In-class Exercises:

1. A sequentially compact subset $C$ of a metric space is bounded (i.e. there exists $x_0 \in X$ and $R > 0$ such that $C \subset D(x_0, R)$). In particular, a sequentially compact metric space is bounded.

2. Show that a subset of $\mathbb{R}$ is sequentially compact if and only if it is closed and bounded.

3. Exhibit a closed and bounded subset of a metric space which is not sequentially compact.

4. Show that if $C$ is a sequentially compact subset of $\mathbb{R}$, then $\sup C$ exists and lies in $C$.

5. If $f : X \rightarrow \mathbb{R}$ is continuous and $C \subset X$ is sequentially compact, then there exists $c \in C$ such that $f(c) = \sup f(C)$, i.e. $f$ achieves its supremum on $C$.

   Notice that Exercise 5 generalizes:

   **Theorem:** If $[a, b]$ is a closed bounded interval in $\mathbb{R}$ and $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function, then $f$ achieves its supremum.

6. Prove that any finite subset of a metric space is sequentially compact.
Team homework: Due Tuesday October 4

1. Show that every bounded sequence in \( \mathbb{R}^2 \) has a convergent subsequence.

2. Show that a subset \( C \) of \( \mathbb{R}^2 \) is sequentially compact if and only if \( C \) is closed and bounded.

3. Show that if \( X \) is a sequentially compact metric space, then every Cauchy sequence in \( X \) is convergent. (We say that sequentially compact metric spaces are complete.)

4. Prove that \( C([a, b], \mathbb{R}) \) is not sequentially compact with the metric

\[
d_{\infty}(f, g) = \sup\{|f(x) - g(x)| \mid x \in [a, b]\}.
\]