Math 490 Handout: Tuesday October 2, 2011

Suppose that \((X, d_X)\) and \((Y, d_Y)\) are metric spaces, we define

\[ d_2 : (X \times Y) \times (X \times Y) \to [0, \infty) \]

by letting

\[ d_2((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2} \]

for all \((x_1, y_1), (x_2, y_2) \in X \times Y\).

**Fact:** \(d_2\) is a metric on \(X \times Y\).

**Proof:** First notice that \(d_2((x_1, y_1), (x_2, y_2)) = 0\) if and only if \(\sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2} = 0\) which occurs if and only if \(d_X(x_1, x_2) = d_Y(y_1, y_2) = 0\) which occurs if and only if \(x_1 = x_2\) and \(y_1 = y_2\) (since \(d_X\) and \(d_Y\) are both metrics). Therefore, \(d_2((x_1, y_1), (x_2, y_2)) = 0\) if and only if \((x_1, y_1) = (x_2, y_2)\) which establishes (M1).

If \((x, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y\), then

\[ d_2((x_1, y_1), (x_2, y_2)) + d_2((x_2, y_2), (x_3, y_3)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2} + \sqrt{d_X(x_2, x_3)^2 + d_Y(y_2, y_3)^2} \]

which establishes property (M2). (The middle equality follows since the fact that \(d_X\) and \(d_Y\) are metrics guarantee that \(d_X(x_1, x_2) = d_X(x_2, x_1)\) and \(d_Y(y_1, y_2) = d_Y(y_2, y_2)\).)

If \((x, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y\), then

\[ d_2((x_1, y_1), (x_2, y_2)) + d_2((x_2, y_2), (x_3, y_3)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2} + \sqrt{d_X(x_2, x_3)^2 + d_Y(y_2, y_3)^2} \]

The triangle inequality in \(\mathbb{R}^2\) guarantees that if \(a, b, c, d \in \mathbb{R}\), then

\[ \sqrt{a^2 + b^2 + c^2 + d^2} = ||(a, b) - (0, 0)|| + ||(0, 0) - (c, -d)|| \geq ||(a, b) - (c, -d)|| = \sqrt{(a + c)^2 + (b + d)^2} \]

Applying this to our situation, we see that

\[ \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2 + d_X(x_2, x_3)^2 + d_Y(y_2, y_3)^2} \geq \sqrt{d_X(x_1, x_2)^2 + d_X(x_2, x_3)^2 + (d_Y(y_1, y_2) + d_Y(y_2, y_3))^2} \]

Since \(d_X\) and \(d_Y\) are metrics, \(d_X(x_1, x_2) + d_X(x_2, x_3) \geq d_X(x_1, x_3)\) and \(d_Y(y_1, y_2) + d_Y(y_2, y_3) \geq d_Y(y_1, y_3)\), so we see that

\[ d_X(x_1, x_2) + d_Y(y_1, y_2) + d_Y(y_2, y_3) \geq d_X(x_1, x_3) + d_Y(y_1, y_3) \]

We may combine the above inequalities to see that

\[ d_2((x_1, y_1), (x_2, y_2)) + d_2((x_2, y_2), (x_3, y_3)) \geq d_2((x_1, y_1), (x_3, y_3)) \]

which establishes the triangle inequality for \(d_2\).

Since we have established properties (M1), (M2) and (M3), \(d_2\) is a metric on \(X \times Y\).
In-class Exercises: In all these exercises, assume that \((X,d_X)\) and \((Y,d_Y)\) are metric spaces and that \(X \times Y\) is given the metric \(d_2\) which is defined above.

1. Prove that if \(\{x_n\}\) is a convergent sequence in \(X\) and \(\{y_n\}\) is a convergent sequence in \(Y\), then \(\{(x_n,y_n)\}\) is a convergent sequence in \(X \times Y\) and \(\lim(x_n,y_n) = (\lim x_n,\lim y_n)\).

2. Show that the projection map \(\pi_X : (X \times Y,d_2) \to (X,d_X)\) given by \(\pi_X(x,y) = x\) is continuous. Observe that the same argument gives that the obvious projection map \(\pi_Y : X \times Y \to Y\) is continuous.

3. Show that \(X \times Y\) is sequentially compact if and only both \(X\) and \(Y\) are sequentially compact.

4. Suppose that \((Z,d_Z)\) is a metric space and \(f : Z \to X\) and \(g : Z \to Y\) are continuous functions. Prove that \(h : Z \to X \times Y\) given by \(h(z) = (f(z),g(z))\) is continuous.

5. Suppose that \((X,d_X)\) and \((Y,d_Y)\) are metric spaces. Define \(d_\infty : (X \times Y) \times (X \times Y) \to [0,\infty)\)

\[
d_\infty((x_1,y_1),(x_2,y_2)) = \max\{d_X(x_1,x_2),d_Y(y_1,y_2)\}
\]

for all \((x_1,y_1),(x_2,y_2)\) \(\in X \times Y\). Prove that \(d_\infty\) is a metric on \(X \times Y\).

Individual homework: Due Thursday October 4

1. Suppose that \((X,d_X)\) and \((Y,d_Y)\) are metric spaces. Show that if \(U\) is open in \(X\) and \(V\) is open in \(Y\), then \(U \times V\) is open in \((X \times Y,d_2)\).

2. If \(f : X \to \mathbb{R}\) is continuous and \(C \subset X\) is sequentially compact, then there exists \(c \in C\) such that \(f(c) = \sup f(C)\), i.e. \(f\) achieves its supremum on \(C\).

3. Suppose that \((X,d_X)\) and \((Y,d_Y)\) are metric spaces. Define \(d_1 : (X \times Y) \times (X \times Y) \to [0,\infty)\)

\[
d_1((x_1,y_1),(x_2,y_2)) = d_X(x_1,x_2) + d_Y(y_1,y_2)
\]

for all \((x_1,y_1),(x_2,y_2)\) \(\in X \times Y\). Prove that \(d_1\) is a metric on \(X \times Y\).