On Tuesday, we proved that compact metric spaces are sequentially compact. Today we will prove the converse. On Tuesday we established the Lebesgue Number Lemma which is the first step in the proof of the converse.

**Lebesgue Number Lemma:** Suppose that \((X,d)\) is a sequentially compact metric space. Every open cover \(U\) of \(X\) has a Lebesgue number.

We recall the definition of a Lebesgue number.

**Definition:** If \(U = \{U_\alpha\}_{\alpha \in \Lambda}\) is an open cover of a metric space \(X\), then \(\delta > 0\) is a \textbf{Lebesgue number} for \(U\) if for all \(x \in X\), there exists \(\alpha_x \in \Lambda\) such that

\[
B(x,\delta) \subset U_{\alpha_x}.
\]

**In-class Exercises:**

1. **Definition:** A subset \(A\) of a metric space \(X\) is said to be an \textbf{\(\epsilon\)-net} if \(\{B(a,\epsilon) \mid a \in A\}\) is an open cover of \(X\).

   Prove that if \(\epsilon > 0\), then every sequentially compact metric space has a finite \(\epsilon\)-net.

2. Prove that a sequentially compact metric space is compact.

3. Let \((X,T)\) be a compact, Hausdorff topological space.

   Prove that if \(C\) is a closed subset of \(X\) and that \(x \in X - C\), then there exist disjoint open subsets \(U\) and \(V\) of \(X\) so that \(x \in U\) and \(C \subset V\). (Spaces with this property are said to be \textbf{regular}.)

4. Suppose that \((X,T_X)\) and \((Y,T_Y)\) are topological spaces and that we give \(X \times Y\) the product topology. Prove that if \(X \times Y\) is compact, then both \(X\) and \(Y\) are compact.

5. Suppose that \((X,T'_X)\) and \((Y,T'_Y)\) are compact topological spaces and that we give \(X \times Y\) the product topology.

   Let \(x_0 \in X\) and let \(N\) be an open set in \(X \times Y\) which contains \(\{x_0\} \times Y\). Prove that there exists an open neighborhood \(W\) of \(x_0\) in \(X\) such that \(W \times Y \subset N\).
Team homework: Due Tuesday December 4

1. Let \((X, T)\) be a compact, Hausdorff topological space. Prove that if \(C\) and \(D\) are disjoint closed subsets of \(X\), then there exist disjoint open sets \(U\) and \(V\) such that \(C \subset U\) and \(D \subset V\). Topological spaces with this property are called normal. So you have just proven that compact, Hausdorff spaces are normal.

2. Definition: Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. A function

\[ f : (X, d_X) \rightarrow (Y, d_Y) \]

is said to be uniformly continuous if given any \(\epsilon > 0\) there exists \(\delta > 0\) such that if \(x_1, x_2 \in X\) and \(d_X(x_1, x_2) < \delta\), then \(d_Y(f(x_1), f(x_2)) < \epsilon\).

a) Prove that a uniformly continuous function is continuous.

b) Exhibit a continuous function which is not uniformly continuous.

3. Suppose that \((X, d_X)\) and \((Y, d_Y)\) are metric spaces. Prove that if \(X\) is sequentially compact and \(f : X \rightarrow Y\) is continuous, then \(f\) is uniformly continuous. (Hint: Think about the cover of \(Y\) by \(\epsilon\)-balls and consider its pre-image under the map \(f\).)

4. Let \((X, T)\) be a compact topological space. Suppose that \(\{C_\alpha\}_{\alpha \in \Lambda}\) is a collection of non-empty closed subsets of \(X\). Show that if the intersection

\[ C_{\alpha_1} \cap \cdots \cap C_{\alpha_n} \]

of any finite subcollection of \(\{C_\alpha\}_{\alpha \in \Lambda}\) is non-empty, then the total intersection

\[ \bigcap_{\alpha \in \Lambda} C_\alpha \]

is non-empty. (Hint: If \(\bigcap_{\alpha \in \Lambda} C_\alpha = \emptyset\), then \(\{X - C_\alpha\}_{\alpha \in \Lambda}\) is an open cover of \(X\).)