**Definition:** A function $f : (X_1, T_1) \to (X_2, T_2)$ between topological spaces is continuous if whenever $U \in T_2$, then $f^{-1}(U) \in T_1$. More informally, we say that a function between topological spaces is continuous if pre-images of open sets are open.

**New in-class exercises:**

1. If $(X_1, T_1)$ and $(X_2, T_2)$ are topological spaces and $f : (X_1, T_1) \to (X_2, T_2)$ is a constant map, then $f$ is continuous.

2. If $X$ is a set and $T_{\text{disc}}$ is the discrete topology on $X$, then any function $f : (X, T_{\text{disc}}) \to (Y, T_Y)$ to any topological space is continuous.

3. If $Y$ is a set and $T_{\text{indisc}}$ is the indiscrete topology on $Y$, then any function $f : (X, T_X) \to (Y, T_{\text{indisc}})$ with domain any topological space is continuous.

4. **Definition:** If $T$ is a topology on a set $X$, then $B$ is a basis for $T$ if $B \subset T$ and every element of $T$ is a union of elements of $B$. (By convention, we regard the empty set $\emptyset$ as the union of an empty collection of elements of $B$.)

Suppose that $f : X \to Y$ is a function between topological spaces $(X, T_X)$ and $(Y, T_Y)$ and that $B_Y$ is a basis for $T_Y$.

Prove that $f$ is continuous if and only if whenever $B$ is an element of $B_Y$, then $f^{-1}(B)$ is open in $X$ (i.e. $f^{-1}(B) \in T_X$.)

5. Let $X$ be a set and let $B$ be a collection of sets such that

$$\bigcup_{B \in B} B = X$$

and if $B_1$ and $B_2$ are elements of $B$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in B$ such that

$$x \in B_3 \subset B_1 \cap B_2.$$ 

Let $T$ be the collection of all subsets of $X$ which are unions of collections of elements of $B$ (where again the empty set is the union of the empty collection of elements of $B$.)

Prove that $T$ is a topology on $X$ and that $B$ is a basis for $T$.

6. Exhibit all possible topologies on the set $X = \{0, 1, 2\}$. (You need not prove that each one is a topology.)
**Reminder:** The Exam will be held in-class on Tuesday October 23. There will be a review session Monday October 21 from 6pm to 8pm in 4088 East Hall.

**Individual homework:** Due Thursday October 18

1. We define a set to be **closed** in the topological space $(X, \mathcal{T})$ if $X \setminus C \in \mathcal{T}$.

   Prove that a function $f : (X_1, \mathcal{T}_1) \rightarrow (X_2, \mathcal{T}_2)$ between topological spaces is continuous if whenever $C$ is closed in $X_2$, then $f^{-1}(C)$ is closed in $X_1$.

2. Suppose that $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous maps between topological spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, and $(Z, \mathcal{T}_Z)$.

   Prove that $g \circ f : X \rightarrow Z$ is continuous.

3. We say that a topological space $(X, \mathcal{T})$ is **Hausdorff** if given any two distinct points $x$ and $y$ in $X$, there exists disjoint open sets $U$ and $V$ so that $x \in U$ and $y \in V$.

   Prove that if $X$ is an infinite set and $\mathcal{T}$ is the finite complement topology, then $(X, \mathcal{T})$ is not Hausdorff. Explain why this implies that $(X, \mathcal{T})$ is not metrizable.