Math 490 Extra Handout on Complete Metric Spaces

**Definition:** A metric space \((X, d)\) is said to be *complete* if every Cauchy sequence in \(X\) converges to a point in \(X\).

We recall the following basic fact from Real Analysis:

**Fact:** The real line (with the Euclidean metric) is complete.

**Exercises:**

1. Euclidean space \(\mathbb{R}^n\) is complete for all \(n\).

2. We say that a function \(f : X \to X\) from a metric space to itself is a *contraction* if there exists \(\alpha < 1\) such that

\[
d(f(x), f(y)) \leq \alpha d(x, y)
\]

for all \(x, y \in X\).

   a) Prove that any contraction is continuous.

   b) Prove that if \(X\) is complete and \(f : X \to X\) is a contraction, then there is a unique point \(x_0 \in X\) such that \(f(x_0) = x_0\). (This is sometimes called Banach’s Fixed Point Theorem and is a key step in the proof of the existence and uniqueness of solutions of classes of differential equations.)

3. Use results from real analysis to prove that \(C([a, b], \mathbb{R})\) with the sup metric \(d_\infty\) is complete.

4. Let \((X, d)\) be a metric space and let \(\mathcal{B}(X, \mathbb{R})\) be the space of bounded functions from \(X\) to \(\mathbb{R}\). (A function \(f : X \to \mathbb{R}\) is bounded if \(f(X)\) is a bounded subset of \(\mathbb{R}\).) We may define a metric \(d_\infty\) on \(\mathcal{B}(X, \mathbb{R})\) by setting

\[
d_\infty(f, g) = \sup\{|f(x) - g(x)| \mid x \in X\}
\]

for all \(f, g \in \mathcal{B}(X, \mathbb{R})\). Prove that \((\mathcal{B}(X, \mathbb{R}), d_\infty)\) is a complete metric space.

5. If \(Y\) is a complete metric space, define \(\mathcal{B}(X, Y)\) and \(d_\infty\) and prove that \((\mathcal{B}(X, Y), d_\infty)\) is a complete metric space.

6. Prove that a closed subset of a complete metric space is a complete metric space (in the subspace metric).
7. Let \((X, d)\) be a metric space and let \(x_0 \in X\).

a) If \(a \in X\), we define \(\phi_a : X \to \mathbb{R}\) by setting
\[
\phi_a(x) = d(x, a) - d(x, x_0)
\]
for all \(x \in X\). Prove that \(\phi_a \in \mathcal{B}(X, \mathbb{R})\).

b) Define a function
\[
h : X \to (\mathcal{B}(X, \mathbb{R}), d_\infty)
\]
by letting \(h(a) = \phi_a\) for all \(a \in X\). Prove that \(h\) is an isometry, i.e. \(d_\infty(h(a), h(b)) = d(a, b)\) for all \(a, b \in X\).

c) Prove that \(h\) is an embedding, i.e. \(h\) is continuous and injective and if we give \(h(X)\) the subspace metric, then \(h^{-1} : h(X) \to X\) is continuous.

d) The closure \(\overline{h(X)}\) of \(h(X)\) in \(\mathcal{B}(X, \mathbb{R})\) is called the completion of \(X\), since it is a complete metric space containing a subspace isometric to \(X\).

e) Prove that every point in the completion of \(X\) is the limit of a Cauchy sequence in \(X\).