite if  $z + \delta > d(f)$ , i.e. if  $z > d(f) - \delta$ . Similarly, if  $x \in \Sigma^+$ , then

$$(-zg - \delta f)(x) \leq -(z + \delta)f(x) + Cz \leq \sup(-(z + \delta)f) + Cz < +\infty$$

if  $z + \delta > 0$ . The function  $z \rightarrow P(-zg - \delta f)$  is monotone decreasing by definition and analytic by Theorem 2.4. We have established (1).

(2) is then an immediate consequence of (1) and Theorem 2.5.  $\square$

#### 4. RENEWAL THEOREMS

Our main tool will be the Renewal Theorem of Kesseböhmer and Kombrink [31]. Their result generalized a result of Lalley [34] for finite Markov shifts.

Consider a locally Hölder continuous potential  $f : \Sigma^+ \rightarrow \mathbb{R}$ . If  $\phi : \Sigma^+ \rightarrow \mathbb{R}$  is a non-negative, bounded, locally Hölder continuous function, we define the *renewal function*

$$N_f(\phi, x, t) := \sum_{n=0}^{\infty} \sum_{y \in \sigma^{-n}(x)} \phi(y) \mathbf{1}_{\{S_n f(y) \leq t\}}(y).$$

We recall that  $N_f(\phi, x, t)$  satisfies the *renewal equation*

$$(4.1) \quad N_f(\phi, x, t) = \left( \sum_{y \in \sigma^{-1}(x)} N_f(\phi, y, t - f(y)) \right) + \phi(x) \mathbf{1}_{\{t \geq 0\}}(t)$$

**Theorem 4.1.** (Renewal theorem; Kesseböhmer-Kombrink [31, Theorem 3.1]) *Suppose that  $\Sigma^+$  is a topologically mixing, one-sided, countable Markov shift with (BIP) and  $f : \Sigma^+ \rightarrow \mathbb{R}$  is a strictly positive, non-arithmetic, locally Hölder continuous function so that there exists a unique  $\delta > 0$  so that  $P(-\delta f) = 0$  and an equilibrium state  $\mu_{-\delta f} = h_{-\delta f} \nu_{-\delta f}$  for  $-\delta f$  so that  $\int_{\Sigma^+} t f \, d\mu_{-\delta f} < +\infty$  for all  $t$  in some neighborhood of  $\delta$ .*

*If  $\phi : \Sigma^+ \rightarrow \mathbb{R}$  is non-negative, bounded, not identically zero, and locally Hölder continuous and there exists  $c > 0$  such that*

$$N_f(\phi, x, t) \leq ce^{t\delta},$$

*then*

$$N_f(\phi, x, t) \sim \frac{e^{t\delta}}{\delta} h_{-\delta f}(x) \frac{\int_{\Sigma^+} \phi \, d\nu_{-\delta f}}{\int_{\Sigma^+} f \, d\mu_{-\delta f}}$$

*as  $t \rightarrow \infty$ , uniformly for  $x \in \Sigma^+$ .*

*Remark 4.2.* The Renewal Theorem we state above is a special case of [31, Theorem 3.1 (i)].

Following the notations in [31], in our case  $\eta = 0$  and  $f_y(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$ . Kesseböhmer and

Kombrink [31] in place of our assumption of non-arithmeticity only require the weaker assumption that  $f$  is not a lattice, i.e. that  $f$  is not cohomologous to a function so that  $\{S_n f(x) \mid x \in \Sigma^+\}$  does not lie in a discrete subgroup of  $\mathbb{R}$ . Moreover, since  $f_y(t) \geq 0$ ,  $\int_{-\infty}^{\infty} e^{-T\delta} f_y(T) dT = \frac{1}{\delta}$ , and  $N_f(\phi, x, t) = 0$  for  $t < 0$  when  $f$  is strictly positive, their conditions (B) and (D) are satisfied. So, it only remains to check that their condition (C) is satisfied, which translates to the existence of  $c > 0$  such that

$$N_f(\phi, x, t) \leq ce^{t\delta}.$$

We first check that a weak entropy gap at infinity implies such a bound on  $N_f(\mathbf{1}, x, t)$ .

**Lemma 4.3.** *Suppose that  $\Sigma^+$  is a topologically mixing, one-sided, countable Markov shift with (BIP) and  $f : \Sigma^+ \rightarrow \mathbb{R}$  is a strictly positive, locally Hölder continuous function with a weak entropy gap at infinity. Let  $\delta > d(f)$  be the unique constant such that  $P(-\delta f) = 0$ . Then there exists  $C > 0$  such that*

$$N_f(\mathbf{1}, x, t) = \sum_{n=0}^{\infty} \sum_{y \in \sigma^{-n}(x)} \mathbf{1}_{\{S_n f(y) \leq t\}}(y) \leq Ce^{t\delta}$$

for all  $x \in \Sigma^+$  and  $t > 0$ .

We adopt the strategy of Lalley [34, Lemma 8.1].

*Proof.* Define for all  $x \in \Sigma^+$  and  $t > 0$

$$G(x, t) = e^{-t\delta} \frac{N_f(\mathbf{1}, x, t)}{h_{-\delta f}(x)}$$

where  $h_{-\delta f}$  is the eigenfunction for the transfer operator given by Theorem 2.5. Let

$$\widehat{G}(t) = \sup\{G(x, s) \mid x \in \Sigma^+, s \leq t\}.$$

Notice that  $\widehat{G}(t)$  is finite for all  $t > 0$ , since  $h_{-\delta f}$  is bounded away from 0, and for any fixed  $t > 0$  there exists only finitely many  $a \in \mathcal{A}$  so that  $I(f, a) \leq t$  (which implies that there are only finitely many  $n$  and only finitely many  $y \in \sigma^{-n}(x)$ , for each  $n$ , so that  $S_n f(y) \leq t$ ). Since  $h_{-\delta f}$  is bounded away from 0 and  $\infty$ , it remains to show that there exists  $\widehat{C}$  so that

$$\widehat{G}(t) \leq \widehat{C}$$

for all  $t > 0$ .

The renewal equation (4.1) implies that

$$G(x, t) = \sum_{y: \sigma(y)=x} G(y, t - f(y)) e^{-\delta f(y)} \frac{h_{-\delta f}(y)}{h_{-\delta f}(x)} + \frac{e^{-t\delta}}{h_{-\delta f}(x)}.$$

for all  $t > 0$ . Notice that since  $h_{-\delta f}(x)$  is the eigenfunction of  $\mathcal{L}_{-\delta f}$  with eigenvalue  $1 = e^{P(-\delta f)}$ ,

$$\sum_{y: \sigma(y)=x} e^{-\delta f(y)} \frac{h_{-\delta f}(y)}{h_{-\delta f}(x)} = \frac{(\mathcal{L}_{-\delta f} h_{-\delta f})(x)}{h_{-\delta f}(x)} = 1.$$

If  $c = c(f) = \inf_{x \in \Sigma^+} f(x) > 0$ , then

$$(4.2) \quad G(x, t) \leq \widehat{G}(t - c) + \frac{e^{-t\delta}}{h_{-\delta f}(x)}$$

for all  $x \in \Sigma^+$  and  $t \geq c$ . Therefore,

$$\widehat{G}(mc) \leq \widehat{G}(c) + \widehat{H} \sum_{n=1}^m e^{-cn\delta}$$

for all  $m \in \mathbb{N}$ , where

$$\widehat{H} = \sup \left\{ \frac{1}{h_{-\delta f}(x)} \mid x \in \Sigma^+ \right\}.$$

Since  $\widehat{G}$  is increasing,

$$\widehat{G}(t) \leq \widehat{C} = \widehat{G}(c) + \widehat{H} \sum_{n=1}^{\infty} e^{-cn\delta}$$

for all  $t > 0$ , which completes the proof.  $\square$

If  $\phi : \Sigma^+ \rightarrow \mathbb{R}$  is bounded, non-negative and locally Hölder continuous, then

$$N_f(\phi, x, t) \leq \left( \sup_{x \in \Sigma^+} \phi(x) \right) N_f(\mathbf{1}, x, t),$$

so Lemmas 3.3, 3.4 and 4.3 together imply that we can apply the Renewal Theorem to  $\phi$  when  $f$  is strictly positive and has a weak entropy gap at infinity.

**Corollary 4.4.** *Suppose that  $\Sigma^+$  is a topologically mixing, one-sided, countable Markov shift with (BIP) and  $f : \Sigma^+ \rightarrow \mathbb{R}$  is a strictly positive, non-arithmetic, locally Hölder continuous function with a weak entropy gap at infinity,  $P(-\delta f) = 0$  and  $h_{-\delta f} \nu_{-\delta f} = \mu_{-\delta f}$  is the equilibrium state for  $-\delta f$ . If  $\phi : \Sigma^+ \rightarrow \mathbb{R}$  is bounded, non-negative, not identically zero and locally Hölder continuous, then*

$$N_f(\phi, x, t) \sim \frac{e^{t\delta}}{\delta} h_{-\delta f}(x) \frac{\int_{\Sigma^+} \phi \, d\nu_{-\delta f}}{\int_{\Sigma^+} f \, d\mu_{-\delta f}}$$

as  $t \rightarrow \infty$ , uniformly for  $x \in \Sigma^+$ .

## 5. PREPARING TO COUNT

In this section we develop the technical tools needed in the proofs of our counting result. The majority of these results bound the size of various subsets of the shift space. Most importantly, we show that if  $y \in \sigma^n(x)$  and  $S_n f(y)$  is “large,” then “typically”  $S_n f(x)$  is close to  $n \int_{\Sigma^+} f \, d\mu_{-\delta f}$ . These results and their proofs generalize Lalley [34, Theorem 6]. The fact that our Markov shift is countable requires more delicate control of error estimates.

For each cylinder  $p$ , we choose a sample point  $z_p \in p$  which is not periodic. We then define

$$W(n, p, t) = \sum_{y \in \sigma^{-n}(z_p)} \mathbf{1}_p(y) \mathbf{1}_{\{x \mid S_n f(x) \leq t\}}(y) = \# \left( p \cap \sigma^{-n}(z_p) \cap \{x \mid S_n f(x) \leq t\} \right).$$

We show that the  $W(n, p, t)$  may be used to approximate the size of  $\mathcal{M}_f(n, t)$ . This allows us to replace the counting of fixed points with counting of pre-images of our sample points.

If  $k \in \mathbb{N}$ , let  $\Lambda_k$  be the countable partition of  $\Sigma^+$  into  $k$ -cylinders.

**Lemma 5.1.** *Suppose that  $\Sigma^+$  is a topologically mixing, one-sided countable Markov shift with (BIP),  $f : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous strictly positive and has a weak entropy gap at infinity. If  $P(-\delta f) = 0$  and  $\mu_{-\delta f}$  is the equilibrium state for  $-\delta f$ , then*

- (i) *If  $v_k = \inf \{ \mu_{-\delta f}(p) \mid p \in \Lambda_k \}$ , then  $\lim_{k \rightarrow \infty} v_k = 0$ .*
- (ii) *For any  $p \in \Lambda_k$  and  $n \geq k$  there exists a bijection*

$$\Psi_p^n : \text{Fix}^n \cap p \rightarrow \sigma^{-n}(z_p) \cap p.$$



(iii) *There exists a sequence  $\{\epsilon_k\}$  such that  $\lim \epsilon_k = 0$  and if  $y \in \text{Fix}^n \cap p$  and  $n \geq k$ , then*

$$|S_n f(y) - S_n f(\Psi_p^n(y))| \leq \epsilon_k.$$

(iv) *If  $n \geq k$ , then*

$$(5.1) \quad \sum_{p \in \Lambda_k} W(n, p, t - \epsilon_k) \leq \#\mathcal{M}_f(n, t) \leq \sum_{p \in \Lambda_k} W(n, p, t + \epsilon_k).$$

*Moreover, for all  $k \in \mathbb{N}$  and  $s \in (d(f), \delta)$ , there exists  $C(k, s) > 0$  such that for any  $n < k$  and  $t > 0$ ,*

$$\sum_{p \in \Lambda_k} W(n, p, t) \leq C(k, s)e^{st} \quad \text{and} \quad \#\mathcal{M}_f(k, t) \leq C(k, s)e^{st}.$$

*Proof.* Recall that since  $\mu_{-\delta f}$  is a Gibbs state for  $-\delta f$  (see Theorem 2.2) and  $P(-\delta f) = 0$ , there exists  $B > 1$  such that for every  $p \in \Lambda_k$ , and  $x \in p$

$$\mu_{-\delta f}(p) \leq B e^{-\delta S_k f(x)}.$$

Since  $f$  is strictly positive,  $\lim_{k \rightarrow \infty} \inf\{S_k(x) \mid x \in \Sigma^+\} = +\infty$ , so (i) holds.

Given  $p \in \Lambda_k$ , we define an explicit bijection

$$\Psi_p^n : \text{Fix}^n \cap p \rightarrow \sigma^{-n}(z_p) \cap p$$

If  $y = \overline{y_1 y_2 \dots y_n} \in \text{Fix}^n \cap p$ , then let

$$\Psi_p^n(y) = y_1 \cdots y_n z_1 \cdots z_m \cdots.$$

Notice that since  $y_1 = z_1$  and  $\overline{y_1 \cdots y_n} \in \Sigma^+$ , we must have  $t_{y_n y_1} = t_{y_n z_1} = 1$ , so  $\Psi_p^n(y) \in \Sigma^+$ . The map  $\Psi_p^n$  is injective by definition. If  $x \in \sigma^{-n}(z_p) \cap p$ , then, since  $n \geq k$ ,  $x_{n+1} = z_1 = x_1$ , which implies that  $\overline{x_1 \cdots x_n} \in \text{Fix}^n \cap p$ , so  $\Psi_p^n$  is also surjective. Thus, we have established (ii).

Since  $f$  is locally Hölder continuous, there exists  $B > 0$  and  $r \in (0, 1)$  so that

$$|f(x) - f(y)| \leq B r^l$$

if  $x_i = y_i$  for all  $i \leq l$ . Therefore, if  $y \in \text{Fix}^n \cap p$ , then, since  $z_p \in p$ ,  $y_i = \Psi_p^n(y)_i$  for all  $i \leq n+k$ , so

$$|S_n f(y) - S_n f(\Psi_p^n(y))| \leq \epsilon_k = B \sum_{l=k}^{\infty} r^l.$$

The first statement in (iv) follows immediately from (ii) and (iii). Choose  $b \in (d(f), z)$ . Lemma 3.1 implies that there exists  $D$  so that

$$B_1(f, t) = \#\{a \in \mathcal{A} \mid I(f, a) \leq t\} \leq D e^{bt}.$$

If

$$c = c(f) = \inf_{x \in \Sigma^+} f(x) = \inf_{a \in \mathcal{A}} I(f, a) > 0$$

and  $r \in \mathbb{N}$ , then

$$\begin{aligned} B_2(f, rc) &= \#\{(a_1, a_2) \in \mathcal{A} \times \mathcal{A} \mid I(f, a_1) + I(f, a_2) \leq rc\} \\ &\leq \sum_{s=1}^r B_1(f, rc - sc) B_1(f, sc) \leq \sum_{s=1}^r D^2 e^{brc} = r D^2 e^{brc}. \end{aligned}$$

We may use the argument above to inductively show that

$$B_k(f, rc) = \#\left\{(a_i) \in \mathcal{A}^k \mid \sum_{i=1}^k I(f, a_i) \leq rc\right\} \leq r^{k-1} D^k e^{brc}.$$

Notice that

$$\sum_{p \in \Lambda_k} W(n, p, rc) \leq B_n(f, rc) \quad \text{and} \quad \#\mathcal{M}_f(k, rc) \leq B_k(f, rc)$$

so (iv) follows.  $\square$

We set up some convenient notation. If  $x \in \Sigma^+$ , let

$$\mathcal{W}(x, t) = \left\{ y \in \Sigma^+ \mid \sigma^n(y) = x, S_n f(y) \leq t \text{ for some } n \geq 1 \right\}$$

Observe that if  $x$  is not periodic and  $y \in \mathcal{W}(x, t)$ , then there is a unique  $n(y)$  so that  $\sigma^{n(y)}(y) = x$ . If  $x$  is not periodic and  $\epsilon > 0$ , we let

$$\mathcal{W}(x, t, \leq \epsilon) = \left\{ y \in \mathcal{W}(x, t) \mid \left| \frac{t}{n(y)} - \bar{f} \right| \leq \epsilon \right\}, \quad \text{and}$$

$$\mathcal{W}(x, t, > \epsilon) = \left\{ y \in \mathcal{W}(x, t) \mid \left| \frac{t}{n(y)} - \bar{f} \right| > \epsilon \right\} = \mathcal{W}(x, t) - \mathcal{W}(x, t, \leq \epsilon)$$

where  $\bar{f} = \int_{\Sigma^+} f d\mu_{-sf}$ . Moreover, let

$$W(x, t) = \#\mathcal{W}(x, t), \quad W(x, t, < \epsilon) = \#\mathcal{W}(x, t, < \epsilon) \quad \text{and} \quad W(x, t, > \epsilon) = \#\mathcal{W}(x, t, > \epsilon) = W(x, t) - W(x, t, \leq \epsilon).$$

The crucial technical result we need for the proof of our counting result is a uniform bound on the growth of  $W(x, t, > \epsilon)$ .

**Proposition 5.2.** *Suppose that  $\Sigma^+$  is a topologically mixing, one-sided, countable Markov shift with (BIP) and  $f : \Sigma^+ \rightarrow \mathbb{R}$  is a strictly positive, locally Hölder continuous function with a weak entropy gap at infinity. Let  $\delta > d(f)$  be the unique constant such that  $P(-\delta f) = 0$ . Given  $\epsilon > 0$ , there exist  $D > 0$  and  $b < \delta$  so that*

$$W(x, t, > \epsilon) \leq D e^{bt}$$

for any non-periodic  $x \in \Sigma^+$ .

*Proof.* Fix, for the entire proof,  $\epsilon \in (0, \bar{f}/2)$ .

Theorem 2.6 implies that if  $s > d(f)$ , then there exist  $R_s > 0$  and  $\eta_s \in (0, 1)$  so that

$$(5.2) \quad \left\| e^{-nP(-sf)} \mathcal{L}_{-sf}^n \mathbf{1}(x) - h_{-sf}(x) \int \mathbf{1} d\nu_{-sf} \right\| \leq R_s \eta_s^n.$$

If  $s > \delta$ , then  $P(-sf) < 0$ , since  $P(-\delta f) = 0$  and  $s \rightarrow P(-sf)$  is monotone decreasing and continuous on  $(d(f), \infty)$  (by Lemma 3.3). Then, for any  $m \in \mathbb{N}$  and  $t > 0$

$$\begin{aligned} \sum_{n \geq m} \sum_{y \in \sigma^{-n}(x)} \mathbf{1}_{\{S_n f(y) \leq t\}}(y) &\leq \sum_{n \geq m} \sum_{y \in \sigma^{-n}(x)} e^{-s(S_n f(y) - t)} \\ &= e^{st} \sum_{n \geq m} (\mathcal{L}_{-sf}^n \mathbf{1})(x) \\ &\leq e^{st} \sum_{n \geq m} e^{nP(-sf)} (h_{-sf}(x) + R_s \eta_s^n) \\ &\leq e^{st} \left( \frac{e^{mP(-sf)}}{1 - e^{P(-sf)}} (H_s + R_s) \right). \end{aligned}$$

where  $H_s = \sup\{h_{-sf}(x) \mid x \in \Sigma^+\}$ .

If  $\frac{t}{n(y)} - \bar{f} < -\epsilon$ , then  $n(y)\bar{f} > t + n(y)\epsilon$  and  $n(y) > \frac{t}{\bar{f}-\epsilon}$ , so  $n(y)\bar{f} > t(1 + \epsilon_1)$  where  $\epsilon_1 = \frac{\epsilon}{\bar{f}-\epsilon}$ . Given  $t > 0$ , let  $m_t = \left\lfloor \frac{t(1+\epsilon_1)}{\bar{f}} \right\rfloor$ . Then

$$\begin{aligned} \#\{y \in \mathcal{W}(x, t) \mid \frac{t}{n(y)} - \bar{f} < -\epsilon\} &\leq \sum_{n \geq m_t} \sum_{y \in \sigma^{-n}(x)} \mathbf{1}_{\{S_n f(y) \leq t\}}(y) \\ &\leq e^{st} \left( \frac{e^{m_t P(-sf)}}{1 - e^{P(-sf)}} (H_s + R_s) \right). \\ &\leq D_0 e^{st + m_t P(-sf)} \end{aligned}$$

where  $D_0 = D_0(s, f, \epsilon) = \frac{H_s + R_s}{1 - e^{P(-sf)}}$ .

Since  $\frac{d}{ds} \Big|_{s=\delta} P(-sf) = -\bar{f} < 0$  (by Theorem 2.4), we may also choose  $s > \delta$  so that

$$b_0 := s + \frac{1 + \epsilon_1}{\bar{f}} P(-sf) < \delta.$$

Notice that  $b_0$  does depend on  $\epsilon$ .

With this choice of  $s$ ,

$$\#\{y \in \mathcal{W}(x, t) \mid \frac{t}{n(y)} - \bar{f} < -\epsilon\} \leq D_0 e^{b_0 t}.$$

One can similarly show that there exist  $D_1 > 0$  and  $b_1 \in (d(f), \delta)$  so that

$$\#\{y \in \mathcal{W}(x, t) \mid \frac{t}{n(y)} - \bar{f} > \epsilon\} \leq D_1 e^{b_1 t}.$$

(In this case, we choose  $r \in (d(f), \delta)$  so that

$$b_1 := r + \frac{1 - \epsilon_2}{\bar{f}} P(-rf) < \delta$$

where  $\epsilon_2 = \frac{\epsilon}{\bar{f} + \epsilon} > 0$ . We then use Equation (5.2) and an analysis similar to the one above to show that

$$\#\{y \in \mathcal{W}(x, t) \mid \frac{t}{n(y)} - \bar{f} > \epsilon\} \leq D_1 e^{t(r + \frac{1 - \epsilon_2}{\bar{f}} P(-rf))}$$

where  $D_1 = D_1(r, f, \epsilon) = e^{P(-rf)}(H_r + R_r)$ .

So,

$$W(x, t, > \epsilon) \leq D_0 e^{b_0 t} + D_1 e^{b_1 t} \leq D e^{bt}$$

where  $D = D_0 + D_1$  and  $b = \max\{b_1, b_2\} < \delta$ .  $\square$

**Corollary 5.3.** *Suppose that  $\Sigma^+$  is a topologically mixing, one-sided, countable Markov shift with (BIP) and  $f : \Sigma^+ \rightarrow \mathbb{R}$  is a strictly positive, locally Hölder continuous function with a weak entropy gap at infinity. Let  $\delta > d(f)$  be the unique constant such that  $P(-\delta f) = 0$ . Then, given any  $\epsilon > 0$ , there exists a  $a > 0$  so that*

(1) *There exists  $\hat{D} > 0$  so that*

$$\frac{W(x, t, > \epsilon)}{W(x, t)} \leq \hat{D} e^{-at}$$

*for any non-periodic  $x \in \Sigma^+$ .*

(2) *Given any cylinder  $p$ , there exists  $D_p$  so that*

$$\frac{\#(\mathcal{W}(x, t, > \epsilon) \cap p)}{\#(\mathcal{W}(x, t) \cap p)} \leq D_p e^{-at}$$

*for any non-periodic  $x \in \Sigma^+$ .*

*Proof.* By Corollary 4.4 we can apply the Renewal Theorem with  $\phi = \mathbf{1}$  to see that

$$(5.3) \quad N_f(\mathbf{1}, x, t) = W(x, t) + 1 = \sum_{n \geq 0} \sum_{\sigma^n(y)=x} \mathbf{1}_{\{S_n f(y) \leq t\}}(y) \sim \frac{h_{-\delta f}(x)}{\delta \bar{f}} e^{t\delta}$$

uniformly in  $x \in \Sigma^+$ , where  $\sim$  indicates that the ratio goes to 1 as  $t \rightarrow \infty$ . Since there exist  $b < \delta$  and  $D > 0$  so that  $W(x, t, > \epsilon) \leq D e^{bt}$ , (1) holds with  $a = \delta - b$  and some  $\hat{D} > 0$ .

We can similarly apply the Renewal Theorem with  $\phi = \mathbf{1}_p$  to conclude that

$$N_f(\mathbf{1}_p, x, t) = \#(\mathcal{W}(x, t) \cap p) + 1 = \sum_{n \geq 0} \sum_{\sigma^n(y)=x} \mathbf{1}_p \mathbf{1}_{\{S_n f(y) \leq t\}}(y) \sim \frac{\nu(p) h_{-\delta f}(x)}{\delta \bar{f}} e^{t\delta}$$

uniformly in  $x \in \Sigma^+$ . Since  $\nu(p) > 0$  and

$$\#(\mathcal{W}(x, t, > \epsilon) \cap p) \leq W(x, t, > \epsilon) \leq D e^{bt},$$

(2) holds for some  $D_p$  depending on the cylinder  $p$ .  $\square$

The following result will allow us to bound the error terms in our approximations. Given  $T > 0$ , let

$$P_T^k = \{p \in \Lambda_k \mid S_k f(z_p) \leq T\} \quad \text{and} \quad Q_T^k = \Lambda_k - P_T^k.$$

Notice that  $P_T^k$  is finite for all  $k$  and  $T$ .

**Corollary 5.4.** *Suppose that  $\Sigma^+$  is a topologically mixing, one-sided, countable Markov shift with (BIP) and  $f : \Sigma^+ \rightarrow \mathbb{R}$  is a strictly positive, locally Hölder continuous function with a weak entropy gap at infinity. Let  $\delta > d(f)$  be the unique constant such that  $P(-\delta f) = 0$ .*

(1) *There exists  $G > 0$  so that*

$$\sum_{n \geq 1} \sum_{\{y \in \sigma^{-n}(x)\}} \frac{1}{n} \mathbf{1}_{\{S_n f(y) \leq t\}}(y) \leq G \frac{e^{t\delta}}{t}$$

*for any  $x \in \Sigma^+$  and all  $t > 0$ .*

(2) *If  $k \in \mathbb{N}$  and  $t > T > 0$ , then*

$$\sum_{n > k} \sum_{\{y \in \sigma^{-n}(x)\}} \frac{1}{n} \mathbf{1}_{Q_T^k}(y) \mathbf{1}_{\{S_n f(y) \leq t\}}(y) \leq G e^{-T\delta} \frac{e^{t\delta}}{t - T}.$$

*Proof.* Fix some  $\epsilon > 0$ . Recall from Lemma 4.3 that  $W(x, t) \leq C e^{t\delta}$  for all  $x \in \Sigma^+$ . Then

$$\begin{aligned} \sum_{n \geq 1} \sum_{y \in \sigma^{-n}(x)} \frac{1}{n} \mathbf{1}_{\{S_n f(y) \leq t\}}(y) &= \sum_{y \in \mathcal{W}(x, t, \leq \epsilon)} \frac{1}{n(y)} + \sum_{y \in \mathcal{W}(x, t, > \epsilon)} \frac{1}{n(y)} \\ &\leq \sum_{y \in \mathcal{W}(x, t, \leq \epsilon)} \left( \frac{\bar{f} + \epsilon}{t} \right) \mathbf{1}(y) + \sum_{y \in \mathcal{W}(x, t, > \epsilon)} \mathbf{1}(y). \\ &\leq C e^{t\delta} \left( \frac{\bar{f} + \epsilon}{t} \right) + (\hat{D} e^{-at}) C e^{t\delta}. \end{aligned}$$

So, (1) holds for some  $G > 0$ .

Now notice that

$$\begin{aligned}
\sum_{n>k} \sum_{y \in \sigma^{-n}(x)} \frac{1}{n} \mathbf{1}_{Q_T^k}(y) \mathbf{1}_{\{S_n f(y) \leq t\}}(y) &\leq \sum_{n>k} \frac{1}{n} \sum_{y \in \sigma^{k-n}(x)} \mathbf{1}_{\{S_{n-k} f(y) \leq t-T\}}(y) \\
&= \sum_{m \geq 1} \sum_{w \in \sigma^{-m}(x)} \frac{1}{m+k} \mathbf{1}_{\{S_m f(w) \leq t-T\}}(w) \\
&\leq \sum_{m \geq 1} \sum_{w \in \sigma^{-m}(x)} \frac{1}{m} \mathbf{1}_{\{S_m f(w) \leq t-T\}}(w) \\
&\leq G e^{-\delta T} \frac{e^{t\delta}}{t-T}
\end{aligned}$$

which completes the proof of (2).  $\square$

## 6. COUNTING

*Proof of Theorem A.* First notice that Lemma 3.2 implies that we may assume that  $f$  is strictly positive and has a weak entropy gap at infinity. We simplify notation by setting  $\mu = \mu_{-\delta f}$ ,  $\nu = \nu_{-\delta f}$ ,  $h = h_{-\delta f}$ , and  $\bar{f} = \int f d\mu$ .

Suppose that  $p \in \Lambda_k$ . Corollary 4.4 implies that we can apply the Renewal Theorem (Theorem 4.1) with  $\phi = \mathbf{1}_p$ . Therefore,

$$L(p, t) := \#(\mathcal{W}(z_p, t) \cap p) = \sum_{n \geq 1} \sum_{y \in \sigma^{-n}(z_p)} \mathbf{1}_p(y) \mathbf{1}_{\{S_n f(y) \leq t\}}(y) \sim C(p) e^{t\delta}$$

where

$$C(p) = \frac{h(z_p) \nu(p)}{\delta \bar{f}}.$$

Fix, for the moment,  $p \in \Lambda_k$ . We define

$$\widehat{L}(p, t) := \sum_{n \geq 1} \frac{1}{n} W(n, p, t) = \sum_{y \in \mathcal{W}(z_p, t)} \frac{1}{n(y)} \mathbf{1}_p(y).$$

Then

$$\begin{aligned}
\widehat{L}(p, t) &= \sum_{y \in \mathcal{W}(z_p, t, \leq \epsilon)} \frac{1}{n(y)} \mathbf{1}_p(y) + \sum_{y \in \mathcal{W}(z_p, t, > \epsilon)} \frac{1}{n(y)} \mathbf{1}_p(y) \\
&\leq \sum_{y \in \mathcal{W}(z_p, t, \leq \epsilon)} \left( \frac{\bar{f} + \epsilon}{t} \right) \mathbf{1}_p(y) + \sum_{y \in \mathcal{W}(z_p, t, > \epsilon)} \mathbf{1}_p(y).
\end{aligned}$$

Since, by Corollary 5.3,

$$\#(\mathcal{W}(z_p, t, > \epsilon) \cap p) \leq D_p e^{-at} \#(\mathcal{W}(z_p, t) \cap p)$$

for some  $D_p, a > 0$ , it follows that

$$\limsup_{t \rightarrow \infty} \frac{t \widehat{L}(p, t)}{L(p, t)} \leq \bar{f} + \epsilon.$$

Similarly,

$$\widehat{L}(p, t) = \sum_{n \geq 1} \frac{1}{n} W(n, p, t) \geq \sum_{y \in \mathcal{W}(z_p, t, \leq \epsilon)} \left( \frac{\bar{f} - \epsilon}{t} \right) \mathbf{1}_p(y)$$

so

$$\liminf_{t \rightarrow \infty} \frac{t \widehat{L}(p, t)}{L(p, t)} \geq \bar{f} - \epsilon.$$

By letting  $\epsilon \rightarrow 0$ , we see that

$$\widehat{L}(p, t) \sim \frac{\bar{f}L(p, t)}{t} \sim \frac{C(p)\bar{f}}{t}e^{t\delta}.$$

Now suppose that  $P$  is a subset of  $\Lambda_k$  and define

$$L(P, t) = \sum_{p \in P} L(p, t) \quad \text{and} \quad \widehat{L}(P, t) = \sum_{p \in P} \widehat{L}(p, t).$$

The above analysis implies that if  $P$  is **finite**, then

$$L(P, t) \sim \sum_{p \in P} C(p)e^{t\delta} \quad \text{and} \quad \widehat{L}(P, t) \sim \sum_{p \in P} \frac{C(p)\bar{f}}{t}e^{t\delta}.$$

Notice that if  $T > 0$  and  $t > T$ , then Corollary 5.4 and Lemma 5.1 imply that there exists  $C_k > 0$  so that

$$\frac{t\widehat{L}(P_T^k, t)}{e^{t\delta}} \leq \frac{t\widehat{L}(\Lambda_k, t)}{e^{t\delta}} \leq \frac{t\widehat{L}(P_T^k, t)}{e^{t\delta}} + tC_k e^{(s-\delta)t} + Ge^{-\delta T} \frac{t}{t-T}$$

for some  $s \in (d(f), \delta)$ , so

$$\bar{f} \sum_{p \in P_T^k} C(p) \leq \liminf_{t \rightarrow \infty} \frac{t\widehat{L}(\Lambda_k, t)}{e^{t\delta}} \leq \limsup_{t \rightarrow \infty} \frac{t\widehat{L}(\Lambda_k, t)}{e^{t\delta}} \leq \bar{f} \sum_{p \in P_T^k} C(p) + Ge^{-\delta T}$$

Applying the above inequality to the sequence  $\{P_T^k\}_{T \in \mathbb{N}}$ , we conclude that

$$\widehat{L}(\Lambda_k, t) \sim \sum_{p \in \Lambda_k} \frac{C(p)\bar{f}}{t}e^{t\delta}.$$

Lemma 5.1 implies that, given  $k \in \mathbb{N}$  there exists  $s < \delta$  and  $C_k > 0$ , so that

$$\sum_{p \in \Lambda_k} \sum_{n=1}^k \frac{1}{n} W(n, p, t) \leq C_k e^{st} \quad \text{and} \quad \sum_{n=1}^k \frac{1}{n} \#(\mathcal{M}_f(n, t)) \leq C_k e^{st}$$

and

$$\sum_{p \in \Lambda_k} \sum_{n=k}^{\infty} \frac{1}{n} W(n, p, t - \epsilon_k) \leq \sum_{n=k}^{\infty} \frac{1}{n} \#(\mathcal{M}_f(n, t)) \leq \sum_{p \in \Lambda_k} \sum_{n=k}^{\infty} \frac{1}{n} W(n, p, t + \epsilon_k).$$

Therefore, recalling that  $M_f(t) = \sum_{n \geq 1} \frac{1}{n} \#(\mathcal{M}_f(n, t))$ , we see that

$$\widehat{L}(\Lambda_k, t - \epsilon_k) - C_k e^{st} \leq M_f(t) \leq \widehat{L}(\Lambda_k, t + \epsilon_k) + C_k e^{st},$$

so

$$e^{-\delta\epsilon_k} \bar{f} \sum_{p \in \Lambda_k} C(p) \leq \liminf_{t \rightarrow \infty} \frac{tM_f(t)}{e^{t\delta}} \leq \limsup_{t \rightarrow \infty} \frac{tM_f(t)}{e^{t\delta}} \leq e^{\delta\epsilon_k} \bar{f} \sum_{p \in \Lambda_k} C(p)$$

Since  $h$  is bounded and continuous and  $v_k = \sup\{\mu(p) \mid p \in \Lambda_k\} \rightarrow 0$  as  $k \rightarrow \infty$ , by Lemma 5.1 (i),

$$\sum_{p \in \Lambda_k} C(p) = \frac{1}{\delta\bar{f}} \sum_{p \in \Lambda_k} h(z_p) \nu(p) \rightarrow \frac{\int h \, d\nu}{\delta\bar{f}} = \frac{1}{\delta\bar{f}}.$$

as  $k \rightarrow \infty$ . Moreover,  $\lim \epsilon_k = 0$ . So, finally, we may conclude that

$$M_f(t) \sim \frac{e^{t\delta}}{t\delta}$$

as desired.  $\square$

## 7. EQUIDISTRIBUTION

We are almost ready to prove our equidistribution result, but first we must develop one more bound in the spirit of [34, Theorem 6].

**7.1. Preparing to equidistribute.** Suppose that  $f : \Sigma^+ \rightarrow \mathbb{R}$  and  $g : \Sigma^+ \rightarrow \mathbb{R}$  are both strictly positive,  $f$  has a weak entropy gap at infinity and  $P(-\delta f) = 0$ . We simplify notation, throughout the section, by letting  $\mu = \mu_{-\delta f}$  denote the equilibrium state of  $-\delta f$  and setting  $\bar{f} := \int f d\mu$  and  $\bar{g} := \int g d\mu$ . Since  $f$  and  $g$  are strictly positive,

$$c(f) = \inf\{f(x) \mid x \in \Sigma^+\} > 0 \quad \text{and} \quad c(g) = \inf\{g(x) \mid x \in \Sigma^+\} > 0.$$

**Proposition 7.1.** *Suppose that  $\Sigma^+$  is a topologically mixing, one-sided, countable Markov shift with (BIP) and  $f : \Sigma^+ \rightarrow \mathbb{R}$  is a strictly positive, locally Hölder continuous function with a weak entropy gap at infinity. Let  $\delta > d(f)$  be the unique constant such that  $P(-\delta f) = 0$ . Further suppose that  $g : \Sigma^+ \rightarrow \mathbb{R}$  is strictly positive and that there exists  $C > 0$  so that  $|f(x) - g(x)| \leq C$  for all  $x \in \Sigma^+$ . Given  $\epsilon > 0$ , there exist  $A > 0$  and  $a < \delta$  so that*

$$\#\left\{y \in \mathcal{W}(x, t) : \left| \frac{S_n g(y)}{n(y)} - \bar{g} \right| > \epsilon, \left| \frac{t}{n(y)} - \bar{f} \right| \leq \epsilon \right\} \leq A e^{at}$$

for any non-periodic  $x \in \Sigma^+$ .

*Proof.* Fix  $\epsilon > 0$ . We may assume that  $\epsilon < \min\{c(f), c(g)\}$ .

If  $\frac{S_n g(y)}{n(y)} - \bar{g} < -\epsilon$ , then  $S_n g(y) < n(y)\bar{g} - n(y)\epsilon$ . If, in addition,  $\left| \frac{t}{n(y)} - \bar{f} \right| \leq \epsilon$ , then  $t \leq n(y)(\bar{f} + \epsilon)$ , so

$$S_n g(y) < n(y)\bar{g} - n(y)\epsilon \leq n(y)\bar{g} - n(y)\frac{\epsilon}{2} - \frac{t\epsilon}{2(\bar{f} + \epsilon)} \leq n(y)(\bar{g} - \epsilon_3) - t\epsilon_3$$

where  $\epsilon_3 = \max\{\frac{\epsilon}{2}, \frac{\epsilon}{2(\bar{f} + \epsilon)}\} > 0$ .

Proposition 3.6 implies that  $s \rightarrow P(-sg - \delta f)$  is monotone decreasing and well-defined on  $(d(f) - \delta, \infty)$ . So, if  $s > 0$ , then  $P(-sg - \delta f) < 0$ . Moreover, there exist an equilibrium state  $\mu_{-sg - \delta f}$  for  $-sg - \delta f$  and an eigenfunction  $h_{-sg - \delta f}$  for  $\mathcal{L}_{-sg - \delta f}$  with eigenvalue  $e^{P(-sg - \delta f)} < 1$ . Furthermore, since  $\frac{d}{ds}|_{s=0} P(-sg - \delta f) = -\bar{g} < 0$  (by Theorem 2.4) we may choose  $s > 0$  so that

$$-d_0 := s(\bar{g} - \epsilon_3) + P(-sg - \delta f) < 0.$$

Theorem 2.6 implies that there exist  $\bar{R}_s > 0$  and  $\bar{\eta}_s \in (0, 1)$  so that

$$(7.1) \quad \left\| e^{-nP(-sg - \delta f)} \mathcal{L}_{-sg - \delta f} \mathbf{1} - h_{-sg - \delta f}(x) \int \mathbf{1} d\nu_{-sg - \delta f} \right\| \leq \bar{R}_s \bar{\eta}_s^n$$

for all  $n \in \mathbb{N}$ . Therefore,

$$\begin{aligned} \#\left\{y \in \mathcal{W}(x, t) : \frac{S_n g(y)}{n(y)} - \bar{g} < -\epsilon, \left| \frac{t}{n(y)} - \bar{f} \right| \leq \epsilon \right\} &\leq \sum_{n \geq 0} \sum_{\sigma^n(y)=x} \mathbf{1}_{\{y \mid S_n g(y) \leq n \cdot (\bar{g} - \epsilon_3) - t\epsilon_3, S_n f(y) \leq t\}}(y) \\ &\leq \sum_{n \geq 0} \sum_{\sigma^n(y)=x} e^{-s(S_n g(y) - n(\bar{g} - \epsilon_3) + t\epsilon_3) - \delta(S_n f(y) - t)} \\ &= e^{t\delta - st\epsilon_3} \sum_{n \geq 0} e^{n(s(\bar{g} - \epsilon_3) + P(-sg - \delta f))} \left( e^{-nP(-sg - \delta f)} \mathcal{L}_{-sg - \delta f}^n \mathbf{1} \right) \\ &\leq e^{t\delta - st\epsilon_3} \sum_{n \geq 0} (h_{-sg - \delta f}(x) + \bar{R}_s \bar{\eta}_s^n) e^{-nd_0} \\ &\leq D_0 e^{t\delta - st\epsilon_3} \end{aligned}$$

for all  $x \in \Sigma^+$ , and some  $D_0 > 0$  (which depends on  $\epsilon, s, g$  and  $f$ ).

One may similarly show that there exist  $\epsilon_4 > 0$ ,  $r < 0$  and  $D_1 > 0$  so that

$$\#\left\{y \in \mathcal{W}(x, t) : \frac{S_n g(y)}{n(y)} - \bar{g} > \epsilon, \left| \frac{t}{n(y)} - \bar{f} \right| \leq \epsilon \right\} \leq D_1 e^{t\delta + rt\epsilon_4}.$$

Therefore, our result holds with  $A = D_0 + D_1$  and  $a = \max\{\delta - s\epsilon_3, \delta + r\epsilon_4\}$ .  $\square$

**7.2. Proof of Theorem B.** Lemma 3.2 again implies that we may assume that  $f$  and  $g$  are strictly positive and  $f$  has a weak entropy gap at infinity. Recall, from Lemma 5.1, that there exists a sequence  $\{\epsilon_k\}$  so that  $\lim \epsilon_k = 0$ , and, for any  $p \in \Lambda_k$  and  $n \geq k$ , there exists a bijection

$$\Psi_p^n : \text{Fix}^n \cap p \rightarrow \sigma^{-n}(z_p) \cap p$$

so that

$$|S_n f(x) - S_n f(\Psi_p^n(x))| \leq \epsilon_k \quad \text{and} \quad |S_n g(x) - S_n g(\Psi_p^n(x))| \leq \epsilon_k$$

for all  $x \in \text{Fix}^n \cap p$ . Since  $\lim \epsilon_k = 0$ , there exists  $k_0$  so that if  $n \geq k \geq k_0$ , then

$$c = \min\{c(f), c(g)\} > 2\epsilon_k.$$

We assume from now on that  $k \geq k_0$ . Then, if  $p \in \Lambda_k$

$$(7.2) \quad \sum_{n \geq k} \frac{1}{n} \sum_{\substack{x \in \text{Fix}^n \cap p \\ S_n f(x) \leq t}} \frac{S_n g(x)}{S_n f(x)} \leq \sum_{y \in \mathcal{W}(z_p, t + \epsilon_k) \cap p} \frac{1}{n(y)} \left( \frac{S_n g(y) + \epsilon_k}{S_n f(y) - \epsilon_k} \right) \mathbf{1}_{\{n(y) \geq k\}}(y)$$

and

$$(7.3) \quad \sum_{n \geq k} \frac{1}{n} \sum_{\substack{x \in \text{Fix}^n \cap p \\ S_n f(x) \leq t}} \frac{S_n g(x)}{S_n f(x)} \geq \sum_{y \in \mathcal{W}(z_p, t - \epsilon_k) \cap p} \frac{1}{n(y)} \left( \frac{S_n g(y) - \epsilon_k}{S_n f(y) + \epsilon_k} \right) \mathbf{1}_{\{n(y) \geq k\}}(y).$$

Since there exists  $C > 0$  so that  $|f(x) - g(x)| \leq C$  for all  $x \in \Sigma^+$ ,  $S_n(y)f(y) \geq cn(y)$  for all  $y \in \Sigma^+$  and  $c > 2\epsilon_k$ , we see that

$$\frac{S_n g(y)}{S_n f(y)} \leq \frac{nC + S_n f(y)}{S_n f(y)} \leq \hat{C} = \frac{C}{c} + 1 \quad \text{and} \quad \frac{S_n(y)g(y) + \epsilon_k}{S_n(y)f(y) - \epsilon_k} \leq 3\hat{C}.$$

Let

$$\mathcal{V}(x, t, \leq \epsilon) = \left\{ y \in \mathcal{W}(x, t) : \left| \frac{S_n f(y)}{n(y)} - \bar{f} \right| \leq \epsilon, \left| \frac{S_n g(y)}{n(y)} - \bar{g} \right| \leq \epsilon \right\}.$$

Given  $\epsilon > 0$  so that  $2\epsilon + 2\epsilon_k < \bar{f}$ . Proposition 5.2 together with Proposition 7.1, applied to both  $f$  and  $g$ , imply that there exist  $\hat{A} > 0$  and  $\hat{a} < \delta$  so that

$$\#(\mathcal{W}(x, t) \setminus \mathcal{V}(x, t, \leq \epsilon)) \leq \hat{A}e^{\hat{a}t}$$

for all  $t > 0$ . Further recall that we saw in the proof of Theorem A that

$$\hat{L}(p, t) = \sum_{y \in \mathcal{W}(z_p, t + \epsilon_k) \cap p} \frac{1}{n(y)} \sim C(p) \bar{f} \frac{e^{t\delta}}{t}.$$

Notice that

$$\begin{aligned} U(p, t + \epsilon_k) &:= \sum_{y \in \mathcal{W}(z_p, t + \epsilon_k) \cap p} \frac{1}{n(y)} \left( \frac{S_n(y)g(y) + \epsilon_k}{S_n(y)f(y) - \epsilon_k} \right) \\ &\leq \left( \sum_{y \in \mathcal{V}(z_p, t + \epsilon_k, \leq \epsilon) \cap p} \frac{1}{n(y)} \left( \frac{\bar{g} + \epsilon + \frac{\epsilon_k}{n(y)}}{\bar{f} - \epsilon - \frac{\epsilon_k}{n(y)}} \right) \right) + 3\hat{C} \#(\mathcal{W}(z_p, t + \epsilon_k) \setminus \mathcal{V}(z_p, t + \epsilon_k, \leq \epsilon)) \\ &\quad + 3\hat{C} \sum_{n=1}^{k-1} W(n, p, t) \end{aligned}$$



and recall, from Lemma 5.1, that given  $s \in (d(f), \delta)$ , there exists  $C(k, s)$  so that

$$W(n, p, t) \leq C(k, s)e^{st} \quad \text{and} \quad \#\mathcal{M}_f(t) \leq C(k, s)e^{st}$$

for all  $n < k$ . Therefore,

$$\limsup_{t \rightarrow \infty} \frac{U(p, t + \epsilon_k)}{\widehat{L}(p, t + \epsilon_k)} \leq \frac{\bar{g} + \epsilon + \epsilon_k}{\bar{f} - \epsilon - \epsilon_k}.$$

Letting  $\epsilon \rightarrow 0$ , we see that

$$\limsup_{t \rightarrow \infty} \frac{U(p, t + \epsilon_k)}{\widehat{L}(p, t + \epsilon_k)} \leq \frac{\bar{g} + \epsilon_k}{\bar{f} - \epsilon_k}.$$

We can similarly show that if

$$Z(p, t - \epsilon_k) = \sum_{y \in \mathcal{W}(z_p, t - \epsilon_k)} \frac{1}{n(y)} \mathbf{1}_p(y) \left( \frac{S_n g(y) - \epsilon_k}{S_n f(y) + \epsilon_k} \right),$$

then

$$\liminf_{t \rightarrow \infty} \frac{Z(p, t - \epsilon_k)}{\widehat{L}(p, t - \epsilon_k)} \geq \frac{\bar{g} - \epsilon_k}{\bar{f} + \epsilon_k}.$$

Therefore,

$$\frac{\bar{g} - \epsilon_k}{\bar{f} + \epsilon_k} \leq \liminf_{t \rightarrow \infty} \frac{1}{\widehat{L}(p, t - \epsilon_k)} \sum_{n \geq k} \frac{1}{n} \sum_{\substack{x \in \text{Fix}^n \cap p \\ S_n f(x) \leq t}} \frac{S_n g(x)}{S_n f(x)} \leq \limsup_{t \rightarrow \infty} \frac{1}{\widehat{L}(p, t + \epsilon_k)} \sum_{n \geq k} \frac{1}{n} \sum_{\substack{x \in \text{Fix}^n \cap p \\ S_n f(x) \leq t}} \frac{S_n g(x)}{S_n f(x)} \leq \frac{\bar{g} + \epsilon_k}{\bar{f} - \epsilon_k}.$$

Since  $P_T^k$  is a finite set of cylinders, for any  $T$  and  $k$ , we see that

$$\frac{\bar{g} - \epsilon_k}{\bar{f} + \epsilon_k} \leq \liminf_{t \rightarrow \infty} \frac{1}{\widehat{L}(P_T, t - \epsilon_k)} \sum_{n \geq k} \frac{1}{n} \sum_{\substack{x \in \text{Fix}^n \cap P_T^k \\ S_n f(x) \leq t}} \frac{S_n g(x)}{S_n f(x)} \leq \limsup_{t \rightarrow \infty} \frac{1}{\widehat{L}(P_T^k, t + \epsilon_k)} \sum_{n \geq k} \frac{1}{n} \sum_{\substack{x \in \text{Fix}^n \cap P_T^k \\ S_n f(x) \leq t}} \frac{S_n g(x)}{S_n f(x)} \leq \frac{\bar{g} + \epsilon_k}{\bar{f} - \epsilon_k}.$$

Now notice that if  $t > T > 0$ , Corollary 5.4 implies that

$$(7.4) \quad \sum_{n \geq k} \frac{1}{n} \sum_{\substack{x \in \text{Fix}^n \cap Q_T^k \\ S_n f(x) \leq t}} \frac{S_n g(x)}{S_n f(x)} \leq 3\widehat{C}\widehat{L}(Q_T^k, t) \leq 3\widehat{C}G e^{-\delta T} \frac{e^{t\delta}}{t - T}$$

Therefore, as in the proof of Theorem A, we conclude that

$$\frac{\bar{g} - \epsilon_k}{\bar{f} + \epsilon_k} \leq \liminf_{t \rightarrow \infty} \frac{1}{\widehat{L}(\Lambda_k, t - \epsilon_k)} \sum_{n \geq k} \frac{1}{n} \sum_{\substack{x \in \text{Fix}^n \\ S_n f(x) \leq t}} \frac{S_n g(x)}{S_n f(x)} \leq \limsup_{t \rightarrow \infty} \frac{1}{\widehat{L}(\Lambda_k, t + \epsilon_k)} \sum_{n \geq k} \frac{1}{n} \sum_{\substack{x \in \text{Fix}^n \\ S_n f(x) \leq t}} \frac{S_n g(x)}{S_n f(x)} \leq \frac{\bar{g} + \epsilon_k}{\bar{f} - \epsilon_k}.$$

Recall that  $\lim \epsilon_k = 0$ ,

$$\widehat{L}(\Lambda_k, t - \epsilon_k) - C_k e^{st} \leq M_f(t) \leq \widehat{L}(\Lambda_k, t + \epsilon_k) + C_k e^{st},$$

for all  $t > 0$ , and that

$$\lim_{t \rightarrow \infty} M_f(t) \frac{t\delta}{e^{\delta t}} = 1,$$

so we see that

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\substack{x \in \text{Fix}^n \\ S_n f(x) \leq t}} \frac{S_n g(x)}{S_n f(x)} \sim \frac{\bar{g}}{\bar{f}} \frac{e^{t\delta}}{t\delta}$$

as desired. This completes the proof of Theorem B.

## 8. THE MANHATTAN CURVE

Suppose that  $f : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous, strictly positive and has a strong entropy gap at infinity and that  $g : \Sigma^+ \rightarrow \mathbb{R}$  is also strictly positive and locally Hölder continuous and there exists  $C > 0$  so that  $|f(x) - g(x)| < C$  for all  $x \in \Sigma^+$ . In this case,  $c(f) = \inf\{f(x) \mid x \in \Sigma^+\} > 0$  and  $c(g) = \inf\{g(x) \mid x \in \Sigma^+\} > 0$ .

In this case we define, the enlarged *Manhattan curve*

$$\mathcal{C}_0(f, g) = \{(a, b) \in \mathcal{D}(f, g) \mid P(-af - bg) = 0\}$$

where

$$\mathcal{D}(f, g) = \{(a, b) \in \mathbb{R}^2 \mid ac(f) + bc(g) > 0 \text{ and } a + b > 0\}.$$

Notice that if  $f : \Sigma^+ \rightarrow \mathbb{R}$  and  $g : \Sigma^+ \rightarrow \mathbb{R}$  are both eventually positive and locally Hölder continuous,  $f$  has a strong entropy gap at infinity and there exists  $C$  so that  $|f(x) - g(x)| \leq C$  for all  $x \in \Sigma^+$ , Lemma 3.2 implies that  $f$  and  $g$  are cohomologous to  $\hat{f} : \Sigma^+ \rightarrow \mathbb{R}$  and  $\hat{g} : \Sigma^+ \rightarrow \mathbb{R}$  (respectively) which are both strictly positive and locally Hölder continuous,  $\hat{f}$  has a strong entropy gap at infinity and there exists  $\hat{C}$  so that  $|\hat{f}(x) - \hat{g}(x)| \leq \hat{C}$  for all  $x \in \Sigma^+$ . Since  $\mathcal{C}(f, g) = \mathcal{C}(\hat{f}, \hat{g})$ , Theorem C follows from the following stronger statement for strictly positive functions.

**Theorem C\*:** *Suppose that  $(\Sigma^+, \sigma)$  is a topologically mixing, one-sided countable Markov shift with (BIP),  $f : \Sigma^+ \rightarrow \mathbb{R}$  is locally Hölder continuous, strictly positive and has a strong entropy gap at infinity and  $g : \Sigma^+ \rightarrow \mathbb{R}$  is also strictly positive and locally Hölder continuous. If there exists  $C > 0$  so that  $|f(x) - g(x)| < C$  for all  $x \in \Sigma^+$ , then*

- (1)  $(\delta(f), 0), (0, \delta(g)) \in \mathcal{C}_0(f, g)$ .
- (2) If  $(a, b) \in \mathcal{D}(f, g)$ , there exists a unique  $t > \frac{d(f)}{a+b}$  so that  $(ta, tb) \in \mathcal{C}_0(f, g)$ .
- (3)  $\mathcal{C}_0(f, g)$  is an analytic curve.
- (4)  $\mathcal{C}_0(f, g)$  is strictly convex, unless

$$(8.1) \quad S_n f(x) = \frac{\delta(g)}{\delta(f)} S_n g(x)$$

for all  $x \in \text{Fix}^n$  and  $n \in \mathbb{N}$ .

Moreover, the tangent line to  $\mathcal{C}_0(f, g)$  at  $(a, b)$  has slope

$$s(a, b) = -\frac{\int_{\Sigma^+} g \, d\mu_{-af-bg}}{\int_{\Sigma^+} f \, d\mu_{-af-bg}}.$$

*Proof.* By definition,  $(\delta(f), 0)$  and  $(0, \delta(g))$  lie on  $\mathcal{C}_0(f, g)$  so (1) holds.

Notice that, since  $|I(f, a) - I(g, a)| \leq C$  for all  $a \in \mathcal{A}$ ,  $d(f) = d(g)$  and  $g$  also has a strong entropy gap at infinity. Moreover, if  $(a, b) \in \mathcal{D}(f, g)$ , then  $af + bg$  is strictly positive, has a strong entropy gap at infinity and

$$d(af + bg) = \frac{d(f)}{a + b}.$$

Lemma 3.3 then implies that if  $(a, b) \in \mathcal{D}(f, g)$ , then  $t \rightarrow P(-t(af + bg))$  is proper and strictly decreasing on  $(\frac{d(f)}{a+b}, \infty)$  so there exists a unique  $t > \frac{d(f)}{a+b}$  so that  $P(-t(af + bg)) = 0$ . Thus, (2) holds.

Lemma 3.4 implies that there is an equilibrium state  $\mu_{-af-bg}$  for  $-af - bg$  and that  $\int_{\Sigma^+} (-af - bg) \, d\mu_{-af-bg}$  is finite. Notice that if  $(c, d) \in \mathcal{D}(f, g)$ , then the ratio  $\frac{cf+dg}{af+bg}$  is bounded, this implies that  $\int_{\Sigma^+} (cf + dg) \, d\mu_{-af-bg}$  is also finite. Theorem 2.4 then implies that if  $(a, b) \in \mathcal{D}(f, g)$ , then

$$\frac{\partial}{\partial a} P(-af - bg) = \int_{\Sigma^+} -f \, d\mu_{-af-bg}$$

and

$$\frac{\partial}{\partial b} P(-af - bg) = \int_{\Sigma^+} -g d\mu_{-af-bg}.$$

Since  $f$  is strictly positive,  $\int_{\Sigma^+} -f d\mu_{-af-bg}$  is non-zero, so  $P$  is a submersion on  $\mathcal{D}(f, g)$ . The implicit function theorem then implies that

$$\mathcal{C}_0(f, g) = \{(a, b) \in \mathcal{D}(f, g) \mid P(-af - bg) = 0\}$$

is an analytic curve and that if  $(a, b) \in \mathcal{C}_0(f, g)$  then the slope of the tangent line to  $\mathcal{C}_0(f, g)$  at  $(a, b)$  is given by

$$s(a, b) = -\frac{\int_{\Sigma^+} g d\mu_{-af-bg}}{\int_{\Sigma^+} f d\mu_{-af-bg}}.$$

Since  $P$  is convex, see Sarig [60, Proposition 4.4],  $\mathcal{C}_0(f, g)$  is convex. A convex analytic curve is strictly convex if and only if it is not a line. So it remains to show that  $f$  and  $g$  satisfy equation (8.1) if and only if  $\mathcal{C}_0(f, g)$  is a straight line.

If  $\mathcal{C}_0(f, g)$  is a straight line, then by (1) it has slope  $-\frac{\delta(f)}{\delta(g)}$ . In particular,

$$(8.2) \quad -s(\delta(f), 0) = \frac{\delta(f)}{\delta(g)} = \frac{\int_{\Sigma^+} g d\mu_{-\delta(f)f}}{\int_{\Sigma^+} f d\mu_{-\delta(f)f}} = \frac{\int_{\Sigma^+} g d\mu_{-\delta(g)g}}{\int_{\Sigma^+} f d\mu_{-\delta(g)g}}.$$

By definition,

$$h_\sigma(\mu_{-\delta(g)g}) - \delta(g) \int_{\Sigma^+} g d\mu_{-\delta(g)g} = 0$$

so, applying equation (8.2), we see that

$$h_\sigma(\mu_{-\delta(g)g}) - \delta(f) \int_{\Sigma^+} f d\mu_{-\delta(g)g} = \delta(g) \int_{\Sigma^+} g d\mu_{-\delta(g)g} - \delta(f) \int_{\Sigma^+} f d\mu_{-\delta(g)g} = 0$$

Since  $P(-\delta(f)f) = 0$ , this implies that  $\mu_{-\delta(g)g}$  is an equilibrium state for  $-\delta(f)f$ . Therefore, by uniqueness of equilibrium states we see that  $\mu_{-\delta(f)f} = \mu_{-\delta(g)g}$ . Sarig [60, Thm. 4.8] showed that this only happens when  $-\delta(f)f$  and  $-\delta(g)g$  are cohomologous, so the Livsic Theorem (Theorem 2.1) implies that this occurs if and only if

$$S_n f(x) = \frac{\delta(g)}{\delta(f)} S_n g(x)$$

for all  $x \in \text{Fix}^n$  and  $n \in \mathbb{N}$ . We have completed the proof.  $\square$

## 9. BACKGROUND FOR APPLICATIONS

In this section, we recall the background material that we will need to construct the roof functions described in Theorem D. We will also recall the more general definition of cusped Anosov representations of geometrically finite Fuchsian groups into  $\text{SL}(d, \mathbb{R})$ . In the next section, we will see that Theorem D also extends to this setting.

**9.1. Linear algebra.** It will be useful to first recall some standard Lie-theoretic notation. Let

$$\mathfrak{a} = \{(a_1, \dots, a_d) \in \mathbb{R}^d \mid a_1 + \dots + a_d = 0\}$$

be the standard Cartan algebra for  $\text{SL}(d, \mathbb{R})$  and let

$$\mathfrak{a}^+ = \{(a_1, \dots, a_d) \in \mathfrak{a} \mid a_1 \geq \dots \geq a_d\}$$

be the standard choice of positive Weyl chamber. Let  $\mathfrak{a}^*$  be the space of linear functionals on  $\mathfrak{a}$ . For all  $k \in \{1, \dots, d-1\}$ , let  $\alpha_k : \mathfrak{a} \rightarrow \mathbb{R}$  be given by  $\alpha_k(\vec{a}) = a_k - a_{k+1}$ . Then  $\{\alpha_1, \dots, \alpha_{d-1}\}$  span  $\mathfrak{a}^*$  and are the simple roots determining the Weyl chamber  $\mathfrak{a}^+$ . It is also natural to consider the fundamental weights  $\omega_k \in \mathfrak{a}^*$  given by  $\omega_k(\vec{a}) = a_1 + \dots + a_k$ . Notice that  $\{\omega_1, \dots, \omega_{d-1}\}$  is also a basis for  $\mathfrak{a}^*$ .

If  $A \in \mathrm{SL}(d, \mathbb{R})$ , let

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_d(A)$$

denote the moduli of the generalized eigenvalues of  $A$  and let

$$\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_d(A)$$

be the singular values of  $A$ . The *Jordan projection*

$$\ell : \mathrm{SL}(d, \mathbb{R}) \rightarrow \mathfrak{a}^+ \text{ is given by } \ell(A) = (\log \lambda_1(A), \dots, \log \lambda_d(A))$$

and the *Cartan projection*

$$\kappa : \mathrm{SL}(d, \mathbb{R}) \rightarrow \mathfrak{a}^+ \text{ is given by } \kappa(A) = (\log \sigma_1(A), \dots, \log \sigma_d(A)).$$

If  $\alpha_k(\ell(A)) > 0$ , then there is a well-defined *attracting  $k$ -plane* which is the plane spanned by the generalized eigenspaces with eigenvalues of modulus at least  $\lambda_k(A)$ . Recall that the Cartan decomposition of  $A \in \mathrm{SL}(d, \mathbb{R})$  has the form  $A = KDL$  where  $K, L \in \mathrm{SO}(d)$  and  $D$  is the diagonal matrix with diagonal entries  $d_{ii} = \sigma_i(A)$ . If  $\alpha_k(A) > 0$ , then the  $k$ -flag  $U_k(A) = K(\langle e_1, \dots, e_k \rangle)$  is well-defined, and is the  $k$ -plane spanned by the  $k$  longest axes of the ellipsoid  $A(S^{d-1})$ . (Notice that  $U_k(A)$  is not typically the attracting  $k$ -plane even when  $\alpha_k(\ell(A)) > 0$ .)

**9.2. Cusped Anosov representations of geometrically finite Fuchsian groups.** Suppose that  $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$  is a torsion-free geometrically finite Fuchsian group, which is not convex cocompact, and let  $\Lambda(\Gamma)$  be its limit set in  $\partial\mathbb{H}^2$ .

Consider a representation  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  such that if  $\alpha \in \Gamma$  is parabolic, then  $\rho(\alpha)$  is unipotent. If  $1 \leq k \leq d-1$ , we will say that  $\rho$  is *cusped  $P_k$ -Anosov* if there exist continuous  $\rho$ -equivariant maps  $\xi_\rho^k : \Lambda(\Gamma) \rightarrow \mathrm{Gr}_k(\mathbb{R}^d)$  and  $\xi_\rho^{d-k} : \Lambda(\Gamma) \rightarrow \mathrm{Gr}_{d-k}(\mathbb{R}^d)$  so that

- (1)  $\xi_\rho^k$  and  $\xi_\rho^{d-k}$  are *transverse*, i.e. if  $x \neq y \in \Lambda(\Gamma)$ , then

$$\xi_\rho^k(x) \oplus \xi_\rho^{d-k}(y) = \mathbb{R}^d.$$

- (2)  $\xi_\rho^k$  and  $\xi_\rho^{d-k}$  are *strongly dynamics preserving*, i.e. if  $j$  is  $k$  or  $d-k$  and  $\{\gamma_n\}$  is a sequence in  $\Gamma$  so that  $\gamma_n(0) \rightarrow x \in \Lambda(\Gamma)$  and  $\gamma_n^{-1}(0) \rightarrow y \in \Lambda(\Gamma)$ , then if  $V \in \mathrm{Gr}_j(\mathbb{R}^d)$  and  $V$  is transverse to  $\xi_\rho^{d-j}(y)$ , then  $\rho(\gamma_n)(V) \rightarrow \xi_\rho^j(x)$ .

The original definition of a cusped  $P_k$ -Anosov representation in [13] is given in terms of a flow space, as in Labourie's original definition [32]. The characterization we give here is a natural generalization of characterizations of Guéritaud-Guichard-Kassel-Wienhard [21], Kapovich-Leeb-Porti [29] and Tsouvalas [66] in the traditional setting. Our cusped  $P_k$ -Anosov representations are examples of the relatively Anosov representations considered by Kapovich-Leeb [28] and the relatively dominated representations considered by Zhu [68].

The following crucial properties of cusped  $P_k$ -Anosov representations are established in Canary-Zhang-Zimmer [13]. (Several of these properties also follow from work of Kapovich-Leeb [28] and Zhu [68] once one establishes that our representations fit into their framework.) If  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  is cusped  $P_k$ -Anosov, we define the space of type-preserving deformations

$$\mathrm{Hom}_{tp}(\rho) \subset \mathrm{Hom}(\Gamma, \mathrm{SL}(d, \mathbb{R}))$$

to be the space of representations  $\sigma$  such that if  $\alpha \in \Gamma$  is parabolic, then  $\sigma(\alpha)$  is conjugate to  $\rho(\alpha)$ .

**Theorem 9.1.** (Canary-Zhang-Zimmer [13]) *If  $\Gamma$  is a geometrically finite Fuchsian group and  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  is a cusped  $P_k$ -Anosov representation, then*

- (1) *There exist  $A, a > 0$  so that if  $\gamma \in \Gamma$ , then*

$$Ae^{ad(b_0, \gamma(b_0))} \geq e^{\alpha_k(\kappa(\rho(\gamma)))} \geq \frac{1}{A} e^{\frac{d(b_0, \gamma(b_0))}{a}}$$

where  $b_0$  is a basepoint for  $\mathbb{H}^2$ .

(2) There exist  $B, b > 0$  so that if  $\gamma \in \Gamma$ , then

$$Be^{bt(\gamma)} \geq e^{\alpha_k(\ell(\rho(\gamma)))} \geq \frac{1}{B} e^{\frac{t(\gamma)}{b}}$$

where  $t(\gamma)$  is the translation length of  $\gamma$  on  $\mathbb{H}^2$ .

(3) The limit maps  $\xi_\rho^k$  and  $\xi_\rho^{d-k}$  are Hölder continuous.

(4) There exists an open neighborhood  $U$  of  $\rho$  in  $\text{Hom}_{\text{tp}}(\rho)$ , so that if  $\sigma \in U$ , then  $\sigma$  is cusped  $P_k$ -Anosov.

(5) If  $v \in \Gamma$  is parabolic and  $j \in \{1, \dots, d-1\}$ , then there exists  $c_j(\rho, v) \in \mathbb{Z}$  and  $C_j(\rho, v) > 0$  so that

$$|\alpha_j(\kappa(\rho(v^n))) - c_j(\rho, v) \log n| < C_j(\rho, v)$$

for all  $n \in \mathbb{N}$ . Moreover, if  $\eta \in \text{Hom}_{\text{tp}}(\rho)$ , then  $c_j(\rho, v) = c_j(\eta, v)$ .

(6)  $\rho$  has the  $P_k$ -Cartan property, i.e. whenever  $\{\gamma_n\}$  is a sequence of distinct elements of  $\Gamma$  such that  $\gamma_n(b_0)$  converges to  $z \in \Lambda(\Gamma)$ , then  $\xi_\rho^k(z) = \lim U_k(\rho(\gamma_n))$ .

(7)  $\rho$  is  $P_{d-k}$ -Anosov.

**9.3. Cusped Hitchin representations.** Canary, Zhang and Zimmer [13] also prove that cusped Hitchin representations are cusped  $P_k$ -Anosov for all  $k$ , i.e they are cusped Borel Anosov, in analogy with work of Labourie [32] in the uncusped case. We say that  $A \in \text{SL}(d, \mathbb{R})$  is *unipotent and totally positive* with respect to a basis  $b = (b_1, \dots, b_d)$  for  $\mathbb{R}^d$ , if its matrix representative with respect to this basis is unipotent, upper triangular, and all the minors which could be positive are positive. Let  $U_{>0}(b)$  denote the set of all such maps. One crucial property here is that  $U_{>0}(b)$  is a semi-group (see Lusztig [39]).

We say that a basis  $b = (b_1, \dots, b_d)$  is *consistent* with a pair  $(F, G)$  of transverse flags if  $\langle b_i \rangle = F^i \cap G^{d-i+1}$  for all  $i$ . A  $k$ -tuple  $(F_1, \dots, F_k)$  in  $\mathcal{F}_d$  is *positive* if there exists a basis  $b$  consistent with  $(F_1, F_k)$  and there exists  $\{u_2, \dots, u_k\} \in U(b)_{>0}$  so that  $F_i = u_i \cdots u_2 F_1$  for all  $i = 2, \dots, k$ .

If  $X$  is a subset of  $S^1$ , we say that a map  $\xi : X \rightarrow \mathcal{F}_d$  is *positive* if whenever  $(x_1, \dots, x_k)$  is a consistently ordered  $k$ -tuple in  $X$  (ordered either clockwise or counter-clockwise), then  $(\xi(x_1), \dots, \xi(x_k))$  is a positive  $k$ -tuple of flags.

A *cusped Hitchin representation* is a representation  $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$  such that if  $\gamma \in \Gamma$  is parabolic, then  $\rho(\gamma)$  is a unipotent element with a single Jordan block and there exists a  $\rho$ -equivariant positive map  $\xi_\rho : \Lambda(\Gamma) \rightarrow \mathcal{F}_d$ . (In fact, it suffices to define  $\xi_\rho$  on the subset  $\Lambda_{\text{per}}(\Gamma)$  consisting of fixed points of peripheral elements of  $\Gamma$ .)

**Theorem 9.2.** (Canary-Zhang-Zimmer [13]) *If  $\Gamma$  is a geometrically finite Fuchsian group and  $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$  is a cusped Hitchin representation, then*

- (1)  $\rho$  is  $P_k$ -Anosov for all  $1 \leq k \leq d-1$ .
- (2)  $\rho$  is irreducible.
- (3) If  $\alpha \in \Gamma$  is parabolic and  $1 \leq k \leq d-1$ , then  $c_k(\rho, \alpha) = 2$ .

We remark that Sambarino [56] has independently established that  $\rho$  is irreducible and that Kapovich-Leeb indicate in [28] that they can prove  $\rho$  is Borel Anosov.

**9.4. Codings for geometrically finite Fuchsian groups.** A torsion-free convex cocompact Fuchsian group admits a finite Markov shift which codes the recurrent portion of its geodesic flow. The most basic such coding is the Bowen-Series coding [7]. However, if the group is geometrically finite, but not convex cocompact, this coding is not well-behaved. In this case one must instead consider the countable Markov shifts constructed by Dal'bo-Peigné [19], if the quotient has infinite area, and Stadlbauer [64] and Ledrappier-Sarig [36], if the quotient has finite area.

We summarize the crucial properties of these Markov shifts in the following theorem and will give a brief description of each coding.

**Theorem 9.3.** (Dal’bo-Peigné [19], Ledrappier-Sarig [36], Stadlbauer [64]) *Suppose that  $\Gamma$  is a torsion-free geometrically finite, but not cocompact, Fuchsian group. There exists a topologically mixing Markov shift  $(\Sigma^+, \mathcal{A})$  with countable alphabet  $\mathcal{A}$  with (BIP) which codes the recurrent portion of the geodesic flow on  $T^1(\mathbb{H}^2/\Gamma)$ . There exist maps*

$$G : \mathcal{A} \rightarrow \Gamma, \quad \omega : \Sigma^+ \rightarrow \Lambda(\Gamma), \quad r : \mathcal{A} \rightarrow \mathbb{N}, \quad \text{and} \quad s : \mathcal{A} \rightarrow \Gamma$$

with the following properties.

- (1)  $\omega$  is locally Hölder continuous, injective and  $\omega(\Sigma^+) = \Lambda_c(\Gamma)$ , i.e. the complement in  $\Lambda(\Gamma)$  of the set of fixed points of parabolic elements of  $\Gamma$ . Moreover,  $\omega(x) = G(x_1)\omega(\sigma(x))$  for every  $x \in \Sigma^+$ .
- (2) If  $x \in \text{Fix}^n$ , then  $\omega(x)$  is the attracting fixed point of  $G(x_1) \cdots G(x_n)$ . Moreover, if  $\gamma \in \Gamma$  is hyperbolic, then there exists  $x \in \text{Fix}^n$  (for some  $n$ ) so that  $\gamma$  is conjugate to  $G(x_1) \cdots G(x_n)$  and  $x$  is unique up to shift.
- (3) There exists  $Q \in \mathbb{N}$  such that  $1 \leq \#(r^{-1}(n)) \leq Q$  for all  $n \in \mathbb{N}$ .
- (4) There exists a finite collection  $\mathcal{P}$  of parabolic elements of  $\Gamma$ , a finite collection  $\mathcal{R}$  of elements of  $\Gamma$  such that if  $a \in \mathcal{A}$ , then  $s(a) \in \mathcal{P} \cup \{id\}$  and  $G(a) = s(a)^{r(a)-2}g_a$  where  $g_a \in \mathcal{R}$ .
- (5) Given a basepoint  $b_0 \in \mathbb{H}^2$ , there exists  $L > 0$  so that if  $x \in \Sigma^+$  and  $n \in \mathbb{N}$ , then

$$d(G(x_1) \cdots G(x_n)(b_0), \overrightarrow{b_0\omega(x)}) \leq L.$$

If  $\Gamma$  is convex cocompact, then one may use the Bowen-Series [7] coding  $(\Sigma^+, \sigma)$  which we briefly recall to set the scene for the more complicated codings we will need in the non-convex cocompact setting. One begins with a fundamental domain  $D_0$  for  $\Gamma$ , containing the basepoint  $b_0$ , all of whose vertices lie in  $\partial\mathbb{H}^2$ , so that the set of face pairings  $\mathcal{A}$  of  $D_0$  is a minimal symmetric generating set for  $\Gamma$ . The classical Bowen-Series coding on the alphabet  $\mathcal{A}$  can be constructed from a “cutting sequence” which records the intersections  $(t_k)$  of a geodesic ray  $\overrightarrow{b_0z}$  which intersects  $D_0$ , where  $z \in \Lambda(\Gamma)$ , with edges of translates of  $D_0$  so that the geodesic is entering  $\gamma_k(D_0)$  as it passes through  $t_k$ . The classical Bowen-Series coding for  $\overrightarrow{b_0z}$  is given by  $(x_k) = (\gamma_k\gamma_{k-1}^{-1})$ . Each  $\gamma_k\gamma_{k-1}^{-1}$  is a face-pairing, hence this alphabet  $\mathcal{A}$  is a finite generating set for  $\Gamma$ . Thus one obtains a map  $G : \mathcal{A} \rightarrow \Gamma$ , the map  $\omega$  simply takes the word encoding the geodesic ray  $\overrightarrow{b_0z}$  to  $z$ . Moreover,  $r(a) = 1$  and  $s(a) = id$  for all  $a \in \mathcal{A}$ . A word  $x$  in  $\mathcal{A}$  is allowable in this coding if and only if  $G(x_{i+1}) \neq G(x_i)^{-1}$  for any  $i$ .

If  $\Gamma$  is geometrically finite and has infinite area quotient, then we may use the Dal’bo-Peigné coding [19]. Roughly, the Dal’bo-Peigné coding coalesces all powers of a parabolic generator in the Bowen-Series coding. This alteration allows  $\omega$  to be locally Hölder continuous. Here we may begin with fundamental domain  $D_0$  for  $\Gamma$ , containing the origin 0 in the Poincaré disk model, all of whose vertices lie in  $\partial\mathbb{H}^2$ , so that the set of face pairings  $\mathcal{A}_0$  of  $D_0$  is a minimal symmetric generating set for  $\Gamma$  and such that every parabolic element of  $\Gamma$  is conjugate to an element of  $\mathcal{A}_0$ . Let  $\mathcal{P}$  denote the parabolic elements of  $\mathcal{A}_0$ . We let

$$\mathcal{A} = \mathcal{A}_0 \cup \{p^n \mid n \geq 2, p \in \mathcal{P}\}.$$

In all cases,  $G(a) = a$ . If  $a = p^n$  for some  $p \in \mathcal{P}$ , then  $r(a) = n + 1$ ,  $s(a) = p$  and  $g_a = p$ , while if not we set  $r(a) = 1$ ,  $s(a) = id$  and  $g_a = a$ . A word  $x$  in  $\mathcal{A}$  is allowable in this coding if and only if for any  $i$ ,  $G(x_{i+1}) \neq G(x_i)^{-1}$  and if  $s(x_i) \in \mathcal{P}$ , then  $s(x_{i+1}) \notin \{s(x_i), s(x_i)^{-1}\}$ . For a discussion of this coding in our language, see Kao [26].

If  $\Gamma$  is geometrically finite and has a finite area quotient then one cannot use the Dal’bo-Peigné coding, since there is not a minimal symmetric generating set which contains elements conjugate to every primitive parabolic element of  $\Gamma$ . Stadlbauer [64] and Ledrappier-Sarig [36] construct a (more complicated) coding in this setting which has the same flavor and coarse behavior as the Dal’bo-Peigné coding. One begins with a Bowen-Series coding of  $\Gamma$  with alphabet  $\mathcal{A}_0$ . Let  $\mathcal{C}$  denote a set of minimal length conjugates of primitive parabolic elements. They then choose

a sufficiently large even number  $2N$  so that the length of every element of  $\mathcal{C}$  divides  $2N$  and let  $\mathcal{P}$  be the collection of powers of elements of  $\mathcal{C}$  of length exactly  $2N$ . Let  $\mathcal{A}_1$  be the set of all strings  $(b_0, b_1, \dots, b_{2N})$  in  $\mathcal{A}_0$  so that  $b_0 b_1 \cdots b_{2N}$  is freely reduced in  $\mathcal{A}_0$  and so that neither  $b_1 b_2 \cdots b_{2N}$  or  $b_0 b_1 \cdots b_{2N-1}$  lies in  $\mathcal{P}$ . Let  $\mathcal{A}_2$  be the set of all freely reduced strings of the form  $(b, v^t, v_1, \dots, v_{k-1}, c)$  where  $b \in \mathcal{A}_0 - \{v_{2N}\}$ ,  $v = v_1 \cdots v_{2N} \in \mathcal{P}$ ,  $v_i \in \mathcal{A}_0$  for all  $i$ ,  $t \in \mathbb{N}$  and  $c \in \mathcal{A}_0 - \{v_k\}$ . Let  $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ . If  $a = (b_0, b_1, \dots, b_{2N}) \in \mathcal{A}_1$ , then  $G(a) = b_1$ ,  $r(a) = 1$ ,  $s(a) = id$  and  $g_a = b_1$ , while if  $a = (b, v^t, v_1 \cdots v_{k-1}, c)$ , then let  $G(a) = v^{t-1} v_1 \cdots v_{k-1}$ ,  $r(a) = t+1$ ,  $s(a) = v$  and  $g_a = v_1 \cdots v_{k-1}$ . The set of allowable words is defined so that if  $x \in \text{Fix}^n$ , then  $G(x_1) \cdots G(x_n)$  cannot be a parabolic element of  $\Gamma$ . (For a more detailed description see Stadlbauer [64], Ledrappier-Sarig [36] or Bray-Canary-Kao [9].)

**9.5. Busemann and Iwasawa cocycles.** We will use the Busemann cocycle to define our roof functions. We first develop the theory we will need in the simpler case where  $\rho$  is cusped  $P_k$ -Anosov for all  $k$ . This theory will suffice for all our application to cusped Hitchin representations, so one may ignore the discussion of partial flag varieties and partial Iwasawa cocycles on a first reading.

Quint [49] introduced a vector valued smooth cocycle, called the *Iwasawa cocycle*,

$$B : \text{SL}(d, \mathbb{R}) \times \mathcal{F}_d \rightarrow \mathfrak{a}$$

where  $\mathcal{F}_d$  is the space of (complete) flags in  $\mathbb{R}^d$ . Let  $F_0$  denote the standard flag

$$F_0 = (\langle e_1 \rangle, \langle e_1, e_2 \rangle, \dots, \langle e_1, \dots, e_{d-1} \rangle).$$

We can write any  $F \in \mathcal{F}_d$  as  $F = K(F_0)$  where  $K \in \text{SO}(d)$ . If  $A \in \text{SL}(d, \mathbb{R})$  and  $F \in \mathcal{F}_d$ , the Iwasawa decomposition of  $AK$  has the form  $QZU$  where  $Q \in \text{SO}(d)$ ,  $Z$  is a diagonal matrix with non-negative entries, and  $U$  is unipotent and upper triangular. Then  $B(A, F) = (\log z_{11}, \dots, \log z_{dd})$ .

One may check that it satisfies the following cocycle property (see Quint [49, Lemma 6.2]):

$$B(ST, F) = B(S, TF) + B(T, F).$$

If  $A$  is loxodromic (i.e.  $\alpha_k(\ell(A)) > 0$  for all  $k$ ), then the set of attracting  $k$ -planes forms a flag  $F_A$ , called the attracting flag of  $A$ . In this case,

$$(9.1) \quad B(A, F_A) = \ell(A)$$

since if  $F_A = K_A(F_0)$ , then  $AK_A$  is upper triangular and the diagonal entries are the eigenvalues with their moduli in descending order. (See Lemma 7.5 in Sambarino [53].)

The Iwasawa cocycle is also closely related to the singular value decomposition, also known as the Cartan decomposition. If  $A$  is Cartan loxodromic (i.e.  $\alpha_k(\kappa(A)) > 0$  for all  $k$ ), then the flag  $U(A) = \{U_k(A)\}$  is well-defined. If  $W$  is the involution taking  $e_i$  to  $e_{d-i+1}$  and  $A$  has Cartan decomposition  $A = KDL$ , then  $A^{-1}$  has Cartan decomposition

$$A^{-1} = (L^{-1}W) (WD^{-1}W) (WK^{-1}).$$

So if  $S(A) = U(A^{-1})$ , one may check that  $B(A, S(A)) = \kappa(A)$ . Moreover, the Cartan decomposition bounds the Iwasawa cocycle, specifically

$$\|B(A, F)\| \leq \|\kappa(A)\|$$

(see Benoist-Quint [2, Corollary 8.20]).

We will make use of the following close relationship between the Iwasawa cocycle and the Cartan projection.

**Lemma 9.4.** (Quint [49, Lemma 6.5]) *For any  $\epsilon \in (0, 1)$ , there exists  $C > 0$  so that if  $A \in \text{SL}(d, \mathbb{R})$ ,  $F \in \mathcal{F}_d$ ,  $\sigma_k(A) > \sigma_{k+1}(A)$  and  $\angle(F^k, U_{d-k}(A^{-1})) \geq \epsilon$ , then*

$$|\omega_k(B(A, F)) - \omega_k(\kappa(A))| \leq C.$$

Given a representation  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  of a geometrically finite Fuchsian group  $\Gamma$  and a  $\rho$ -equivariant map  $\xi_\rho : \Lambda(\Gamma) \rightarrow \mathcal{F}_d$  we define its associated *Busemann cocycle*

$$\beta_\rho : \Gamma \times \Lambda(\Gamma) \rightarrow \mathfrak{a}$$

by letting

$$\beta_\rho(\gamma, x) = B(\rho(\gamma), \rho(\gamma^{-1})(\xi_\rho(x))).$$

The Busemann cocycle was first defined by Quint [49] and was previously used to powerful effect in the setting of uncusped Hitchin representations by Sambarino [54], Martone-Zhang [41] and Potrie-Sambarino [48].

**Lemma 9.5.** *If  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  is a representation of a geometrically finite Fuchsian group  $\Gamma$  and  $\xi_\rho : \Lambda(\Gamma) \rightarrow \mathcal{F}_d$  is a  $\rho$ -equivariant map, then  $\beta_\rho$  satisfies the cocycle property*

$$\beta_\rho(\alpha\gamma, z) = \beta_\rho(\alpha, z) + \beta_\rho(\gamma, \alpha^{-1}(z))$$

for all  $\alpha, \gamma \in \Gamma$  and  $z \in \Lambda(\Gamma)$ .

Moreover, if  $\rho(\gamma)$  is loxodromic and  $\xi_\rho(\gamma^+)$  is the attracting flag of  $\rho(\gamma)$ , then

$$\beta_\rho(\gamma, \gamma^+) = \ell(\rho(\gamma)).$$

*Proof.* First notice that

$$\begin{aligned} \beta_\rho(\alpha\gamma, z) &= B(\rho(\alpha)\rho(\gamma), \rho(\gamma^{-1})\rho(\alpha^{-1})(\xi_\rho(z))) \\ &= B(\rho(\alpha), \rho(\alpha)^{-1}(\xi_\rho(z))) + B(\rho(\gamma), \rho(\gamma^{-1})\rho(\alpha^{-1})(\xi_\rho(z))) \\ &= \beta_\rho(\alpha, z) + \beta_\rho(\gamma, \alpha^{-1}(z)). \end{aligned}$$

Then observe that

$$\beta_\rho(\gamma, \gamma^+) = B(\rho(\gamma), \rho(\gamma^{-1})(\xi_\rho(\gamma^+))) = B(\rho(\gamma), \xi_\rho(\gamma^+)).$$

Since we have assumed that  $\xi_\rho(\gamma^+)$  is the attracting flag of  $\rho(\gamma)$ , we may apply Equation (9.1).  $\square$

We now generalize the theory developed above to the setting of partial flag varieties. If  $\theta = \{i_1 < \dots < i_r\} \subset \{1, \dots, d\}$ , then a  $\theta$ -flag is a nested collection of vector subspaces of dimension  $i_j$  of the form

$$F = \{0 \subset F^{i_1} \subset \dots \subset F^{i_r} \subset \mathbb{R}^d\}.$$

The  $\theta$ -flag variety  $\mathcal{F}_\theta$  is the set of all  $\theta$ -flags. Let

$$\mathfrak{a}_\theta = \{\vec{a} \in \mathfrak{a} \mid \alpha_k(\vec{a}) = 0 \text{ if } k \notin \theta\}.$$

There is a unique projection

$$p_\theta : \mathfrak{a} \rightarrow \mathfrak{a}_\theta$$

invariant by  $\{w \in W : w(\mathfrak{a}_\theta) = \mathfrak{a}_\theta\}$  where  $W$  is the Weyl group acting on  $\mathfrak{a}$  by coordinate permutations. Benoist and Quint [2, Section 8.6] describe a partial Iwasawa cocycle

$$B_\theta : \mathrm{SL}(d, \mathbb{R}) \times \mathcal{F}_\theta \rightarrow \mathfrak{a}_\theta$$

such that  $p_\theta \circ B$  factors through  $B_\theta$ .

We say that  $A \in \mathrm{SL}(d, \mathbb{R})$  is  $\theta$ -proximal if  $\alpha_k(\ell(A)) > 0$  for all  $k \in \theta$ . In this case,  $A$  has a well-defined attracting  $\theta$ -flag  $F_A^\theta$ , and

$$B_\theta(A, F_A^\theta) = p_\theta(\ell(A))$$

In particular,

$$(9.2) \quad \omega_k(B_\theta(A, F_A^\theta)) = \omega_k(\ell(A))$$

for all  $k \in \theta$ .



Given a representation  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  of a geometrically finite Fuchsian group  $\Gamma$  and a  $\rho$ -equivariant map  $\xi_\rho : \Lambda(\Gamma) \rightarrow \mathcal{F}_\theta$  we define its associated  $\theta$ -Busemann cocycle

$$\beta_\rho^\theta : \Gamma \times \Lambda(\Gamma) \rightarrow \mathfrak{a}_\theta$$

by letting

$$\beta_\rho^\theta(\gamma, z) = B_\theta(\rho(\gamma), \rho(\gamma^{-1})(\xi_\rho(z))).$$

Since  $p_\theta$  is linear, Lemma 9.5 immediately generalizes to give

**Lemma 9.6.** *If  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  is a representation of a geometrically finite Fuchsian group  $\Gamma$  and  $\xi : \Lambda(\Gamma) \rightarrow \mathcal{F}_\theta$  is a  $\rho$ -equivariant map, then  $\beta_\rho^\theta$  satisfies the cocycle property*

$$\beta_\rho^\theta(\alpha\gamma, z) = \beta_\rho^\theta(\alpha, z) + \beta_\rho^\theta(\gamma, \alpha^{-1}(z))$$

for all  $\alpha, \gamma \in \Gamma$  and  $z \in \Lambda(\Gamma)$ .

Moreover, if  $\rho(\gamma)$  is  $\theta$ -proximal and  $\xi_\rho(\gamma^+)$  is the attracting  $\theta$ -flag of  $\rho(\gamma)$ , then

$$\beta_\rho^\theta(\gamma, \gamma^+) = p_\theta(\ell(\rho(\gamma))).$$

In particular,

$$\omega_k(\beta_\rho^\theta(\gamma, \gamma^+)) = \omega_k(\ell(\rho(\gamma)))$$

if  $k \in \theta$ .

## 10. ROOF FUNCTIONS FOR ANOSOV REPRESENTATIONS

If  $\theta \subset \{1, \dots, d-1\}$  is non-empty, we will say that  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  is cusped  $\theta$ -Anosov if it is cusped  $P_k$ -Anosov for all  $k \in \theta$ . We say that  $\theta$  is *symmetric* if  $k \in \theta$  if and only if  $d-k \in \theta$ . It will be natural to always assume that  $\theta$  is symmetric, since  $\rho$  is cusped  $P_k$ -Anosov if and only if it is cusped  $P_{d-k}$ -Anosov. If  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  is a cusped  $\theta$ -Anosov representation of a geometrically finite Fuchsian group, we define a vector valued roof function

$$\tau_\rho : \Sigma^+ \rightarrow \mathfrak{a}_\theta$$

by setting

$$\tau_\rho(x) = \beta_\rho^\theta(G(x_1), \omega(x)) = B_\theta\left(\rho(G(x_1)), \rho(G(x_1))^{-1}(\xi_\rho(\omega(x)))\right).$$

If  $\phi$  is a linear functional on  $\mathfrak{a}_\theta$  we define the  $\phi$ -roof function  $\tau_\rho^\phi = \phi \circ \tau_\rho$ . If  $\rho$  is cusped Borel Anosov, i.e. if  $\theta = \{1, \dots, d-1\}$ , then  $\mathfrak{a}_\theta = \mathfrak{a}$  and  $B_\theta = B$  so we are in the simpler setting described in the first part of Section 9.5.

Recall that the *Benoist limit cone* of a representation  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  is given by

$$\mathcal{B}(\rho) = \overline{\bigcap_{n \geq 0} \bigcup_{\|\kappa(\rho(\gamma))\| \geq n} \mathbb{R}_+ \kappa(\rho(\gamma))} \subset \mathfrak{a}^+.$$

Benoist [1] showed that if  $\Gamma$  is Zariski dense, then  $\mathcal{B}(\rho)$  is convex and has non-empty interior. It is natural to consider linear functionals which are positive on the Benoist limit cone

$$\mathcal{B}(\rho)^+ = \left\{ \phi \in \mathfrak{a}^* \mid \phi\left(\mathcal{B}(\rho) - \{\vec{0}\}\right) \subset (0, \infty) \right\}.$$

Note that if  $\phi \in \mathcal{B}(\rho)^+$ , then there is a constant  $c$  such that  $\phi(v) > c\|v\|$  for all  $v \in \mathcal{B}(\rho)$ .

We will in general consider roof functions associated to linear functionals in  $\mathfrak{a}_\theta^* \cap \mathcal{B}(\rho)^+$ . Recall that  $\mathfrak{a}_\theta^*$  is spanned by  $\{\omega_k \mid k \in \theta\}$ . So if  $\{1, d-1\} \subset \theta$  and  $\rho$  is cusped  $\theta$ -Anosov (i.e. if  $\rho$  is cusped  $P_1$ -Anosov), then  $\omega_1$  and the Hilbert length functional  $\alpha_H = \omega_1 + \omega_{d-1}$  both lie in  $\mathfrak{a}_\theta^* \cap \mathcal{B}(\rho)^+$ . If  $\{1, 2\} \subset \theta$ , then  $\alpha_1 = \omega_2 - 2\omega_1 \in \mathfrak{a}_\theta^* \cap \mathcal{B}(\rho)^+$ , and, more generally, if  $\{k-1, k, k+1\} \subset \theta$ , then

$\alpha_k = -\omega_{k+1} + 2\omega_k - \omega_{k-1} \in \mathfrak{a}_\theta^* \cap \mathcal{B}(\rho)^+$ , if  $\rho$  is cusped  $\theta$ -Anosov. Finally, if  $\theta = \{1, \dots, d-1\}$  (i.e.  $\rho$  is cusped Borel Anosov), then

$$\Delta = \left\{ a_1\alpha_1 + \dots + a_{d-1}\alpha_{d-1} \mid a_i \geq 0 \ \forall i, \sum_{i=1}^{d-1} a_i > 0 \right\} \subset \mathfrak{a}_\theta^* \cap \mathcal{B}(\rho)^+ = \mathcal{B}(\rho)^+.$$

**Theorem D\*:** *Suppose that  $\Gamma$  is a torsion-free geometrically finite, but not convex cocompact, Fuchsian group,  $\theta \subset \{1, \dots, d-1\}$  is non-empty and symmetric, and  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  is cusped  $\theta$ -Anosov. If  $\phi \in \mathfrak{a}_\theta^* \cap \mathcal{B}(\rho)^+$ , then  $\tau_\rho^\phi : \Sigma^+ \rightarrow \mathbb{R}$  is a locally Hölder continuous function such that*

(1) *If  $x = \overline{x_1 \cdots x_n}$  is a periodic element of  $\Sigma^+$ , then*

$$S_n \tau_\rho^\phi(x) = \phi\left(\ell(\rho(G(x_1) \cdots G(x_n)))\right).$$

(2)  *$\tau_\rho^\phi$  is eventually positive.*

(3) *There exists  $C_\rho > 0$  such that if  $j \in \theta$ , then*

$$\left| \tau_\rho^{\omega_j}(x) - c_j(\rho, s(x_1)) \log r(x_1) \right| \leq C_\rho$$

(with the convention that  $c_j(\rho, \gamma) = 0$  if  $\gamma$  is not parabolic).

(4)  *$\tau_\rho^\phi$  has a strong entropy gap at infinity. Moreover, if  $\phi = \sum_{k \in \theta} a_k \omega_k$ , then*

$$d(\tau_\rho^\phi) = \frac{1}{c(\rho, \phi)}$$

where

$$c(\rho, \phi) = \inf \left\{ \sum_{k \in \theta} a_k c_k(\rho, v) \mid v \in \Gamma \text{ parabolic} \right\}.$$

(5) *If  $\eta \in \mathrm{Hom}_{\mathrm{tp}}(\rho)$  is also  $P_k$ -Anosov and  $\phi \in \mathcal{B}(\eta)^+$ , then there exists  $C > 0$  so that*

$$|\tau_\rho^\phi(x) - \tau_\eta^\phi(x)| \leq C$$

for all  $x \in \Sigma^+$ .

(6)  *$\tau_\rho^\phi$  is non-arithmetic.*

*Proof of Theorem D\*.* It follows immediately from Lemma 9.6 and Theorem 9.3 (1) that if  $x \in \Sigma^+$ , then

$$(10.1) \quad S_n \tau_\rho(x) = \sum_{j=0}^{n-1} \tau_\rho(\sigma^j(x)) = \beta_\rho^\theta(G(x_1) \cdots G(x_n), \omega(x)).$$

In particular, if  $x = \overline{x_1 \cdots x_n} \in \Sigma^+$  is periodic, then, by Lemma 9.6 and Theorem 9.3 (2),

$$(10.2) \quad \omega_k(S_n \tau_\rho(x)) = \omega_k\left(\ell(\rho(G(x_1) \cdots G(x_n)))\right),$$

for all  $k \in \theta$ , since  $\xi_\rho(\omega(x))$  is the attracting  $\theta$ -flag of  $\rho(G(x_1) \cdots G(x_n))$ . Thus, (1) holds since  $\{\omega_k \mid k \in \theta\}$  is a basis for  $\mathfrak{a}_\theta^*$  and the map  $\phi \rightarrow \tau_\phi$  is linear.

If  $\phi \circ \tau_\rho$  is not eventually positive, then there exist sequences  $\{x_n\}$  in  $\Sigma^+$  and  $\{m_n\}$  in  $\mathbb{N}$  so that  $m_n \rightarrow \infty$  and  $\phi(S_{m_n} \tau_\rho(x_n)) < 1$  for all  $n$ . Let  $\gamma_n = G((x_n)_1) \cdots G((x_n)_{m_n})$  and  $z_n = \omega(x_n)$ . Then

$$\phi(\beta_\rho^\theta(\gamma_n, z_n)) < 1 \quad \text{for all } n \in \mathbb{N}.$$

We may assume that  $\{z_n\}$  converges to  $z \in \Lambda(\Gamma)$ . Theorem 9.3 (5) implies that there exists  $L$  so that  $d(\gamma_n(b_0), \overrightarrow{b_0 z_n}) \leq L$  for all  $n$ . After passing to another subsequence, we may assume that  $\{\gamma_n^{-1}(b_0)\}$  converges to some  $w \in \Lambda(\Gamma)$ . We pass to another subsequence, so that  $\{\gamma_n^{-1}(z_n)\}$  converges to some  $x \in \Lambda(\Gamma)$ . Notice that  $x \neq w$ , since  $\overrightarrow{\gamma_n^{-1}(b_0) \gamma_n^{-1}(z_n)}$  converges to a bi-infinite geodesic joining  $w$  to  $x$  which lies within  $L$  of the basepoint  $b_0$ .

Since  $\lim \gamma_n^{-1}(b_0) = w$  and  $\rho$  has the  $P_k$ -Cartan property for all  $k \in \theta$  by Theorem 9.2(6),

$$\lim U_k(\rho(\gamma_n^{-1})) = \xi_\rho^k(w).$$

Since  $\xi_\rho^{d-k}(x)$  and  $\xi_\rho^{d-k}(w)$  are transverse, there exist  $N \in \mathbb{N}$  and  $\epsilon > 0$  so that if  $n > N$ , then

$$\angle(\xi_\rho^k(\gamma_n^{-1}z_n), U_{d-k}(\rho(\gamma_n)^{-1})) \geq \epsilon.$$

Lemma 9.4 and the  $\rho$ -equivariance of the limit map  $\xi_\rho$  then imply that there exists  $C$  so that

$$|\omega_k(\beta_\rho^\theta(\gamma_n, \xi_\rho(z_n))) - \omega_k(\kappa(\rho(\gamma_n)))| = |\omega_k(B_\theta(\rho(\gamma_n), \rho(\gamma_n^{-1})(\xi_\rho(z_n)))) - \omega_k(\kappa(\rho(\gamma_n)))| \leq C$$

for all  $k \in \theta$  and all  $n \geq N$ . Since  $\phi \in \mathfrak{a}_\theta^*$  this implies that there exists  $\hat{C} > 0$  such that

$$|\phi(\beta_\rho^\theta(\gamma_n, \xi_\rho(z_n))) - \phi(\kappa(\rho(\gamma_n)))| \leq \hat{C}$$

for all  $n \geq N$ .

By Theorem 9.1(1),  $\phi(\kappa(\rho(\gamma_n))) \rightarrow \infty$ , so we have achieved a contradiction. Therefore,  $\tau_\rho^\phi$  is eventually positive, so (2) holds.

In order to establish (3), we first notice that, since  $\|B_\theta(A, F)\| \leq \|\kappa(A)\|$  for all  $F \in \mathcal{F}_\theta$ ,

$$|\tau_\rho^{\omega_j}(x)| \leq C_{x_1} = j \|\kappa(\rho(G(x_1)))\|$$

for all  $x \in \Sigma^+$  and  $j \in \theta$ . Since our alphabet is infinite and  $C_{x_1} \rightarrow \infty$  as  $r(x_1) \rightarrow \infty$ , there is more work to be done.

If  $x \in \Sigma^+$  and  $r(x_1) \geq 2$ , then  $G(x_1) = v^n g_a$  for some  $v \in \mathcal{P}$  and  $g_a \in \mathcal{R}$ , where  $n = r(x_1) - 2$ , then

$$\begin{aligned} \tau_\rho(x) &= \beta_\rho^\theta(v^n g_a, \omega(x)) = B_\theta(\rho(v^n g_a), \rho(v^n g_a)^{-1}(\xi_\rho(\omega(x)))) \\ &= B_\theta(\rho(v^n), \rho(v^{-n})(\xi_\rho(\omega(x)))) + B_\theta(\rho(g_a), \rho(v^n g_a)^{-1}(\xi_\rho(\omega(x)))). \end{aligned}$$

Notice that

$$\left| \omega_j \left( B_\theta(\rho(g_a), \rho(v^n g_a)^{-1}(\xi_\rho(\omega(x)))) \right) \right| \leq R = \max \{ d \|\kappa(\rho(g_a))\| \mid g_a \in \mathcal{R} \}$$

for all  $j \in \theta$ .

Let  $p$  be the fixed point of  $v$  in  $\Lambda(\Gamma)$ . Notice that, by construction, there exists  $\hat{a} \in \mathcal{A}$  so that  $G(\hat{a}) = v g_a$ . Then  $X = \omega([\hat{a}])$  is a compact subset of  $\Lambda(\Gamma) - \{p\}$ . Therefore, if  $G(x_1) = v^n g_a$ ,  $\omega(x) \in v^{n-1}(X)$ , so  $v^{-n}(\omega(x)) \in v^{-1}(X)$ . It follows that there exists  $\epsilon = \epsilon(v) > 0$  so that if  $G(x_1) = v^n g_a$  and  $n \in \mathbb{N}$ , then

$$\angle(\rho(v^{-n})(\xi_\rho^j(\omega(x))), \xi_\rho^{d-j}(p)) \geq \epsilon$$

for all  $j \in \theta$ . Lemma 9.4 then implies that there exists  $D = D(v, g_a) > 0$  so that

$$\left| \omega_j(B_\theta(\rho(v^n), \rho(v^{-n})(\xi_\rho(\omega(x)))) - \omega_j(\kappa(\rho(v^n))) \right| \leq D.$$

for all  $n \in \mathbb{N}$  and  $j \in \theta$ . Theorem 9.1 implies that there exists  $C = C(v, g_a) > 0$  so that

$$|\omega_j(\kappa(\rho(v^n))) - c_j(\rho, v) \log n| < C$$

for all  $n \in \mathbb{N}$ . By combining, we see that

$$\left| \omega_j \left( B_\theta(\rho(v^n), \rho(v^{-n})(\xi_\rho(\omega(x)))) \right) - c_j(\rho, v) \log n \right| \leq C + D$$

and hence that

$$\left| \tau_\rho^{\omega_j}(x) - c_j(\rho, v) \log(r(x_1) - 2) \right| \leq C + D + R$$

for all  $n \in \mathbb{N}$  and  $j \in \theta$ . Since there are only finitely many  $v$  in  $\mathcal{P}$ , and only finitely many elements of  $\mathcal{A}$  so that  $r(a) \leq 2$  we have completed the proof of (3).

We next check that  $\tau_\rho^\phi$  is locally Hölder continuous. Since  $\omega : \Sigma^+ \rightarrow \Lambda(\Gamma)$  is locally Hölder continuous, there exist  $Z > 0$  and  $\zeta > 0$  so that if  $x_j = y_j$  for all  $j \leq n$ , then

$$d(\omega(x), \omega(y)) \leq Ze^{-\zeta n}.$$

Since  $\xi_\rho : \Lambda(\Gamma) \rightarrow \mathcal{F}_d$  is Hölder, there exist  $D > 0$  and  $\iota > 0$ , so that if  $z, w \in \Lambda(\Gamma)$ , then

$$d(\xi_\rho(z), \xi_\rho(w)) \leq Dd(z, w)^\iota$$

Therefore,  $\xi_\rho \circ \omega$  is locally Hölder continuous, i.e. there exists  $C$  and  $\beta > 0$  so that

$$d(\xi_\rho(\omega(x)), \xi_\rho(\omega(y))) \leq Ce^{-\beta n}$$

if  $x_j = y_j$  for all  $j \leq n$ .

If  $a \in \mathcal{A}$ , let

$$D_a = \sup \left\{ \|D_F B_\theta(\rho(G(a)), \cdot)\| \mid F \in \mathcal{F}_\theta \right\}$$

where  $D_F B_\theta(\rho(G(a)), \cdot)$  is the derivative at  $F$  of  $B_\theta(\rho(G(a)), \cdot) : \mathcal{F}_\theta \rightarrow \mathfrak{a}_\theta$ . It follows that if  $x_j = y_j$  for all  $j \leq n$  and  $x_1 = y_1 = a$ , then

$$|\tau_\rho^\phi(x) - \tau_\rho^\phi(y)| \leq \|\phi\| D_a C e^{-\beta n}$$

Recall that if  $x \in \Sigma^+$  and  $G(x_1) = v^m g_a$ , then

$$\tau_\rho(x) = B_\theta(\rho(v^m), \rho(v^{-m})(\xi_\rho(\omega(x)))) + B_\theta(\rho(g_a), \rho(v^m g_a)^{-1}(\xi_\rho(\omega(x))))$$

and that  $v^{-m}(\omega(x))$  lies in a compact subset  $v^{-1}(X)$  of  $\Lambda(\Gamma) - \{p\}$  (where  $p$  is the fixed point of  $v$ ).

There exists  $c > 0$  so that if  $x, y \in v^{-1}(X)$  and  $r \in \mathbb{N}$ , then

$$d(v^r(x), v^r(y)) \leq \frac{c}{r^2} d(x, y).$$

Notice that, by the cocycle property for  $B_\theta$ ,

$$B_\theta(\rho(v^m), F) = \sum_{j=1}^m B_\theta(\rho(v), v^{j-1}(F)).$$

Thus, if

$$\hat{D} = \hat{D}(v) = \sup \left\{ \|D_F B_\theta(\rho(v), \cdot)\| \mid F \in \mathcal{F}_\theta \right\}$$

then

$$\|B_\theta(\rho(v^m), x) - B_\theta(\rho(v^m), y)\| \leq \sum_{s=1}^m \hat{D} \frac{c}{s^2} d(x, y)$$

if  $x, y \in v^{-1}(X)$ . Notice that there exists  $T = T(v) > 0$  so that this series can be bounded above by  $Td(x, y)$ . Therefore, if  $x_j = y_j$  for all  $j = 1, \dots, n$  and  $G(x_1) = v^s g_a$  where  $s \geq 1$ , then

$$|(\phi \circ \tau_\rho)(x) - (\phi \circ \tau_\rho)(y)| \leq (T + R)C \|\phi\| e^{-\beta n}$$

where

$$R = \sup \left\{ \|D_F B_\theta(\rho(g_a), \cdot)\| \mid F \in \mathcal{F}_d, g_a \in \mathcal{R} \right\}.$$

Since there are only finitely many  $v$  in  $\mathcal{P}$  and only finitely many elements of  $\mathcal{A}$  so that  $r(a) \leq 2$ ,  $\tau_\rho^\phi$  is locally Hölder continuous.

If  $\phi = \sum_{k \in \theta} a_k \omega_k$  and  $v \in \mathcal{P}$ , let

$$c(\rho, \phi, v) = \sum_{k \in \theta} a_k c_k(\rho, v) \quad \text{and} \quad c(\rho, \phi) = \inf \{c(\rho, \phi, v) \mid v \in \mathcal{P}\}.$$

Notice that  $c(\rho, \phi)$  must be positive, since  $\phi \in \mathcal{B}(\rho)^+$ . Property (3) then implies that

$$|\tau_\rho^\phi(x) - c(\rho, \phi, s(x_1)) \log(r(x_1))| \leq C_\rho \|\phi\|$$

for all  $x \in \Sigma^+$ . Therefore,

$$\sum_{n=1}^{\infty} e^{-sC_\rho\|\phi\|} \frac{1}{n^{sc(\rho,\phi)}} = \sum_{n=1}^{\infty} e^{-s(c(\rho,\phi)\log n + C_\rho\|\phi\|)} \leq Z_1(\tau_\rho^\phi, s)$$

and

$$Z_1(\tau_\rho^\phi, s) \leq \sum_{n=1}^{\infty} Q\#(\mathcal{P})e^{-s(c(\rho,\phi)\log n - C_\rho\|\phi\|)} \leq \sum_{n=1}^{\infty} Q\#(\mathcal{P})e^{sC_\rho\|\phi\|} \frac{1}{n^{sc(\rho,\phi)}}$$

if  $s > 0$ . (Recall that if  $n \in \mathbb{N}$ , then  $1 \leq \#\{a \in \mathcal{A} \mid r(a) = n\} \leq Q$ .) Therefore,  $Z_1(\tau_\rho^\phi, s)$  converges if and only if  $s > \frac{1}{c(\rho,\phi)}$ , which establishes (4).

If  $\eta \in \text{Hom}_{tp}(\rho)$  is cusped  $\theta$ -Anosov and  $\phi \in \mathcal{B}(\eta^+)$ , then  $c_j(\rho, v) = c_j(\eta, v)$  for all  $j \in \theta$  and  $v \in \mathcal{P}$ . Property (5) then follows from applying (3) to both  $\tau_\rho$  and  $\tau_\eta$  and the fact that both  $\tau_\rho^\phi$  and  $\tau_\eta^\phi$  are locally Hölder continuous.

We may assume that the Zariski closure  $\mathbf{G}$  of  $\rho(\Gamma)$  is reductive. (If it is not reductive, then Guéritaud-Guichard-Kassel-Wienhard [21, Section 2.5.4] exhibit a representation  $\rho^{ss} : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$  so that the Zariski closure of  $\rho^{ss}(\Gamma)$  is reductive and  $\ell(\rho(\gamma)) = \ell(\rho^{ss}(\gamma))$  for all  $\gamma \in \Gamma$ .) A result of Benoist-Quint [2, Proposition 9.8] then implies that the subgroup  $\mathfrak{h}$  of the Cartan algebra  $\mathfrak{a}_\mathbf{g}$  of  $\mathbf{G}$  generated by  $\lambda_\mathbf{G}(\rho(\Gamma))$  is dense in  $\mathfrak{a}_\mathbf{g}$  (where  $\lambda_\mathbf{G} : \mathbf{G} \rightarrow \mathfrak{a}_\mathbf{g}$  is the Jordan projection of  $\mathbf{G}$ ). Up to conjugation, we may assume that  $\mathfrak{a}_\mathbf{g}$  is a sub-algebra of  $\mathfrak{a}$  (since  $\mathfrak{a}_\mathbf{g}$  is an abelian algebra and thus is contained in a translate of  $\mathfrak{a}$ , which is a maximal abelian sub-algebra of  $\mathfrak{sl}(d, \mathbb{R})$ ). Therefore, the subgroup of  $\mathbb{R}$  generated by  $\{\phi \circ \tau_\rho(x) \mid x \in \text{Fix}^n\}$ , which is just  $\phi(\mathfrak{h})$ , is dense in  $\mathbb{R}$ . Thus, we have established (6).  $\square$

## 11. APPLICATIONS

**11.1. Anosov representations of geometrically finite Fuchsian groups.** Given Theorem D\*, we can apply our main results to the roof functions of Anosov representations.

The following counting result is a strict generalization of Corollary 1.3. It follows immediately from Theorems D\* and A.

**Corollary 11.1.** *Suppose that  $\Gamma$  is a torsion-free, geometrically finite, but not convex cocompact, Fuchsian group,  $\theta \subset \{1, \dots, d-1\}$  is non-empty and symmetric, and  $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$  is cusped  $\theta$ -Anosov. If  $\phi \in \mathfrak{a}_\theta^* \cap \mathcal{B}(\rho)^+$ , then there exists a unique  $\delta_\phi(\rho) > \frac{1}{c(\rho,\phi)}$  so that  $P(-\delta_\phi(\rho)\tau_\rho^\phi) = 0$  and*

$$\lim_{t \rightarrow \infty} M_\phi(t) \frac{t\delta_\phi(\rho)}{e^{t\delta_\phi(\rho)}} = 1$$

where

$$M_\phi(t) = \#\left\{[\gamma] \in [\Gamma] \mid 0 < \phi(\ell(\rho(\gamma))) \leq t\right\}.$$

Similarly, one may combine Theorems C and D\* to obtain a generalization of Corollary 1.4.

**Corollary 11.2.** *Suppose that  $\Gamma$  is a torsion-free, geometrically finite, but not convex cocompact Fuchsian group,  $\theta \subset \{1, \dots, d-1\}$  is non-empty and symmetric, and  $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$  is cusped  $\theta$ -Anosov. If  $\eta \in \text{Hom}_{tp}(\rho)$  is also cusped  $\theta$ -Anosov,  $\phi \in \mathfrak{a}_\theta^* \cap \mathcal{B}(\rho)^+ \cap \mathcal{B}(\eta)^+$ , and*

$$\mathcal{C}^\phi(\rho, \eta) = \{(a, b) \in \mathcal{D}(\rho, \eta) \mid P(-a\tau_\rho^\phi - b\tau_\eta^\phi) = 0\}$$

where

$$\mathcal{D}(\rho, \eta) = \{(a, b) \in \mathbb{R}^2 \mid a + b > c(\rho, \phi)\},$$

then

- (1)  $\mathcal{C}^\phi(\rho, \eta)$  is an analytic curve,

- (2)  $(\delta_\phi(\rho), 0)$  and  $(0, \delta_\phi(\eta))$  lie on  $\mathcal{C}^\phi(\rho, \eta)$ ,  
(3)  $\mathcal{C}^\phi(\rho, \eta)$  is strictly convex, unless

$$\ell^\phi(\rho(\gamma)) = \frac{\delta_\phi(\eta)}{\delta_\phi(\rho)} \ell^\phi(\eta(\gamma))$$

for all  $\gamma \in \Gamma$ ,

- (4) and the tangent line to  $\mathcal{C}^\phi(\rho, \eta)$  at  $(\delta_\phi(\rho), 0)$  has slope

$$s^\phi(\rho, \eta) = -\frac{\int \tau_\eta^\phi dm_{-\delta_\phi(\rho)\tau_\rho^\phi}}{\int \tau_\rho^\phi dm_{-\delta_\phi(\rho)\tau_\rho^\phi}}.$$

In the setting of the previous corollary, we may define the *pressure intersection*  $I^\phi(\rho, \eta) = -s^\phi(\rho, \eta)$  and the *renormalized pressure intersection*

$$J^\phi(\rho, \eta) = \frac{\delta_\phi(\eta)}{\delta_\phi(\rho)} I^\phi(\rho, \eta).$$

We obtain the following intersection rigidity result which will be used crucially in the construction of pressure metrics. The proof follows at once from statements (3) and (4) in Corollary 11.2.

**Corollary 11.3.** *Suppose that  $\Gamma$  is a torsion-free, geometrically finite, but not convex cocompact, Fuchsian group,  $\theta \subset \{1, \dots, d-1\}$  is non-empty and symmetric, and  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  is cusped  $\theta$ -Anosov. If  $\eta \in \mathrm{Hom}_{\mathrm{tp}}(\rho)$  is also cusped  $\theta$ -Anosov and  $\phi \in \mathfrak{a}_\theta^* \cap \mathcal{B}(\rho)^+ \cap \mathcal{B}(\eta)^+$ , then*

$$J^\phi(\rho, \eta) \geq 1$$

with equality if and only if

$$\ell^\phi(\rho(\gamma)) = \frac{\delta_\phi(\eta)}{\delta_\phi(\rho)} \ell^\phi(\eta(\gamma))$$

for all  $\gamma \in \Gamma$ .

Finally, we derive our equidistribution result, which generalizes Corollary 1.6. It follows immediately from Theorems B and D\*.

**Corollary 11.4.** *Suppose that  $\Gamma$  is a torsion-free, geometrically finite, but not convex cocompact, Fuchsian group,  $\theta \subset \{1, \dots, d-1\}$  is non-empty and symmetric, and  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  is cusped  $\theta$ -Anosov. If  $\eta \in \mathrm{Hom}_{\mathrm{tp}}(\rho)$  is also cusped  $\theta$ -Anosov and  $\phi \in \mathfrak{a}_\theta^* \cap \mathcal{B}(\rho)^+ \cap \mathcal{B}(\eta)^+$ , then*

$$I^\phi(\rho, \eta) = \lim_{T \rightarrow \infty} \frac{1}{\#(R_T^\phi(\rho))} \sum_{[\gamma] \in R_T^\phi(\rho)} \frac{\ell^\phi(\eta(\gamma))}{\ell^\phi(\rho(\gamma))}$$

where  $R_T(\rho) = \{[\gamma] \in \Gamma \mid 0 < \ell^\phi(\rho(\gamma)) \leq T\}$ .

**11.2. Traditional Anosov representations.** Andres Sambarino [53, 54, 55] established analogues of our counting and equidistribution results in the setting of traditional “uncusped” Anosov representations. In this section, we will sketch how to establish (mild generalizations of) his results in our framework. We start by recalling a characterization of Anosov representations of word hyperbolic groups established by Kapovich-Leeb-Porti [30] and Bochi-Potrie-Sambarino [4].

If  $\Gamma$  is a word hyperbolic group, then a representation  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$  is  $P_k$ -Anosov if there exist  $A, a > 0$  so that

$$\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geq Ae^{a|\gamma|}$$

for all  $\gamma \in \Gamma$ , where  $|\gamma|$  is the word length of  $\gamma$  with respect to some fixed generating set on  $\Gamma$ . In this case, it is known (see [10] or [15]) that there is a finite Markov shift  $(\Sigma_\Gamma^+, \sigma)$  for the geodesic flow of  $\Gamma$  and a surjective map

$$G : \bigcup_{n \in \mathbb{N}} \text{Fix}^n \rightarrow [\Gamma].$$

Moreover, if  $\theta \subset \{1, \dots, d-1\}$  is non-empty and symmetric,  $\rho$  is  $\theta$ -Anosov, and  $\phi \in \mathfrak{a}_\theta \cap \mathcal{B}(\rho)^+$ , then there exists a Hölder continuous function  $\tau_\rho^\phi : \Sigma_\Gamma^+ \rightarrow \mathbb{R}$  so that if  $x \in \text{Fix}^n \subset \Sigma_\Gamma^+$ , then

$$S_n \tau_\rho^\phi(x) = \phi(\ell(\rho(G(x)))).$$

Lalley [34, Theorems 5 and 7] established analogues of our counting and equidistribution results for finite Markov shifts. Moreover, our proofs generalize his techniques so they go through in the setting of finite Markov shifts without any assumptions on entropy gap.

**Corollary 11.5.** *Suppose that  $\Gamma$  is a word hyperbolic group,  $\theta \subset \{1, \dots, d-1\}$  is non-empty and symmetric, and  $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$  is  $\theta$ -Anosov. If  $\phi \in \mathfrak{a}_\theta^* \cap \mathcal{B}(\rho)^+$ , then there exists a unique  $\delta_\phi(\rho) > 0$  so that  $P(-\delta_\phi(\rho)\tau_\rho^\phi) = 0$  and*

$$\lim_{t \rightarrow \infty} M_\phi(t) \frac{t \delta_\phi(\rho)}{e^{t \delta_\phi(\rho)}} = 1$$

where

$$M_\phi(t) = \#\left\{[\gamma] \in [\Gamma] \mid \phi(\ell(\rho(\gamma))) \leq t\right\}.$$

*Proof.* Our proof of property (6) in Theorem D\* gives immediately that  $\tau_\rho^\phi$  is non-arithmetic, which is the only assumption needed to apply our Theorem A or Theorem 7 in [34] in the setting of a finite Markov shifts.  $\square$

We also obtain a Manhattan Curve theorem, which does not seem to have appeared in print before in this generality, but was certainly well-known to experts. In particular, Sambarino [54, Proposition 4.7] describes a closely related phenomenon for Borel Anosov representations.

**Corollary 11.6.** *Suppose that  $\Gamma$  is a word hyperbolic group,  $\theta \subset \{1, \dots, d-1\}$  is non-empty and symmetric, and that  $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$  and  $\eta : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$  are  $\theta$ -Anosov. If  $\phi \in \mathfrak{a}_\theta^* \cap \mathcal{B}(\rho)^+ \cap \mathcal{B}(\eta)^+$  and*

$$\mathcal{C}^\phi(\rho, \eta) = \{(a, b) \in \mathbb{R}^2 \mid a + b > 0 \text{ and } P(-a\tau_\rho^\phi - b\tau_\eta^\phi) = 0\},$$

then

- (1)  $\mathcal{C}^\phi(\rho, \eta)$  is an analytic curve,
- (2)  $(\delta_\phi(\rho), 0)$  and  $(0, \delta_\phi(\eta))$  lie on  $\mathcal{C}^\phi(\rho, \eta)$ ,
- (3) and  $\mathcal{C}^\phi(\rho, \eta)$  is strictly convex, unless

$$\ell^\phi(\rho(\gamma)) = \frac{\delta_\phi(\eta)}{\delta_\phi(\rho)} \ell^\phi(\eta(\gamma))$$

for all  $\gamma \in \Gamma$ .

Moreover, the tangent line to  $\mathcal{C}^\phi(\rho, \eta)$  at  $(\delta_\phi(\rho), 0)$  has slope

$$-I^\phi(\rho, \eta) = -\frac{\int \tau_\eta^\phi dm_{-\delta_\phi(\rho)\tau_\rho^\phi}}{\int \tau_\rho^\phi dm_{-\delta_\phi(\rho)\tau_\rho^\phi}}$$

The analogues of Corollaries 1.5 and 1.6 appear in [10, Section 8] as consequences of classical Thermodynamical results of Bowen, Pollicott and Ruelle [5, 6, 45, 51].

**Historical Remarks:** In the counting estimates and equidistribution results in his papers, Sambarino assumes that  $\rho$  is irreducible if  $\theta = \{1, d-1\}$  (see [53]) or Zariski dense if  $\rho$  is Borel Anosov

(see [54, 55]) and that  $\Gamma = \pi_1(M)$  where  $M$  is a negatively curved manifold. However, after [10] the generalizations stated here would certainly have been well-known to him. Carvajales [14, Appendix A] uses results from [10] to explain how one can remove the assumption that  $\Gamma = \pi_1(M)$  in Sambarino's work. The removal of the irreducibility assumption follows from the construction of the semi-simplification in [21]. Pollicott and Sharp [46] independently derived related counting results for Hitchin representations.

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