Definition: A topological space \((X, \mathcal{T})\) is disconnected if there exist disjoint non-empty open subsets \(A\) and \(B\) of \(X\) such that \(X = A \cup B\). If \(X\) is not disconnected, it is said to be connected.

A subset \(A\) of \(X\) is said to be connected if it is connected in the subspace topology, i.e. \((A, \mathcal{T}_A)\) is connected. Equivalently, a subset \(A\) of \(X\) is connected if and only if there does not exist open sets \(U\) and \(V\) in \(X\) such that \(A \subset U \cup V\), \((U \cap V) \cap A = \emptyset\), and \(A \cap U\) and \(A \cap V\) are both non-empty.

We recall the Intermediate Value Theorem from Real Analysis:

Intermediate Value Theorem: If \(f: [a, b] \to \mathbb{R}\) is continuous and \(d\) lies between \(f(a)\) and \(f(b)\) (i.e. either \(f(a) \leq d \leq f(b)\) or \(f(b) \leq d \leq f(a)\)), then there exists \(c \in [a, b]\) such that \(f(c) = d\).

In-class Exercises:

1. Suppose \(X\) is a set and \(\mathcal{T}\) is the indiscrete topology.
   
   Prove that \((X, \mathcal{T})\) is connected.

2. Suppose that \(X\) is a set with more than one point and \(\mathcal{T}\) is the discrete topology.
   
   Prove that \((X, \mathcal{T})\) is disconnected.

3. Prove that a topological space \((X, \mathcal{T})\) is disconnected if and only if there exists a continuous onto function \(f: X \to \{0, 1\}\) (where \(\{0, 1\}\) is given the discrete topology).

4. Prove that any interval in \(\mathbb{R}\) is connected. (Hint: You may use the intermediate value theorem.)

5. Prove that any subset of \(\mathbb{R}\) which is not an interval is disconnected. (Hint: A subset \(A \subset R\) is an interval if and only if whenever \(x, y \in A\) and \(z\) lies between \(x\) and \(y\), then \(z \in A\).)

6. Suppose that \((X, \mathcal{T}_X)\) and \((Y, \mathcal{T}_Y)\) are topological spaces.
   
   Prove that if \(f: X \to Y\) is continuous and \(X\) is connected, then \(f(X)\) is connected.
Individual homework: Due Thursday November 8

1. Let $X$ be an infinite set and let $\mathcal{T}$ be the finite complement topology.
   Prove that $(X, \mathcal{T})$ is connected.

2. Prove that $X$ is disconnected if and only if it contains a subset $A$ which is both open and closed such that $A \neq \emptyset$ and $A \neq X$.

3. Suppose that $(X, \mathcal{T})$ is a topological space.
   Prove that if $A$ and $B$ are connected subsets of $X$ and $A \cap B \neq \emptyset$, then $A \cup B$ is connected.