In-class Exercises:

1. Suppose that \((X, \mathcal{T})\) is a topological space, \(A \subset X\) and \(\mathcal{T}_A\) is the subspace topology on \(A\).
   Prove that if \(B\) is a basis for \(\mathcal{T}\), then
   \[ B_A = \{ B \cap A \mid B \in \mathcal{B} \} \]
   is a basis for \(\mathcal{T}_A\).

2. Suppose that \((X, d)\) is a metric space and \(A \subset X\). Let \(d_A = d|_{A \times A}\).
   Prove that the subspace topology \(\mathcal{T}_{d_A}\) on \(A\) associated to \(\mathcal{T}_d\) is the same as \(\mathcal{T}_{d_A}\).

3. Let \(X\) be an infinite set and let \(\mathcal{T}\) be the finite-complement topology on \(X\).
   Prove that if \(A\) is any finite subset of \(X\), then the subspace topology on \(A\) agrees with the discrete topology on \(A\).

**Definition:** Suppose that \(\{x_n\}\) is a sequence of points in a topological space \((X, \mathcal{T})\).
We say that \(\{x_n\}\) converges to \(x \in X\), if given any open neighborhood \(U\) of \(x\), there exists \(N\) such that \(x_n \in U\) if \(n \geq N\). (We recall that an open neighborhood of \(x\) is an open set containing \(x\).)

4. If \((X, \mathcal{T})\) is Hausdorff, then a sequence in \(X\) converges to at most one point in \(X\).

5. Exhibit a sequence in a topological space \((X, \mathcal{T})\) which converges to more than one point. (Hint: It would help if \(X\) had very few open sets.)

6. Suppose that \((X, \mathcal{T}_X)\) and \((Y, \mathcal{T}_Y)\) are topological spaces. Prove that if \(f : X \to Y\) is continuous and one-to-one and \((Y, \mathcal{T}_Y)\) is Hausdorff, then \((X, \mathcal{T}_X)\) is Hausdorff.

7. Exhibit a continuous, onto function \(f : X \to Y\) between topological spaces such that \(X\) is Hausdorff, but \(Y\) is not Hausdorff. This shows that images of Hausdorff spaces need not be Hausdorff.
Individual homework: Due Thursday November 1

1. Suppose that \((X, \mathcal{T})\) is a Hausdorff topological space. Prove that if \(x \in X\), then \(\{x\}\) is a closed subset of \(X\).

2. Let \(C\) be a closed subset of a topological space \((X, \mathcal{T})\). Suppose that \(\{x_n\}\) is a sequence of points in \(C\) which converges to \(x \in X\). Prove that \(x \in C\).

3. Definition: Suppose that \((X, \mathcal{T})\) is a topological space and \(A \subset X\). We say that \(x \in \bar{A}\) if every open neighborhood of \(x\) intersects \(A\) i.e. if \(U\) is open and \(x \in U\), then \(U \cap A \neq \emptyset\). The set \(\bar{A}\) is called the closure of \(A\).

Prove that the closure \(\bar{A}\) of a subset \(A\) of a topological space \((X, \mathcal{T})\) is closed in \(X\).