Math 490 Handout: Thursday December 6, 2012

We will again begin with presentations of exercises from the previous class:

1. Let \((X, T)\) be a compact, Hausdorff topological space.
   
   Prove that if \(C\) is a closed subset of \(X\) and that \(x \in X - C\), then there exist disjoint open subsets \(U\) and \(V\) of \(X\) so that \(x \in U\) and \(C \subset V\). (Spaces with this property are said to be \textbf{regular}.)

2. Suppose that \((X, T_X)\) and \((Y, T_Y)\) are topological spaces and that we give \(X \times Y\) the product topology. Prove that if \(X \times Y\) is compact, then both \(X\) and \(Y\) are compact.

3. Suppose that \((X, T_X)\) and \((Y, T_Y)\) are compact topological spaces and that we give \(X \times Y\) the product topology.
   
   Let \(x_0 \in X\) and let \(N\) be an open set in \(X \times Y\) which contains \(\{x_0\} \times Y\). Prove that there exists an open neighborhood \(W\) of \(x_0\) in \(X\) such that \(W \times Y \subset N\).

4. Suppose that \((X, T_X)\) and \((Y, T_Y)\) are compact topological spaces.
   
   Prove that \(X \times Y\) is compact in the product topology.

\textbf{In-class Exercises:}

1. Prove that \([0, 1)\) and \((0, 1)\) are not homeomorphic.

2. Prove that \(\mathbb{R}\) and \(\mathbb{R}^2\) are not homeomorphic.

3. Prove that \((0, 1)\) and \(\mathbb{R}\) are homeomorphic.

4. Let \(A_0 = [0, 1]\). Let \(A_1\) be obtained from \(A_0\) by deleting its “middle third,” i.e.
   
   \[A_1 = A_0 - \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].\]
   
   In general, let \(A_n\) be obtained from \(A_{n-1}\) by removing the middle third of each interval, i.e. let
   
   \[A_n = A_{n-1} - \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n}\right).\]
   
   The \textbf{Cantor set} is defined to be
   
   \[C = \cap_{n=1}^{\infty} A_n.\]
   
   Prove that \(C\) is compact and \textbf{totally disconnected}, i.e. every connected subset of \(C\) is a single point.

5. Let \(C\) be the Cantor set. Prove that \(C\) is \textbf{perfect}, i.e. every point \(x \in C\) is a limit of a sequence in \(C - \{x\}\). (This implies that \(C\) is uncountable.)

It is known that every perfect, compact, totally disconnected metric space is homeomorphic to the Cantor set.