PRESSURE METRICS FOR CUSPED HITCHIN COMPONENTS

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Abstract. We study the cusped Hitchin component consisting of (conjugacy classes of) cusped Hitchin representations of a torsion-free geometrically finite Fuchsian group Γ into $\mathsf{PSL}(d,\mathbb{R})$. We produce pressure metrics associated to the first fundamental weight and the first simple root. When $d = 3$ we produce a pressure metric associated to the Hilbert length, which is new even when Γ is cocompact.

Contents

1. Introduction

In this paper, we construct pressure metrics on the cusped Hitchin component of Hitchin representations of a torsion-free Fuchsian lattice into $\mathsf{PSL}(d,\mathbb{R})$. The first two metrics are mapping class group invariant, analytic Riemannian metrics. These metrics are associated to the first fundamental weight and the first simple root. Their constructions are based on earlier constructions of Bridgeman, Canary, Labourie and Sambarino [11, 12, 13] in the case of Hitchin components of closed surface groups. Our third pressure metric is new even in the case of closed surface groups. This final pressure metric is based on the Hilbert length and is established only when $d = 3$. It is a mapping class group invariant path metric which is an analytic Riemannian metric off of the Fuchsian locus. We hope that when $d = 3$, the Hilbert length pressure metric may be more natural to study given its connection to Hilbert geometry.

The main new technical difficulties involve the fact that while the geodesic flow of a closed hyperbolic surface may be coded by a finite Markov shift, there is no finite Markov coding of the geodesic flow of a geometrically finite hyperbolic surface. Stadlbauer [49] and Ledrappier-Sarig [29] provide a countable Markov coding of the (recurrent portion

Canary was partially supported by grant DMS-1906441 from the National Science Foundation and grant 674990 from the Simons Foundation.

of the) geodesic flow of a finite area hyperbolic surface. In a previous paper, we used these codings, work of Canary-Zhang-Zimmer [15] on cusped Hitchin representations, and the Thermodynamic Formalism for countable Markov shifts, to establish counting and equidistribution results for cusped Hitchin representations. In this paper, we apply the theory developed in that paper to construct our pressure metrics.

The long-term goal of this project is to realize these metrics as the induced metric on the strata at infinity of the metric completion of the Hitchin component of a closed surface group with its pressure metric. In the classical setting, when $d=2$, Masur [35] showed that the metric completion of Teichmüller space of a closed surface S, with the Weil-Petersson metric, is the augmented Teichmüller space. The strata at infinity in the augmented Teichmüller space come from Teichmüller space of, possibly disconnected, surfaces obtained from pinching S along a multicurve. When $d=3$, the Hitchin component of a closed surface is the space of holonomy maps of convex projective structures on the surface. The strata at infinity of the augmented Hitchin component would then be cusped Hitchin components consisting of finite area convex projective structures obtained from pinching the surface along a multicurve. We hope to eventually establish an analogue of Masur's result in the higher rank setting. (See [14] for a more detailed description of the conjectural geometric picture of the augmented Hitchin component.)

We now discuss our results more precisely. We recall that if Γ is a torsion-free, geometrically finite Fuchsian group (i.e. a discrete finitely generated subgroup of $\mathsf{PSL}(2,\mathbb{R})$), then a Hitchin representation is a representation $\rho:\Gamma\to\mathsf{PSL}(d,\mathbb{R})$ which admits a positive equivariant limit map $\xi:\partial\mathbb{H}^2\to\mathcal{F}_d$ where $\mathcal{F}_d$ is the space of d-dimensional flags. As in the closed case, they all arise as type-preserving deformations of the restriction of an irreducible representation of $\mathsf{PSL}(2,\mathbb{R})$ into $\mathsf{PSL}(d,\mathbb{R})$.

The *Hitchin component* $\mathcal{H}_d(\Gamma)$ is the space of conjugacy classes of Hitchin representations of Γ into $\mathsf{PSL}(d,\mathbb{R})$. Fock and Goncharov, see the discussion in [18, Sec 1.8], show that the Hitchin component is topologically a cell. (When $d=3$, $\mathcal{H}_3(\Gamma)$ is parameterized by Marquis [32], when Γ is a lattice, and more generally by Loftin and Zhang [30]. Bonahon-Dreyer [5, Thm. 2] and Zhang [52, Prop. 3.5] explicitly describe variations of the Fock-Goncharov parametrization when Γ is cocompact, and their analyses should extend to our setting.)

Theorem 1.1 (Fock-Goncharov [18], Hitchin [22])**.** *If $\Gamma\subset\mathsf{PSL}(2,\mathbb{R})$ is torsion-free and geometrically finite, then the cusped Hitchin component $\mathcal{H}_d(\Gamma)$ is an analytic manifold diffeomorphic to $\mathbb{R}^m$ for some $m\in\mathbb{N}$.*

If

$$\mathfrak{a}=\left\{\vec{x}\in\mathbb{R}^d \mid \sum x_i=0\right\}$$

is the standard Cartan algebra for $\mathsf{PSL}(d,\mathbb{R})$, let

$$\Delta=\left\{\phi=\sum_{i=1}^{d-1}a_i\alpha_i \mid a_i\geq 0\ \forall\ i,\ \sum a_i>0\right\}\subset\mathfrak{a}^*$$

where α_i is the simple root given by $\alpha_i(\vec{x})=x_i-x_{i-1}$. Notice that Δ is exactly the collection of linear functionals which are strictly positive on the interior of the Weyl chamber

$$\mathfrak{a}^+ = \{\vec{x} \in \mathfrak{a} \mid x_1 \geqslant \ldots \geqslant x_d\}.$$

Consider the *Jordan projection* $\nu : \mathsf{PSL}(d,\mathbb{R}) \to \mathfrak{a}$ given by

$$\nu(A) = (\log \lambda_1(A), \ldots, \log \lambda_d(A))$$

where $\lambda_1(A) \geq \cdots \geq \lambda_d(A)$ are the (ordered) moduli of generalized eigenvalues of A.

If $\phi \in \Delta$ and $\rho \in \mathcal{H}_d(\Gamma)$, denote by $\ell^\phi_\rho(\gamma) = \phi(\nu(\rho(\gamma)))$ the ϕ-length of $\gamma \in \Gamma$. We may define the ϕ-entropy of ρ as

$$h^\phi(\rho) = \lim_{T\to\infty} \frac{\# R^\phi_T(\rho)}{T}$$

where $[\Gamma_{hyp}]$ is the set of conjugacy classes of hyperbolic elements in Γ, and

$$R^\phi_T(\rho) = \left\{[\gamma] \in [\Gamma_{hyp}] \mid \ell^\phi_\rho(\gamma) \leq T\right\}.$$

Moreover, if $\rho, \eta \in \mathcal{H}_d(\Gamma)$, we may define the ϕ-pressure intersection

$$I^\phi(\rho,\eta) = \lim_{T\to\infty} \frac{1}{|R^\phi_T(\rho)|} \sum_{[\gamma]\in R^\phi_T(\rho)} \frac{\ell^\phi_\eta(\gamma)}{\ell^\phi_\rho(\gamma)},$$

and a renormalized ϕ-pressure intersection

$$J^\phi(\rho,\eta) = \frac{h^\phi(\eta)}{h^\phi(\rho)} I^\phi(\rho,\eta).$$

Our key tool in the construction of the pressure metric will be results of Bray, Canary, Kao and Martone [8] and Canary, Zhang and Zimmer [15] which combine to prove that all these quantities vary analytically. See [11, Section 8.1] for the analogous statement when Γ is cocompact.

Theorem 1.2. *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is torsion-free and geometrically finite and $\phi \in \Delta$, then $h^\phi(\rho)$ varies analytically over $\mathcal{H}_d(\Gamma)$ and I^ϕ and J^ϕ vary analytically over $\mathcal{H}_d(\Gamma)\times\mathcal{H}_d(\Gamma)$. Moreover, if $\rho,\eta \in \mathcal{H}_d(\Gamma)$, then*

$$J^\phi(\rho,\eta) \geq 1$$

and $J^\phi(\rho,\eta) = 1$ if and only if $\ell^\phi_\rho(\gamma) = \frac{h^\phi(\eta)}{h^\phi(\rho)}\ell^\phi_\eta(\gamma)$ for all $\gamma \in \Gamma$.

Given $\phi \in \Delta$, we define a pressure form on the Hitchin component, by letting

$$\mathbb{P}^\phi\big|_{T_\rho\mathcal{H}_d(\Gamma)} = \mathrm{Hess}(J^\phi(\rho,\cdot)).$$

Since J^ϕ achieves its minimum along the diagonal, $\mathbb{P}^\phi$ will always be non-negative. However, it will not always be non-degenerate. Typically, the most difficult portion of the proof of the construction of a pressure metric is to verify non-degeneracy, or, more generally, to characterize which vectors are degenerate.

We first consider the first fundamental weight $\omega_1 \in \Delta$, given by $\omega_1(\vec{x}) = x_1$. As a consequence of a much more general result, Bridgeman, Canary, Labourie and Sambarino [11] prove that $\mathbb{P}^{\omega_1}$ is non-degenerate on the Hitchin component of a convex cocompact Fuchsian group. We will follow the outline of the proof of non-degeneracy in the survey article [13] which makes use of simplifications arising from restricting to the Hitchin setting. We recall that the mapping class group $\mathrm{Mod}(\Gamma)$ is the group of (isotopy classes of) orientation-preserving self-homeomorphisms of $\mathbb{H}^2/\Gamma$.

Theorem 1.3. *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is torsion-free and geometrically finite, then the pressure form $\mathbb{P}^{\omega_1}$ is non-degenerate, so gives rise to a mapping class group invariant, analytic Riemannian metric on $\mathcal{H}_d(\Gamma)$.*

Bridgeman, Canary, Labourie and Sambarino [12] later expanded their techniques to show that the first simple root gives rise to a non-degenerate pressure metric on the Hitchin component of a closed surface group. We implement their outline in the cusped setting. We make crucial use of a result of Canary, Zhang and Zimmer [16] which assures us that simple root entropies are constant on the Hitchin components of Fuchsian lattices (which generalizes a result of Potrie and Sambarino [40] for Hitchin components of closed surface groups).

Theorem 1.4. *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is a torsion-free lattice, then the pressure form $\mathbb{P}^{\alpha_1}$ is non-degenerate, so gives rise to a mapping class group invariant, analytic Riemannian metric on $\mathcal{H}_d(\Gamma)$.*

Finally, we consider the functional ω_H associated to the Hilbert length given by $\omega_H(\vec{x}) = x_1 - x_d$. It is easy to see that if $C : \mathcal{H}_d(\Gamma) \to \mathcal{H}_d(\Gamma)$ is the contragredient involution and $\vec{v} \in T\mathcal{H}_d(\Gamma)$ is *anti-self-dual*, i.e. $\mathrm{D}C(\vec{v}) = -\vec{v}$, then $\mathbb{P}^{\omega_H}(\vec{v},\vec{v}) = 0$ (see [13, Lem. 5.22]). In particular, $\mathbb{P}^{\omega_H}$ is not globally non-degenerate. However, when $d = 3$, we show that these are the only non-trivial vectors which are degenerate for $\mathbb{P}^{\omega_H}$, which allows us to see that $\mathbb{P}^{\omega_H}$ gives rise to a path metric on $\mathcal{H}_d(\Gamma)$. All such vectors lie on the Fuchsian locus, which consists of representations which are images of Fuchsian groups in $\mathsf{PSL}(2,\mathbb{R})$ under the irreducible representation.

Theorem 1.5. *Suppose that $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is torsion-free and geometrically finite. If $\vec{v} \in \mathsf{T}\mathcal{H}_3(\Gamma)$ is non-zero, then $\mathbb{P}^{\omega_H}(\vec{v},\vec{v}) = 0$ if and only if $\vec{v}$ is anti-self-dual. Therefore, $\mathbb{P}^{\omega_H}$ gives rise to a mapping class group invariant path metric on $\mathcal{H}_3(\Gamma)$ which is an analytic Riemannian metric off of the Fuchsian locus.*

When $d = 3$, cusped Hitchin representations of a torsion-free lattice are holonomy maps of finite area convex projective surfaces and the Hilbert length is the translation length with respect to the Hilbert metric. In this case, the analogy with the augmented Teichmüller space is most compelling and we expect that this case may be the easiest case in which to begin the analysis of the augmented Hitchin component. Notice that our proposed augmented Hitchin component would be a proper subspace of the augmented Hitchin component introduced and studied in [30].

We expect that the natural analogue of Theorem 1.5 should hold for all d. In general, the degenerate vectors should be exactly anti-self-dual vectors. In particular, all such vectors should be based at the self-dual locus, i.e. $\{\rho \in \mathcal{H}_d(\Gamma) \mid C(\rho) = \rho\}$.

Finally, we remark that if Γ is geometrically finite but has torsion, then it has a finite index normal subgroup Γ_0 which is torsion-free. One may identify Γ/Γ_0 with a finite index subgroup G of the mapping class group of $\mathbb{H}^2/\Gamma_0$ and then identify $\mathcal{H}_d(\Gamma)$ with the submanifold of $\mathcal{H}_d(\Gamma_0)$ which is stabilized by G. It follows that one obtains mapping class group invariant analytic Riemannian metrics $\mathbb{P}^{\omega_1}$ and $\mathbb{P}^{\alpha_1}$ on $\mathcal{H}_d(\Gamma)$ and a mapping class group invariant path metric on $\mathcal{H}_3(\Gamma)$ which is analytic Riemannian off of the Fuchsian locus.

Historical remarks. Thurston described a metric on Teichmüller space which was the "Hessian of the length of a random geodesic." Wolpert [51] showed that this metric gives a scalar multiple of the classical Weil-Petersson metric. Bonahon [4] reinterpreted Thurston's metric in terms of geodesic currents. McMullen [37] showed that one may interpret Thurston's metric in terms of Thermodynamic formalism, as the Hessian of a pressure intersection function. Bridgeman [9] generalized McMullen's construction to the setting of quasifuchsian space. Bridgeman, Canary, Labourie and Sambarino [11] then showed how to use his construction to produce analytic Riemannian metrics at "generic" smooth points of deformation spaces of projective Anosov representations, and in particular on Hitchin components. Pollicott and Sharp [39] gave an alternate interpretation of this metric.

Kao [23] used countable Markov codings to construct pressure metrics on Teichmüller spaces of punctured surfaces. Bray, Canary and Kao [7] generalized this to the setting of cusped quasifuchsian groups.

2. Background

2.1. **Linear algebra.** The *Cartan projection* $\kappa\colon \mathsf{SL}(d,\mathbb{R}) \to \mathfrak{a}^+$ is

$$\kappa(A) = (\log \sigma_1(A), \ldots, \log \sigma_d(A))$$

where $\{\sigma_i(A)\}_{i=1}^d$ are the singular values of A labelled in decreasing order. Recall that each element of $\mathsf{SL}(d,\mathbb{R})$ may be written as $A = KDL$ where $K, L \in \mathsf{SO}(d)$ and D is the diagonal matrix with (i,i) entry given by $\sigma_i(A)$. If $\alpha_k(\kappa(A)) > 0$, then $U_k(A) = K(\langle e_1, \ldots, e_k\rangle)$ is well-defined and is the k-plane spanned by the first k major axes of $A(S^{d-1})$.

If $\rho \in \mathcal{H}_d(\Gamma)$, the *Benoist limit cone* of ρ is

$$\mathcal{B}(\rho) = \bigcap_{n\geqslant 0} \overline{\bigcup_{\|\kappa(\rho(\gamma))\|\geqslant n} \mathbb{R}_+\kappa(\rho(\gamma))} \subset \mathfrak{a}^+.$$

The positive dual to the Benoist limit cone is given by

$$\mathcal{B}(\rho)^+ = \left\{\phi \in \mathfrak{a}^* \mid \phi\left(\mathcal{B}(\rho) - \{\vec{0}\}\right) \subset (0,\infty)\right\}.$$

Quint [41] introduced a vector valued smooth cocycle, called the *Iwasawa cocycle*,

$$B : \mathsf{SL}(d,\mathbb{R}) \times \mathcal{F}_d \to \mathfrak{a}$$

where $\mathcal{F}_d$ is the space of (complete) flags in $\mathbb{R}^d$. Let F_0 denote the standard flag

$$F_0 = (\langle e_1\rangle, \langle e_1, e_2\rangle, \ldots, \langle e_1, \ldots, e_{d-1}\rangle).$$

We can write any $F \in \mathcal{F}_d$ as $F = K(F_0)$ where $K \in \mathsf{SO}(d)$. If $A \in \mathsf{SL}(d,\mathbb{R})$ and $F \in \mathcal{F}_d$, the Iwasawa decomposition of AK has the form QZU where $Q \in \mathsf{SO}(d)$, Z is a diagonal matrix with non-negative entries, and U is unipotent and upper triangular. Then $B(A,F) = (\log z_{11}, \ldots, \log z_{dd})$.

Note that the Jordan and Cartan projections (resp. Iwasawa cocycle) descend to well-defined functions on $\mathsf{PSL}(d,\mathbb{R})$ (resp. $\mathsf{PSL}(d,\mathbb{R})\times\mathcal{F}_d$).

We say that a basis $b=(b_1,\dots,b_d)$ is *consistent* with a pair (F,G) of transverse flags if $\langle b_i\rangle = F^i\cap G^{d-i+1}$ for all i. We denote by $U(b)_{>0}\subset\mathsf{SL}(d,\mathbb{R})$ the subsemigroup of upper triangular unipotent matrices which are totally positive with respect to b, i.e. $A\in U(b)_{>0}$ in the basis b is upper triangular unipotent and the determinants of all the minors of A are positive, unless they are forced to be zero by the fact that A is upper triangular.

Then, a k-tuple $(F_1,\dots,F_k)$ in $\mathcal{F}_d$ is *positive* if there exists a basis b consistent with (F_1,F_k) and there exists $\{u_2,\dots,u_{k-1}\}\in U(b)_{>0}$ so that $F_i=u_{k-1}\cdots u_iF_k$ for all $i=2,\dots,k-1$. If X is a subset of S^1, we say that a map $\xi: X\to\mathcal{F}_d$ is *positive* if whenever $(x_1,\dots,x_k)$ is a consistently ordered k-tuple in X (ordered either clockwise or counter-clockwise), then $(\xi(x_1),\dots,\xi(x_k))$ is a positive k-tuple of flags.

2.2. **Thermodynamic Formalism.** In this section, we recall the background results we will need from the Thermodynamic Formalism for countable Markov shifts as developed by Gurevich-Savchenko [21], Mauldin-Urbanski [36] and Sarig [48].

Given a countable alphabet $\mathcal{A}$ and a transition matrix $\mathbb{T}=(t_{ab})\in\{0,1\}^{\mathcal{A}\times\mathcal{A}}$ a one-sided Markov shift is

$$\Sigma^+=\{x=(x_i)\in\mathcal{A}^{\mathbb{N}}\mid t_{x_ix_{i+1}}=1\text{ for all } i\in\mathbb{N}\}$$

equipped with a shift map $\sigma:\Sigma^+\to\Sigma^+$ which takes $(x_i)_{i\in\mathbb{N}}$ to $(x_{i+1})_{i\in\mathbb{N}}$. One says that (Σ^+,σ) is *topologically mixing* if for all $a,b\in\mathcal{A}$, there exists $N=N(a,b)$ so that if $n\geq N$, then there exists $x\in\Sigma$ so that $x_1=a$ and $x_n=b$. The shift (Σ^+,σ) has the big images and pre-images property (BIP) if there exists a finite subset $\mathcal{B}\subset\mathcal{A}$ so that if $a\in\mathcal{A}$, then there exists $b_0,b_1\in\mathcal{B}$ so that $t_{b_0,a}=1=t_{a,b_1}$.

Given a one-sided countable Markov shift (Σ^+,σ) and a function $g:\Sigma^+\to\mathbb{R}$, we say that g is *locally Hölder continuous* if there exists $C>0$ and $\theta\in(0,1)$ so that if $x,y\in\Sigma^+$ and $x_i=y_i$ for all $1\leq i\leq n$, then

$$|g(x)-g(y)|\leq C\theta^n.$$

If $n\in\mathbb{N}$, the n^{th} *ergodic sum* of g at $x\in\Sigma^+$ is

$$S_n(g)(x)=\sum_{i=1}^n g(\sigma^{i-1}(x))$$

and $\mathrm{Fix}^n=\{x\in\Sigma^+\mid\sigma^n(x)=x\}$ is the set of periodic words with period n.

The *pressure* of a locally Hölder continuous function $g:\Sigma^+\to\mathbb{R}$ is defined to be

$$P(g)=\sup_{m\in\mathcal{M}} h_\sigma(m)+\int_{\Sigma^+} g\ dm$$

where $\mathcal{M}$ is the space of σ-invariant probability measures on Σ^+ and $h_\sigma(m)$ is the measure-theoretic entropy of σ with respect to the measure m.

A σ-invariant Borel probability measure m on Σ^+ is an *equilibrium measure* for a locally Hölder continuous function $g:\Sigma^+\to\mathbb{R}$ if

$$P(g)=h_\sigma(m)+\int_{\Sigma^+} g\ dm.$$

Mauldin-Urbanski [36, Thm. 2.6.12, Prop. 2.6.13 and 2.6.14] and Sarig ([47, Cor. 4], [48, Thm 5.10 and 5.13]) prove that the pressure function is real analytic in our setting and compute its derivatives. Recall that $\{g_u : \Sigma^+ \to \mathbb{R}\}_{u\in M}$ is a *real analytic family* if M is a real analytic manifold and for all $x \in \Sigma^+$, $u \to g_u(x)$ is a real analytic function on M and that if m is a probability measure on Σ^+ then its variance is given by

$$\mathrm{Var}(f,m) = \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma^+} S_n\Big(\big(f - \int_{\Sigma^+} f\ dm\big)^2\Big)\ dm.$$

Theorem 2.1. (Mauldin-Urbanski, Sarig) *Suppose that (Σ^+,σ) is a one-sided countable Markov shift which has BIP and is topologically mixing. If $\{g_u : \Sigma^+ \to \mathbb{R}\}_{u\in M}$ is a real analytic family of locally Hölder continuous functions such that $P(g_u) < \infty$ for all u, then $u \to P(g_u)$ is real analytic.*

Moreover, if $\vec{v} \in T_{u_0}M$ and there exists a neighborhood U of u_0 in M so that if $u \in U$, then $-\int_{\Sigma^+} g_u dm_{g_{u_0}} < \infty$, then

$$D_{\vec{v}}P(g_u) = \int_{\Sigma^+} D_{\vec{v}}(g_u(x))\ dm_{g_{u_0}}$$

and

$$D^2_{\vec{v}}P(g_u) = \mathrm{Var}(D_{\vec{v}}g_u, m_{g_{u0}}) + \int_{\Sigma^C_r ho+} D^2_{\vec{v}}g_u dm_{g_{u_0}}$$

where $m_{g_{u_0}}$ is the unique equilibrium state for g_{u_0}.

In the case of finite Markov shifts, the assumption that $P(g_u) < \infty$ is automatically satisfied and Theorem 2.1 result is due to Ruelle [42] and Parry-Pollicott [38].

Dal'bo and Peigné [17], when the surface has infinite area, and Stadlbauer [49] and Ledrappier-Sarig [29], when the surface has finite area, constructed countable Markov coding for the recurrent portion of the geodesic flow of a geometrically finite, but not convex cocompact, hyperbolic surface. In the case that Γ is convex cocompact, one can use the finite Markov coding of Bowen and Series [6]. We summarize their crucial properties below (see [7] for a more complete description in our language).

Theorem 2.2. (Bowen-Series [6], Dal'bo-Peigné [17], Ledrappier-Sarig [29], Stadlbauer [49]) *Suppose that Γ is a torsion-free geometrically finite Fuchsian group. There exists a topologically mixing Markov shift $(\Sigma^+,\mathcal{A})$ with countable alphabet $\mathcal{A}$ with (BIP) which codes the recurrent portion of the geodesic flow on T^1S. There exist maps*

$$G : \mathcal{A} \to \Gamma, \quad \omega : \Sigma^+ \to \Lambda(\Gamma), \quad r : \mathcal{A} \to \mathbb{N}, \quad \text{and} \quad s : \mathcal{A} \to \Gamma$$

with the following properties.

1. *ω is locally Hölder continuous, finite-to-one and $\omega(\Sigma^+) = \Lambda_c(\Gamma)$, i.e. the complement in $\Lambda(\Gamma)$ of the set of fixed points of parabolic elements of Γ. Moreover, $\omega(x) = G(x_1)\omega(\sigma(x))$ for every $x \in \Sigma^+$.*
2. *If $x \in \mathrm{Fix}^n$, then $\omega(x)$ is the attracting fixed point of $G(x_1)\cdots G(x_n)$. Moreover, if $\gamma \in \Gamma$ is hyperbolic, then there exists $x \in \mathrm{Fix}^n$ (for some n) so that γ is conjugate to $G(x_1)\cdots G(x_n)$ and x is unique up to cyclic permutation.*
3. *There exists $D \in \mathbb{N}$ so that $1 \le \#(r^{-1}(n)) \le D$ for all $n \in \mathbb{N}$.*

2.3. **Cusped Hitchin representations.** We next recall the definition of a cusped Hitchin representation and the results of Bray-Canary-Kao-Martone [8] and Canary-Zhang-Zimmer [15] which will play a crucial role in our work.

Let Γ be a geometrically finite Fuchsian group and let $\Lambda(\Gamma) \subset \partial\mathbb{H}^2$ be its limit set. Following Fock and Goncharov [18], a *Hitchin representation* $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is a representation such that there exists a ρ-equivariant positive map $\xi_\rho : \Lambda(\Gamma) \to \mathcal{F}_d$. If S is closed, Hitchin representations are just the traditional Hitchin representations introduced by Hitchin [22] and further studied by Labourie [27]. When Γ contains a parabolic element, we sometimes refer to these Hitchin representations as cusped Hitchin representations to distinguish them from the traditional Hitchin representations.

Canary, Zhang and Zimmer establish fundamental properties of Hitchin representations of torsion-free geometrically finite Fuchsian groups which generalize the properties of classical Hitchin representations. (Part (1) was independently established by Sambarino [44].)

Theorem 2.3. (Canary-Zhang-Zimmer [15]) *Suppose that $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is a Hitchin representation.*

(1) *ρ is strongly irreducible.*
(2) *If $\gamma \in \Gamma$ is hyperbolic, then $\rho(\gamma)$ is loxodromic, i.e. $\nu(\rho(\gamma))$ lies in the interior of $\mathfrak{a}^+$.*
(3) *There exist $A, a > 0$ so that if $\gamma \in \Gamma$ and $k \in \{1,\dots,d-1\}$, then*
$$Ae^{ad(b_0,\gamma(b_0))} \geq e^{\alpha_k(\kappa(\rho(\gamma)))} \geq \frac{1}{A}e^{\frac{d(b_0,\gamma(b_0))}{a}}$$
where b_0 is a basepoint for $\mathbb{H}^2$.
(4) *ρ has the P_k-*Cartan property *for every $k \in \{1,\dots,d-1\}$, i.e. whenever $\{\gamma_n\}$ is a sequence of distinct elements of Γ such that $\gamma_n(b_0)$ converges to $z \in \Lambda(\Gamma)$, then the k-plane in the flag $\xi_\rho(z)$ equals $\lim U_k(\rho(\gamma_n))$.*

They also show that limit maps of cusped Hitchin representations vary analytically.

Theorem 2.4. (Canary-Zhang-Zimmer [15]) *If $\{\rho_u : \Gamma \to \mathsf{PSL}(d,\mathbb{R})\}_{u\in M}$ is a real analytic family of Hitchin representations of a torsion-free geometrically finite Fuchsian group and $z \in \Lambda(\Gamma)$, then the map from M to $\mathcal{F}_d$ given by $u \to \xi_{\rho_u}(z)$ is real analytic.*

In previous work [8], we constructed potentials on the Markov shift which encode the spectral properties of cusped Hitchin representations. First define a vector-valued roof function $\tau_\rho : \Sigma^+ \to \mathfrak{a}$ by
$$\tau_\rho(x) = B\big(\rho(G(x_1)), \rho(G(x_1))^{-1}(\xi_\rho(\omega(x)))\big).$$
If $\phi \in \mathcal{B}(\rho)^+$, one defines the roof function $\tau_\rho^\phi = \phi \circ \tau_\rho$. Notice that since $\mathcal{B}(\rho)$ is contained in the interior of the positive Weyl chamber $\mathfrak{a}^+$, Δ is contained in $\mathcal{B}(\rho)^+$. In particular, every fundamental weight ω_i is contained in $\mathcal{B}(\rho)^+$.

We use the Thermodynamic Formalism for countable Markov shifts to analyze these potentials. In particular, we use a renewal theorem of Kesseböhmer and Kombrink [26] to generalize arguments of Lalley [28] to establish counting and equidistribution results in our setting. We summarize the results we will need from their work below.

Theorem 2.5. (Bray-Canary-Kao-Martone [8]) *Suppose that $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is a Hitchin representation and $\phi \in \mathcal{B}(\rho)^+$. Then, there exists a locally Hölder continuous function $\tau_\rho^\phi = \phi \circ \tau_\rho : \Sigma^+ \to \mathbb{R}$ such that*

(1) *τ_ρ^ϕ is eventually positive, i.e. there exist $N \in \mathbb{N}$ and $B > 0$ such that $S_n\tau_\rho^\phi(x) > B$ for all $n \geqslant N$ and $x \in \Sigma^+$.*
(2) *There exists $C_\rho > 0$ so that if $x \in \Sigma^+$ and $i \in \{1, \dots, d-1\}$, then*
$$\left|\tau_\rho^{\alpha_i}(x) - 2\log r(x_1)\right| \leq C_\rho.$$
(3) *If $x = \overline{x_1 \cdots x_n}$ is a periodic element of Σ^+, then*
$$S_n\tau_\rho^\phi(x) = \ell_\rho^\phi\big(G(x_1)\cdots G(x_n)\big).$$
(4) *If $\phi = \sum_{i=1}^{d-1} a_i\alpha_i$, then $P(-h\tau_\rho^\phi)$ is finite if and only if $h > d(\phi) = \frac{1}{2(a_1+\cdots+a_{d-1})}$.*
(5) *The ϕ-entropy $h^\phi(\rho)$ of ρ is the unique solution of $P(-h\tau_\rho^\phi) = 0$. Moreover,*
$$\lim_{T\to\infty} \frac{h^\phi(\rho)TR_T^\phi(\rho)}{e^{h^\phi(\rho)T}} = 1.$$
(6) *There is a unique equilibrium measure m_ρ^ϕ for $-h^\phi(\rho)\tau_\rho^\phi$.*

We also established a rigidity theorem for renormalized pressure intersection and use our equidistribution result to give a thermodynamical reformulation of the pressure intersection.

Theorem 2.6. (Bray-Canary-Kao-Martone [8]) *If $\mathcal{H}_d(\Gamma)$ is a Hitchin component, $\rho, \eta \in \mathcal{H}_d(\Gamma)$ and $\phi \in \Delta$, then*
$$J^\phi(\rho,\eta) \geq 1$$
with equality if and only if
$$\ell_\rho^\phi(\gamma) = \frac{h_\phi(\eta)}{h_\phi(\rho)}\ell_\eta^\phi(\gamma)$$
for all $\gamma \in \Gamma$. Moreover,
$$I^\phi(\rho,\eta) = \frac{\int_{\Sigma^+} \tau_\eta^\phi \; dm_\rho^\phi}{\int_{\Sigma^+} \tau_\rho^\phi \; dm_\rho^\phi}$$
and $-I^\phi(\rho,\eta)$ is the slope of the tangent line at $(h^\phi(\rho), 0)$ to
$$\mathcal{C}^\phi(\rho,\eta) = \{(a,b) \in \mathbb{R}^2 \mid P(-a\tau_\rho^\phi - b\tau_\eta^\phi) = 0, a \geqslant 0, b \geqslant 0, a+b > 0\}.$$

Historical remarks: The results in this subsection generalize earlier results in the case when Γ is convex cocompact. More precisely, when Γ is convex cocompact, Theorem 2.3 follows from work of Labourie [27], Fock and Goncharov [18], Guichard-Wienhard [20]. Kapovich, Leeb and Porti [24, 25], Guérituad, Guichard, Kassel, and Wienhard [19] and Tsouvalas [50], Theorems 2.4 and 2.6 are due to Bridgeman, Canary, Labourie, and Sambarino [11] and Theorem 2.5 is due to Sambarino [43].

3. Entropy, intersection and the pressure form

It is now easy to establish Theorem 1.2 as a consequence of the work described in the previous section. Notice that h^ϕ, I^ϕ and J^ϕ are all invariant under conjugation, so descend to functions defined on $\mathcal{H}_d(\Gamma)$ or $\mathcal{H}_d(\Gamma)\times\mathcal{H}_d(\Gamma)$.

Theorem 1.2. *If $\phi\in\Delta$, then $h^\phi(\rho)$ varies analytically over $\mathcal{H}_d(\Gamma)$ and I^ϕ and J^ϕ vary analytically over $\mathcal{H}_d(\Gamma)\times\mathcal{H}_d(\Gamma)$. Moreover, if $\rho,\eta\in\mathcal{H}_d(\Gamma)$, then*

$$J^\phi(\rho,\eta)\geq 1$$

and $J^\phi(\rho,\eta)=1$ if and only if $\ell^\phi_\rho(\gamma)=\frac{h^\phi(\eta)}{h^\phi(\rho)}\ell^\phi_\eta(\gamma)$ for all $\gamma\in\Gamma$.

Proof. Let $\widetilde{\mathcal{H}}_d(\Gamma)\subset \mathrm{Hom}(\Gamma,\mathsf{PSL}(d,\mathbb{R}))$ denote the component consisting of Hitchin representations. Theorem 2.4 implies that the limit map ξ_ρ varies analytically over $\widetilde{\mathcal{H}}_d(\Gamma)$. Since $\tau_\rho(x)=B(\rho(G(x_1)),\rho(G(x_1))^{-1}(\xi_\rho(\omega(x)))$ and B is analytic we see that τ_ρ, and hence $\tau^\phi_\rho=\phi\circ\tau_\rho$, varies analytically over $\widetilde{\mathcal{H}}_d(\Gamma)$. It then follows from Theorem 2.5 that P is analytic on $(d(\phi),\infty)\times\widetilde{\mathcal{H}}_d(\Gamma)$. Since $P(-h^\phi(\rho)\tau^\phi_\rho)=0$ and

$$\frac{d}{dt}P(-t\tau^\phi_\rho)\Big|_{t=h^\phi(\rho)}=-\int\tau^\phi_\rho\;dm^\phi_\rho<0$$

for all $\rho\in\widetilde{\mathcal{H}}_d(\Gamma)$, the Implicit Function Theorem implies that $h^\phi(\rho)$ varies analytically over $\widetilde{\mathcal{H}}_d(\Gamma)$, and hence over $\mathcal{H}_d(\Gamma)$.

If $\phi=\sum_{k=1}^{d-1}a_k\alpha_k$, then let $c(\phi)=\sum a_k$, so $d(\phi)=\frac{1}{2c(\phi)}$. Let

$$R=\widetilde{\mathcal{H}}_d(\Gamma)\times\widetilde{\mathcal{H}}_d(\Gamma)\times\hat{D}_\phi\qquad\text{where}\qquad \hat{D}_\phi=\left\{(a,b)\in\mathbb{R}^2\;|a+b>d(\phi)\right\}$$

and let $P_R:R\to\mathbb{R}$ be given by $P_R(\rho,\eta,a,b)=P(-a\tau^\phi_\rho-b\tau^\phi_\eta)$. Mauldin and Urbanski [36, Thm. 2.1.9] show that if f is locally Hölder continuous, then $P(f)$ is finite if and only if $Z_1(f)<+\infty$, where

$$Z_1(f)=\sum_{s\in\mathcal{A}}e^{\sup\{f(x)\;:\;x_1=s\}}<+\infty.$$

By grouping the terms so that $r(s)=n$, Theorem 2.5 implies that

$$Z_1(-a\tau^\phi_\rho-b\tau^\phi_\eta)\leq D\sum_{n=1}^{\infty}e^{-(a+b)(2c(\phi)\log n-c(\phi)\max\{C_\rho,C_\eta\})}$$

so $P(-a\tau^\phi_\rho-b\tau^\phi_\eta)<+\infty$ if $a+b>d(\phi)$. Therefore, P_R is finite on R, and hence, by Theorem 2.1, analytic on R. As above, P_R is a submersion on $P_R^{-1}(0)$, so $P_R^{-1}(0)$ is an analytic submanifold of R. Moreover, by Theorem 2.6, $-I^\phi(\rho,\eta)$ is the slope of the tangent line to $P_R^{-1}(0)\cap\{(\rho,\eta)\times\hat{D}_\phi\}$ at the point $(\rho,\eta,(h(\rho),0))$, so $I^\phi(\rho,\eta)$ is analytic. Since entropy is analytic, it follows immediately that $J^\phi(\rho,\eta)$ is analytic.

The final claim follows directly from Theorem 2.6. □

Recall that given $\phi\in\Delta$, we define a pressure form on the Hitchin component by letting

$$\mathbb{P}^\phi\big|_{T_\rho\mathcal{H}_d(\Gamma)}=\mathrm{Hess}(J^\phi(\rho,\cdot))$$

where

$$J^\phi(\rho,\eta) = \frac{h^\phi(\eta)}{h^\phi(\rho)} \lim_{T\to\infty} \frac{1}{|R_T^\phi(\rho)|} \sum_{[\gamma]\in R_T^\phi(\rho)} \frac{\ell_\eta^\phi(\gamma)}{\ell_\rho^\phi(\gamma)}.$$

If $\vec{v} = \frac{d}{dt}\big|_{t=0}[\rho_t]$ where $\{\rho_t\}_{t\in(-\epsilon,\epsilon)}$ is a one-parameter analytic family in $\widetilde{\mathcal{H}}_d(\Gamma)$, then

$$\mathbb{P}^\phi(\vec{v},\vec{v}) = \frac{d^2}{dt^2}\Big|_{t=0} J^\phi(\rho_0,\rho_t) = \frac{\mathrm{Var}\left(\frac{d}{dt}\big|_{t=0}\left(-h^\phi(\rho_t)\tau^\phi_{\rho_t}\right), m^\phi_{\rho_0}\right)}{\int_{\Sigma^+} h^\phi(\rho_0)\tau^\phi_{\rho_0}\, dm^\phi_{\rho_0}}.$$

We note that the exact same definitions apply when Γ is a cocompact lattice, see [13, Sect. 5.4]. We observe the following immediate properties.

Proposition 3.1. *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is torsion-free and geometrically finite and $\phi\in\Delta$, then $\mathbb{P}^\phi$ is analytic, mapping class group invariant and non-negative, i.e. if $\vec{v}\in\mathsf{T}\mathcal{H}_d(\Gamma)$, then $\mathbb{P}^\phi(\vec{v},\vec{v})\geq 0$.*

Proof. If ψ lies in the mapping class group of Γ, then $\ell^\phi_\gamma(\rho\circ\psi) = \ell^\phi_{\psi^{-1}(\gamma)}(\rho)$, so $R^\phi_T(\rho\circ\psi) = \psi^{-1}\big(R^\phi_T(\rho)\big)$ and $h^\phi(\rho) = h^\phi(\psi\circ\rho)$, so $J^\phi(\rho,\eta) = J^\phi(\rho\circ\psi,\eta\circ\psi)$, which implies that $\mathbb{P}^\phi$ is mapping class group invariant. The pressure form $\mathbb{P}^\phi$ is analytic, since J^ϕ is analytic and is non-negative since J^ϕ achieves its minimum along the diagonal, see Theorem 2.6. □

The following degeneracy criterion for pressure metrics is standard in the setting of finite Markov shifts, see for example [13, Cor. 2.5], but requires a little more effort in the setting of countable Markov shifts. In our setting, this criterion is established exactly as in Lemma 8.1 in [7].

Proposition 3.2. *If $\vec{v}\in\mathsf{T}\mathcal{H}_d(\Gamma)$ and $\phi\in\Delta$, then $\mathbb{P}^\phi(\vec{v},\vec{v}) = 0$ if and only if*

$$\mathrm{D}_{\vec{v}}\left(h^\phi\ell^\phi_\gamma\right) = 0$$

for all $\gamma\in\Gamma$.

We next observe that $\mathrm{D}_{\vec{v}}\log\ell^\phi_\gamma$ is independent of γ if $\vec{v}$ is degenerate and γ is hyperbolic.

Lemma 3.3. *If $\vec{v}\in\mathsf{T}\mathcal{H}_d(\Gamma)$ and $\mathbb{P}^\phi(\vec{v},\vec{v}) = 0$, then,*

$$\mathrm{D}_{\vec{v}}\ell^\phi_\gamma = K\ell^\phi_\gamma$$

for all $\gamma\in\Gamma$, where $K = -\frac{\mathrm{D}_{\vec{v}}h^\phi}{h^\phi(\rho)}$.

Proof. By Proposition 3.2, if $\vec{v}$ is degenerate, then

$$h^\phi(\rho)\mathrm{D}_{\vec{v}}\ell^\phi_\gamma = -\big(\mathrm{D}_{\vec{v}}h^\phi\big)\ell^\phi_\gamma$$

for all hyperbolic $\gamma\in\Gamma$, so $\mathrm{D}_{\vec{v}}\ell^\phi_\gamma = K\ell^\phi_\gamma$ for all hyperbolic $\gamma\in\Gamma$. If $\gamma\in\Gamma$ is parabolic, then ℓ^ϕ_γ is the zero function, so the condition holds trivially. □

Recall that if M is a real analytic manifold, an analytic function $f: M\to\mathbb{R}$ has *log-type* K at $v\in\mathsf{T}_uM$ if $f(u)\neq 0$ and

$$\mathrm{D}_u\log(|f|)(v) = K\log(|f(u)|).$$

In this language, Lemma 3.3 implies that if $\Lambda^\phi_\gamma = e^{\ell^\phi_\gamma}$ and $\mathbb{P}^\phi(\vec{v},\vec{v}) = 0$ then there exists K so that Λ^ϕ_γ has log-type K at $\vec{v}$ for all $\gamma\in\Gamma$.

4. Trace identities

In this section, we collect several identities for the trace (and related functions) of $\rho \in \mathcal{H}_d(\Gamma)$ and $\gamma \in \Gamma$. It will be convenient to identify $\mathcal{H}_d(\Gamma)$ with a subset $\widehat{\mathcal{H}}_d(\Gamma)$ of the character variety

$$X_d(\Gamma) = \mathrm{Hom}(\Gamma, \mathsf{SL}(d,\mathbb{C}))//\mathsf{SL}(d,\mathbb{C}).$$

If Γ is cocompact, then Hitchin [22] showed that there is a component, $\widehat{\mathcal{H}}_d(\Gamma)$, of $X_d(\Gamma)$ and an analytic diffeomorphism $F : \mathcal{H}_d(\Gamma) \to \widehat{\mathcal{H}}_d(\Gamma)$, so that $F([\rho])$ is the conjugacy class of a lift to $\mathsf{SL}(d,\mathbb{R})$ of ρ.

Let $\widetilde{\mathcal{H}}_d(\Gamma)$ denote the set of all Hitchin representations of Γ in $\mathsf{PSL}(d,\mathbb{R})$. If Γ is not cocompact, then, since Γ is a free group and $\widetilde{\mathcal{H}}_d(\Gamma)$ is an analytic manifold, it is easy to define an analytic map

$$F : \widetilde{\mathcal{H}}_d(\Gamma) \to \mathrm{Hom}(\Gamma, \mathsf{SL}(d,\mathbb{R}))$$

so that $F(\rho)$ is a lift of ρ. Since Hitchin representations are strongly irreducible, see Theorem 2.3, Schur's Lemma implies that $F(\rho)$ is conjugate to $F(\eta)$ in $\mathsf{SL}(d,\mathbb{C})$ if and only ρ and η are conjugate in $\mathsf{PSL}(d,\mathbb{R})$. Then, again since Hitchin representations are strongly irreducible, it follows that F descends to an analytic embedding $\hat{F} : \mathcal{H}_d(\Gamma) \to X_d(\Gamma)$ whose image lies in the smooth part of $X_d(\Gamma)$. We then let $\widehat{\mathcal{H}}_d(\Gamma) = \hat{F}(\mathcal{H}_d(\Gamma))$. Notice that if d is odd, then $\widehat{\mathcal{H}}_d(\Gamma) = \mathcal{H}_d(\Gamma)$.

If $\gamma \in \Gamma$, there is a complex analytic trace function $\mathrm{Tr}_\gamma : X_d(\Gamma) \to \mathbb{C}$ so that $\mathrm{Tr}_\gamma([\rho])$ is the trace of $\rho(\gamma)$. It is well-known that derivatives of trace functions generate the cotangent space at any smooth point, see for example Lubotzky-Magid [31]. The following consequence will be used to verify the non-degeneracy of pressure forms.

Lemma 4.1. *If $[\rho] \in \widehat{\mathcal{H}}_d(\Gamma)$, then $\{\mathrm{D}_{\vec{v}}\,\mathrm{Tr}_\gamma \mid \gamma \in \Gamma\}$ spans the cotangent space $\mathsf{T}^*_{[\rho]}\widehat{\mathcal{H}}_d(\Gamma)$.*

Even though Tr_γ is not well-defined on $\mathcal{H}_d(\Gamma)$, we will abuse notation by saying that $\mathrm{D}_{\vec{v}}\mathrm{Tr}_\beta = 0$ for some $\vec{v} \in \mathsf{T}\mathcal{H}_d(S)$ if $\mathrm{D}_{DF(\vec{v})}\mathrm{Tr}_\beta = 0$.

If $\gamma \in \Gamma$ is hyperbolic and $[\rho] \in \widehat{\mathcal{H}}_d(\Gamma)$, then $\rho(\gamma)$ is loxodromic, see Theorem 2.3 or [18, Thm. 9.3], so there exists a basis $\{e_1(\rho(\gamma)), \ldots, e_d(\rho(\gamma))\}$ of $\mathbb{R}^d$ consisting of unit length eigenvectors for $\rho(\gamma)$ so that

$$\rho(\gamma)\big(e_i(\rho(\gamma))\big) = \mathsf{L}_i(\rho(\gamma))e_i(\rho(\gamma))$$

for all i, where $\mathsf{L}_i(\rho(\gamma))$ is the eigenvalue of $\rho(\gamma)$ with modulus $\lambda_i(\rho(\gamma))$. We may write

$$\rho(\gamma) = \sum_{i=1}^{d} \mathsf{L}_i(\rho(\gamma))\mathbf{p}_i(\rho(\gamma))$$

where $\mathbf{p}_i(\rho(\gamma))$ is the projection onto the eigenline spanned by $e_i(\rho(\gamma))$ parallel to the hyperplane spanned by the other basis elements.

The following trace identities are a special case of a result of Benoist and Quint [3, Lemma 7.5], see also Benoist [1, Cor 1.6] and [11, Prop. 9.4].

Lemma 4.2. *If $\alpha, \beta \in \Gamma$ are hyperbolic and $[\rho] \in \widehat{\mathcal{H}}_d(\Gamma)$, then*

$$\lim_{n\to\infty} \frac{\lambda_1\big(\rho(\alpha^n\beta^n)\big)}{\lambda_1\big(\rho(\alpha^n)\big)\lambda_1\big(\rho(\beta^n)\big)} = \Big|\,\mathrm{Tr}\,\Big(\mathbf{p}_1\big(\rho(\alpha)\big)\mathbf{p}_1\big(\rho(\beta)\big)\Big)\Big|$$

and

$$\lim_{n\to\infty} \frac{\lambda_1\big(\rho(\alpha^n\beta)\big)}{\lambda_1\big(\rho(\alpha^n)\big)} = \Big|\,\mathrm{Tr}\,\Big(\mathbf{p}_1\big(\rho(\alpha)\big)\rho(\beta)\Big)\Big|.$$

We further observe:

Lemma 4.3. *If $\alpha, \beta \in \Gamma$ are hyperbolic and $[\rho] \in \widehat{\mathcal{H}}_d(\Gamma)$, then*

$$\mathrm{Tr}\,\Big(\mathbf{p}_1\big(\rho(\alpha)\big)\mathbf{p}_1\big(\rho(\beta)\big)\Big) \neq 0 \qquad \textit{and} \qquad \mathrm{Tr}\,\Big(\mathbf{p}_1\big(\rho(\alpha)\big)\rho(\beta)\Big) \neq 0.$$

Moreover, if $\vec{v} \in T_{[\rho]}\widehat{\mathcal{H}}_d(\Gamma)$ is non-zero and $e^{\ell^{\omega_1}(\rho(\gamma))} = \lambda_1(\rho(\gamma))$ is of log-type K at $\vec{v}$ for all $\gamma \in \Gamma$, then

$$\mathrm{Tr}\,\Big(\mathbf{p}_1(\rho(\alpha))\mathbf{p}_1(\rho(\beta))\Big) \qquad \textit{and} \qquad \mathrm{Tr}\,\Big(\mathbf{p}_1(\rho(\alpha))\rho(\beta)\Big)$$

are log-type K at $\vec{v}$.

Proof. If we write $\mathbf{p}_1\big(\rho(\alpha)\big)\mathbf{p}_1\big(\rho(\beta)\big)$ in the basis $\{e_1(\rho(\beta)), \ldots, e_d(\rho(\beta))\}$, one may easily check that

$$\mathrm{Tr}\,\Big(\mathbf{p}_1\big(\rho(\alpha)\big)\mathbf{p}_1\big(\rho(\beta)\big)\Big) = \mathbf{p}_1(\rho(\beta))\Big(\mathbf{p}_1(\rho(\alpha))\big(e_1(\rho(\beta))\big)\Big) \cdot e_1(\rho(\beta))$$

Notice that the transversality of the limit map implies that $e_1(\rho(\beta))$, which lies in $\xi^1_\rho(\beta^+)$, cannot lie in the kernel of $\mathbf{p}_1(\rho(\alpha))$, so $\mathbf{p}_1(\rho(\alpha))(e_1(\rho(\beta)))$ is a non-zero multiple of $e_1(\rho(\alpha))$. Similarly, $e_1(\rho(\alpha))$ cannot lie in the kernel of $\mathbf{p}_1(\rho(\beta))$, so

$$\mathbf{p}_1(\rho(\beta))\Big(\mathbf{p}_1(\rho(\alpha))\big(e_1(\rho(\beta))\big)\Big) \cdot e_1(\rho(\beta)) \neq 0.$$

Next, write $\mathbf{p}_1\big(\rho(\alpha)\big)\rho(\beta)$ with respect to a basis $\{e_1(\rho(\beta)), b_2, \ldots, b_d\}$ where $\{b_2, \ldots, b_d\}$ is a basis for $\rho(\beta)^{-1}\big(\xi^{d-1}_\rho(\alpha^-)\big)$. We then compute that

$$\mathrm{Tr}\,\Big(\mathbf{p}_1\big(\rho(\alpha)\big)\rho(\beta)\Big) = \mathsf{L}_1(\rho(\beta))\mathbf{q}\Big(\mathbf{p}_1(\rho(\alpha))\big(e_1(\rho(\beta))\big)\Big) \cdot e_1(\rho(\beta))$$

where $\mathbf{q}$ is the projection map onto $\langle e_1(\rho(\beta))\rangle = \xi^1_\rho(\beta^+)$ parallel to $\rho(\beta)^{-1}\big(\xi^{d-1}_\rho(\alpha^-)\big) = \xi^{d-1}_\rho\big(\beta^{-1}(\alpha^-)\big)$. The transversality of the limit map again implies that this is non-zero.

The final claim is proved in [11, Prop. 9.4.]. We highlight the main steps of the proof for the function $\mathrm{Tr}\,(\mathbf{p}_1(\rho(\alpha))\mathbf{p}_1(\rho(\beta)))$. Note that by hypothesis, $\mathsf{L}_1(\rho(\gamma))$ is of log-type K at $\vec{v}$. One uses the identity $\rho(\alpha^n\beta^n) = \rho(\alpha^n)\rho(\beta^n)$ to write

$$\mathrm{Tr}(\mathbf{p}_1(\rho(\alpha))\mathbf{p}_1(\rho(\beta))) = \frac{\mathsf{L}_1(\rho(\alpha^n\beta^n))}{\mathsf{L}_1(\rho(\alpha^n))\mathsf{L}_1(\rho(\beta^n))} \cdot \frac{(1+g_n)}{(1+\hat{g}_n)}$$

where $\{g_n\}_{n\in\mathbb{N}}$ and $\{\hat{g}_n\}_{n\in\mathbb{N}}$ are sequences of (explicit) analytic functions such that

$$\lim_{n\to\infty} f_n(z) = 0 \text{ and } \lim_{n\to\infty} \mathrm{D}_z f_n(v) = 0 \qquad \text{for } f_n = g_n \text{ or } \hat{g}_n.$$

Next, observe that ratios and products of (non-zero) log-type K functions at $\vec{v}$ are log-type K at $\vec{v}$. Thus, $G_n = \frac{\mathsf{L}_1(\rho(\alpha^n\beta^n))}{\mathsf{L}_1(\rho(\alpha^n))\mathsf{L}_1(\rho(\beta^n))}$ is of log-type K at $\vec{v}$ for all $n \in \mathbb{N}$.

We just argued that the function $G = \mathrm{Tr}\,(\mathbf{p}_1(\rho(\alpha))\mathbf{p}_1(\rho(\beta)))$ satisfies the hypothesis of Lemma 9.5 of [11] which shows that such a function G is of log-type K at $\vec{v}$. □

Lemmas 4.2 and 4.3 will be used in Section 5 for our discussion of the pressure metric associated to the first fundamental weight. In the other cases we will need a more detailed discussion.

If $\alpha, \beta \in \Gamma$ are hyperbolic, then we say that they have non-intersecting axes if the axes of α and β do not intersect in $\mathbb{H}^2$. The following strengthened transversality property follows immediately from Theorem 3.6 in [10].

Theorem 4.4 (Bridgeman, Canary and Labourie [10, Theorem 3.6]). *If $[\rho] \in \widehat{\mathcal{H}}_d(\Gamma)$, and $\alpha, \beta \in \Gamma$ are hyperbolic elements with non-intersecting axes, then any d elements of*

$$\{e_1(\rho(\alpha)), \ldots, e_d(\rho(\alpha)), e_1(\rho(\beta)), \ldots, e_d(\rho(\beta))\}$$

span $\mathbb{R}^d$. In particular,

$$\mathbf{p}_i(\rho(\alpha))(e_j(\rho(\beta))) \neq 0$$

for any $i, j \in \{1, \ldots, d\}$.

The next lemma collects some of the ingredients needed in Section 7 to study the degeneracy of the Hilbert pressure metric on $\mathcal{H}_3(\Gamma)$.

Lemma 4.5.

(1) *If $\alpha, \beta \in \Gamma$ are hyperbolic and $\rho \in \mathcal{H}_3(\Gamma)$, then*

$$\lim_{n\to\infty} \frac{\lambda_1\big(\rho(\alpha^n\beta^n)\big)}{\lambda_3\big(\rho(\alpha^n\beta^n)\big)} \frac{\lambda_3\big(\rho(\alpha^n)\big)}{\lambda_1\big(\rho(\alpha^n)\big)} \frac{\lambda_3\big(\rho(\beta^n)\big)}{\lambda_1\big(\rho(\beta^n)\big)} = \Big| \mathrm{Tr}\Big(\mathbf{p}_1\big(\rho(\alpha)\big)\mathbf{p}_1\big(\rho(\beta)\big)\Big) \cdot \mathrm{Tr}\Big(\mathbf{p}_3\big(\rho(\alpha)\big)\mathbf{p}_3\big(\rho(\beta)\big)\Big)\Big| \neq 0$$

and

$$\lim_{n\to\infty} \frac{\lambda_1\big(\rho(\alpha^n\beta)\big)}{\lambda_3\big(\rho(\alpha^n\beta)\big)} \frac{\lambda_3\big(\rho(\alpha^n)\big)}{\lambda_1\big(\rho(\alpha^n)\big)} = \Big| \mathrm{Tr}\Big(\mathbf{p}_1\big(\rho(\alpha)\big)\rho(\beta)\Big) \cdot \mathrm{Tr}\Big(\mathbf{p}_3\big(\rho(\alpha)\big)\rho(\beta)\Big)\Big| \neq 0.$$

(2) *If $v \in T_\rho\mathcal{H}_3(\Gamma)$ is non-zero and $e^{\ell^{\omega_H}(\rho(\gamma))} = \frac{\lambda_1(\rho(\gamma))}{\lambda_3(\rho(\gamma))}$ is log-type K at $\vec{v}$ for all hyperbolic $\gamma \in \Gamma$, then*

$$\mathrm{Tr}\Big(\mathbf{p}_1\big(\rho(\alpha)\big)\mathbf{p}_1\big(\rho(\beta)\big)\Big) \cdot \mathrm{Tr}\Big(\mathbf{p}_3\big(\rho(\alpha)\big)\mathbf{p}_3\big(\rho(\beta)\big)\Big) \quad \text{and} \quad \mathrm{Tr}\Big(\mathbf{p}_1\big(\rho(\alpha)\big)\rho(\beta)\Big) \cdot \mathrm{Tr}\Big(\mathbf{p}_3\big(\rho(\alpha)\big)\rho(\beta)\Big)$$

are log-type K at $\vec{v}$.

(3) *If $\rho \in \mathcal{H}_3(\Gamma)$, $\alpha, \beta \in \Gamma$ are hyperbolic elements with non-intersecting axes, then*

$$\mathrm{Tr}(\mathbf{p}_i(\rho(\alpha))\mathbf{p}_j(\rho(\beta))) \neq 0 \qquad \text{for } i, j \in \{1, 2, 3\}.$$

Proof. Part (1) follows from Lemmas 4.2 and 4.3 after observing that for all $\alpha, \gamma \in \Gamma$,

$$\frac{1}{\mathsf{L}_3(\rho(\gamma^n))} = \mathsf{L}_1(\rho(\gamma^{-n})), \quad \mathsf{L}_1(\rho(\alpha^n\gamma)) = \mathsf{L}_1(\rho(\gamma\alpha^n)), \text{ and } \mathbf{p}_3(\rho(\gamma)) = \mathbf{p}_1(\rho(\gamma^{-1})).$$

For part (2) observe that, as in the proof of Lemma 4.3, for all $n \in \mathbb{N}$

$$\mathrm{Tr}\Big(\mathbf{p}_1\big(\rho(\alpha)\big)\mathbf{p}_1\big(\rho(\beta)\big)\Big)\cdot\mathrm{Tr}\Big(\mathbf{p}_3\big(\rho(\alpha)\big)\mathbf{p}_3\big(\rho(\beta)\big)\Big) = \frac{\mathsf{L}_1(\rho(\alpha^n\beta^n))}{\mathsf{L}_3(\rho(\alpha^n\beta^n))}\frac{\mathsf{L}_3(\rho(\alpha^n))}{\mathsf{L}_1(\rho(\alpha^n))}\frac{\mathsf{L}_3(\rho(\beta^n))}{\mathsf{L}_1(\rho(\beta^n))}\frac{1+g_n}{1+\hat{g}_n}$$

where $\{g_n\}_{n\in\mathbb{N}}$ and $\{\hat{g}_n\}_{n\in\mathbb{N}}$ are sequences of (explicit) analytic functions such that

$$\lim_{n\to\infty} f_n(z) = 0 \text{ and } \lim_{n\to\infty} \mathrm{D}_z f_n(v) = 0 \qquad \text{for } f_n = g_n \text{ or } \hat{g}_n.$$

Note that $\frac{\mathsf{L}_1(\rho(\alpha^n\beta^n))}{\mathsf{L}_3(\rho(\alpha^n\beta^n))}\frac{\mathsf{L}_3(\rho(\alpha^n))}{\mathsf{L}_1(\rho(\alpha^n))}\frac{\mathsf{L}_3(\rho(\beta^n))}{\mathsf{L}_1(\rho(\beta^n))}$ is log-type K at $\vec{v}$ by hypothesis, so Lemma 9.5 in [11] gives that

$$\mathrm{Tr}\Big(\mathbf{p}_1\big(\rho(\alpha)\big)\mathbf{p}_1\big(\rho(\beta)\big)\Big)\cdot\mathrm{Tr}\Big(\mathbf{p}_3\big(\rho(\alpha)\big)\mathbf{p}_3\big(\rho(\beta)\big)\Big)$$

is of log-type K at $\vec{v}$. The proof that $\mathrm{Tr}\Big(\mathbf{p}_1\big(\rho(\alpha)\big)\rho(\beta)\Big)\cdot\mathrm{Tr}\Big(\mathbf{p}_3\big(\rho(\alpha)\big)\rho(\beta)\Big)$ is log-type K at $\vec{v}$ is analogous.

By Theorem 4.4, we have that $\mathbf{p}_i(\rho(\alpha))e_j(\rho(\beta))$ spans $\langle e_i(\rho(\alpha))\rangle$ and $\mathbf{p}_j(\rho(\beta))e_i(\rho(\alpha))$ spans $\langle e_j(\rho(\beta))\rangle$, for $i,j\in\{1,2,3\}$. Therefore, we see that

$$\mathrm{Tr}(\mathbf{p}_i(\rho(\alpha))\mathbf{p}_j(\rho(\beta))) = \mathbf{p}_j(\rho(\beta))\Big(\mathbf{p}_i(\rho(\alpha))e_j(\rho(\beta))\Big)\cdot e_j(\rho(\beta)) \neq 0.$$

for $i,j\in\{1,2,3\}$. This concludes the proof of Part (3). □

We now develop some additional notation which will be used here and in Section 6.

If $\mathsf{S}^2\rho : \Gamma \to \mathsf{SL}(\mathsf{S}^2\mathbb{R}^d)$ is the second symmetric product of $\rho \in \mathcal{H}_d(\Gamma)$ and $\gamma \in \Gamma$ is hyperbolic, then

$$\mathsf{S}^2\rho(\gamma) = \sum_{i\leq j}^{d} \mathsf{L}_i(\rho(\gamma))\mathsf{L}_j(\rho(\gamma))\mathbf{p}_{ij}(\rho(\gamma))$$

and if $\mathsf{E}^2\rho : \Gamma \to \mathsf{SL}\big(\mathsf{E}^2\mathbb{R}^d\big)$ is the second exterior product of ρ, then

$$\mathsf{E}^2\rho(\gamma) = \sum_{i<j}^{d} \mathsf{L}_i(\rho(\gamma))\mathsf{L}_j(\rho(\gamma))\mathbf{q}_{ij}(\rho(\gamma))$$

where $\mathbf{p}_{ij}(\rho(\gamma))$ is the projection onto the eigenline spanned by $e_i(\rho(\gamma))\cdot e_j(\rho(\gamma))$ and $\mathbf{q}_{ij}(\rho(\gamma))$ is the projection onto the eigenline $e_i(\rho(\gamma))\wedge e_j(\rho(\gamma))$ parallel to the hyperplane spanned by the other products of basis elements. Explicitly,

$$\mathbf{p}_{ii}(\rho(\gamma))(v\cdot w) = \mathbf{p}_i(\rho(\gamma))(v)\cdot\mathbf{p}_i(\rho(\gamma))(w),$$
$$\mathbf{p}_{ij}(\rho(\gamma))(v\cdot w) = \mathbf{p}_i(\rho(\gamma))(v)\cdot\mathbf{p}_j(\rho(\gamma))(w) + \mathbf{p}_j(\rho(\gamma))(v)\cdot\mathbf{p}_i(\rho(\gamma))(w) \quad \text{for } i\neq j, \text{ and}$$
$$\mathbf{q}_{ij}(\rho(\gamma))(v\wedge w) = \mathbf{p}_i(\rho(\gamma))(v)\wedge\mathbf{p}_j(\rho(\gamma))(w) - \mathbf{p}_j(\rho(\gamma))(v)\wedge\mathbf{p}_i(\rho(\gamma))(w).$$

We use Theorem 4.4 to argue that several trace functions are non-zero. The proof is exactly the same as in [12, Lemma 7.2].

Lemma 4.6. *If $\rho \in \widehat{\mathcal{H}}_d(\Gamma)$, and $\alpha,\beta\in\Gamma$ are hyperbolic elements with non-intersecting axes, then*

(1) $\mathrm{Tr}\big(\mathbf{p}_{ii}(\rho(\alpha))\mathbf{p}_{kk}(\rho(\beta))\big) \neq 0$, *for all* $i,k\in\{1,\dots,d\}$,
(2) $\mathrm{Tr}\big(\mathbf{q}_{ij}(\rho(\alpha))\mathbf{q}_{kl}(\rho(\beta))\big) \neq 0$ *if* $i,j,k,l\in\{1,\dots,d\}$, $i\neq j$ *and* $k\neq l$,
(3) $\mathrm{Tr}\left(\mathbf{p}_{ii}(\rho(\alpha))\mathsf{S}^2\rho(\beta)\right) \neq 0$ *if* $i\in\{1,\dots,d\}$, *and*

(4) $\mathrm{Tr}\left(\mathbf{q}_{ij}(\rho(\alpha))\mathsf{E}^2\rho(\beta)\right) \neq 0$ *if* $i, j \in \{1, \dots, d\}$ *and* $i \neq j$.

We will also make use of the following asymptotic trace equalities, which again follow from [3, Lemma 7.5], see also [12, Lemma 7.3].

Lemma 4.7. *If* $\alpha, \beta \in \Gamma$ *are hyperbolic elements with non-intersecting axes, then*

$$\lim_{n\to\infty} \frac{\frac{\lambda_1(\rho(\alpha^n\beta^n))}{\lambda_2(\rho(\alpha^n\beta^n))}}{\frac{\lambda_1(\rho(\alpha^n))}{\lambda_2(\rho(\alpha^n))}\frac{\lambda_1(\rho(\beta^n))}{\lambda_2(\rho(\beta^n))}} = \left|\frac{\mathrm{Tr}(\mathbf{p}_{11}(\rho(\alpha))\mathbf{p}_{11}(\rho(\beta)))}{\mathrm{Tr}(\mathbf{q}_{12}(\rho(\alpha))\mathbf{q}_{12}(\rho(\beta)))}\right| \neq 0$$

and

$$\lim_{n\to\infty} \frac{\frac{\lambda_1(\rho(\alpha^n\beta))}{\lambda_2(\rho(\alpha^n\beta))}}{\frac{\lambda_1(\rho(\alpha^n))}{\lambda_2(\rho(\alpha^n))}} = \left|\frac{\mathrm{Tr}\left(\mathbf{p}_{11}(\rho(\alpha))\mathsf{S}^2\rho(\beta)\right)}{\mathrm{Tr}\left(\mathbf{q}_{12}(\rho(\alpha)\mathsf{E}^2\rho(\beta)\right)}\right| \neq 0.$$

5. The spectral radius pressure metric

We are now ready to establish the non-degeneracy of the pressure form $\mathbb{P}^{\omega_1}$ associated to the first fundamental weight. Our proof generalizes the argument in [13] (which itself is a simpler version of the more general argument in [11]). We give the argument in nearly full detail so as to make our paper self-contained and for future reference in Section 7.

Theorem 1.3. *If* $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ *is torsion-free and geometrically finite, then the pressure form* $\mathbb{P}^{\omega_1}$ *is non-degenerate, so gives rise to a mapping class group invariant, analytic Riemannian metric on* $\mathcal{H}_d(\Gamma)$.

Proposition 3.1 gives that $\mathbb{P}^{\omega_1}$ is mapping class group invariant, analytic and non-negative. Lemma 4.1 then implies that Theorem 1.3 follows from the following proposition.

Proposition 5.1. *If* $\vec{v} \in \mathsf{T}_{[\eta]}\mathcal{H}_d(S)$ *and* $\mathbb{P}^{\omega_1}(\vec{v},\vec{v}) = 0$, *then*

$$\mathrm{D}_{\vec{v}}\mathrm{Tr}_\gamma = 0$$

for all $\beta \in \Gamma$.

Proof. We abuse notation by identifying $[\rho]$ with $F([\rho]) \in \widehat{\mathcal{H}}_d(\Gamma)$ and $\vec{v}$ with $DF(\vec{v})$. Suppose that $\beta \in \Gamma$ is hyperbolic. Let $\mathsf{L}_{i,\beta} : \mathcal{H}_d(\Gamma) \to \mathbb{R}$ be the analytic function given by $\mathsf{L}_{i,\beta}(\rho) = \mathsf{L}_i(\rho(\beta))$ for all i. Then

$$\rho(\beta^n) = \sum_{k=1}^{d} \mathsf{L}_{i,\beta}(\rho)^n \mathbf{p}_k\big(\rho(\beta)\big)$$

for all $n \in \mathbb{N}$. Lemma 3.3 implies that there exists K so that $\mathsf{L}_1(\rho(\beta))$ has log-type K at $\vec{v}$ for all $\beta \in \Gamma$.

Choose a hyperbolic element $\alpha \in \Gamma$ so that $\langle \alpha, \beta \rangle$ is non-abelian (which implies that it is a free group of rank two) and consider the analytic function $F_n : \mathcal{H}_d(\Gamma) \to \mathbb{R}$ given by

$$\begin{aligned} F_n(\rho) &= \frac{\mathrm{Tr}\Big(\mathbf{p}_1(\rho(\alpha))\rho(\beta^n)\Big)}{\mathsf{L}_{1,\beta}(\rho)^n\,\mathrm{Tr}\Big(\mathbf{p}_1\big(\rho(\alpha)\big)\mathbf{p}_1\big(\rho(\beta)\big)\Big)} \\ &= 1 + \sum_{k=2}^{d} \frac{\mathrm{Tr}\Big(\mathbf{p}_1\big(\rho(\alpha)\big)\mathbf{p}_k\big(\rho(\beta)\big)\Big)}{\mathrm{Tr}\Big(\mathbf{p}_1\big(\rho(\alpha)\big)\mathbf{p}_1\big(\rho(\beta)\big)\Big)} \left(\frac{\mathsf{L}_{k,\beta}(\rho)}{\mathsf{L}_{1,\beta}(\rho)}\right)^n = 1 + \sum_{k=2}^{d} f_k t_k^n \end{aligned}$$

where

$$f_k(\rho) = \frac{\mathrm{Tr}\Big(\mathbf{p}_1\big(\rho(\alpha)\big)\mathbf{p}_k\big(\rho(\beta)\big)\Big)}{\mathrm{Tr}\Big(\mathbf{p}_1\big(\rho(\alpha)\big)\mathbf{p}_1\big(\rho(\beta)\big)\Big)} \neq 0 \qquad \text{and} \qquad t_k(\rho) = \frac{\mathsf{L}_{k,\beta}(\rho)}{\mathsf{L}_{1,\beta}(\rho)} \neq 0.$$

Lemma 4.3 implies that F_n is of log-type K at $\vec{v}$ and is non-zero at every point in $\mathcal{H}_d(\Gamma)$, so

$$\mathrm{D}_{\vec{v}} F_n = \sum_{k=2}^{d} n f_k(\eta) t_k(\eta)^n \frac{\dot{t}_k}{t_k(\eta)} + \sum_{k=2}^{d} \dot{f}_k t_k(\eta)^n = K F_n(\eta) \log(F_n(\eta)), \tag{1}$$

where $\dot{t}_k = \mathrm{D}_{\vec{v}} t_k$ and $\dot{f}_k = \mathrm{D}_{\vec{v}} f_k$. Applying the Taylor series expansion for $\log(1+x)$ and grouping terms, we obtain

$$F_{2n}(\eta) \log\big(F_{2n}(\eta)\big) = \left(1 + \sum_{k=2}^{d} f_k(\eta) t_k(\eta)^{2n}\right) \log\left(1 + \sum_{k=2}^{d} f_k(\eta) t_k(\eta)^{2n}\right) = \sum_{s=1}^{\infty} c_s w_s^n$$

where $\{w_s\}$ is a strictly decreasing sequence of positive terms. Regrouping terms again, we can write this as

$$\sum_{k=2}^{d} 2n \left(\frac{f_k(\eta)\dot{t}_k}{t_k(\eta)}\right) t_k(\eta)^{2n} = \sum_{s=1}^{\infty} c_s w_s^n - \sum_{k=2}^{d} \dot{f}_k t_k(\eta)^{2n} = \sum_{s=1}^{\infty} b_s v_s^n$$

where $\{|v_s|\}$ is a strictly decreasing sequence of positive terms. Setting $u_k = t_k^2$, we see that

$$\sum_{k=2}^{d} n \left(\frac{2 f_k(\eta)\dot{t}_k}{t_k(\eta)}\right) u_k(\eta)^n = \sum_{s=1}^{\infty} b_s v_s^n.$$

We now make use of the following mysterious technical lemma from [13], which implies that $\frac{f_k(\eta)\dot{t}_k}{t_k(\eta)} = 0$ for all k. Since $f_k(\eta) \neq 0$, this implies that $\dot{t}_k = 0$ for all k.

Lemma 5.2. ([13, Lemma 5.16]) *Suppose that $\{a_p\}_{p=1}^q, \{u_p\}_{p=1}^q, \{b_s\}_{s=1}^\infty$, and $\{v_s\}_{s=1}^\infty$ are collections of real numbers so that $\{|u_p|\}_{p=1}^q$ and $\{|v_s|\}_{s=1}^\infty$ are strictly decreasing, each u_p is non-zero,*

$$\sum_{p=1}^{q} n a_p u_p^n = \sum_{s=1}^{\infty} b_s v_s^n$$

for all $n > 0$, *and* $\sum_{s=1}^{\infty} b_s v_s^n$ *is absolutely convergent. Then,* $a_p = 0$ *for all* p.

Since $\dot{t}_k(\eta) = 0$, we see that

$$\frac{\dot{\mathsf{L}}_{k,\beta}\mathsf{L}_{1,\beta}(\eta) - \dot{\mathsf{L}}_{1,\beta}\mathsf{L}_{k,\beta}(\eta)}{\mathsf{L}_{1,\beta}(\eta)^2} = 0$$

for all k where $\dot{\mathsf{L}}_{k,\beta} = \mathrm{D}_{\vec{v}}\mathsf{L}_{k,\beta}$. It follows that

$$\mathrm{D}_{\vec{v}}(\log(|\mathsf{L}_{k,\beta}|)) = \frac{\dot{\mathsf{L}}_{k,\beta}}{\mathsf{L}_{k,\beta}(\eta)} = \frac{\dot{\mathsf{L}}_{1,\beta}}{\mathsf{L}_{1,\beta}(\eta)} = \mathrm{D}_{\vec{v}}(\log(|\mathsf{L}_{1,\beta}|)) = K\log(|\mathsf{L}_{1,\beta}(\eta)|).$$

Since $\mathsf{L}_{d,\beta} = \frac{1}{\mathsf{L}_{1,\beta^{-1}}}$,

$$\begin{aligned} K\log(|\mathsf{L}_{1,\beta^{-1}}(\eta)|) &= \mathrm{D}_{\vec{v}}(\log(|\mathsf{L}_{1,\beta^{-1}}|)) = -\mathrm{D}_{\vec{v}}(\log(|\mathsf{L}_{d,\beta}|)) \\ &= -\mathrm{D}_{\vec{v}}(\log(|\mathsf{L}_{1,\beta}|)) = -K\log(|\mathsf{L}_{1,\beta}(\eta)|). \end{aligned}$$

Since $\log(|\mathsf{L}_{1,\beta^{-1}}(\eta)|)$ and $\log(|\mathsf{L}_{1,\beta}(\eta)|)$ are both positive, this implies that $K = 0$, so $\dot{\mathsf{L}}_{k,\beta} = 0$ for all k. Therefore,

$$\mathrm{D}_{\vec{v}}\,\mathrm{Tr}_\beta = \sum_{k=1}^{d} \dot{\mathsf{L}}_{k,\beta} = 0.$$

On the other hand, if β is parabolic, then Tr_β is constant on $\widehat{\mathcal{H}}_d(\Gamma)$, so $\mathrm{D}_{\vec{v}}\,\mathrm{Tr}_\beta = 0$, which completes the proof. □

Remark: Suppose that $\mathcal{A}$ is an analytic family of (conjugacy classes of) cusped P_1-Anosov representations of a geometrically finite Fuchsian group, in the sense of [15], such that all representations in $\mathcal{A}$ are type-preserving deformations of some fixed representation $\rho_0 \in \mathcal{A}$. One may then define a first fundamental weight pressure form on $\mathcal{A}$ and one expects that a generalization of the more complicated argument given in [11] would allow one to show that this pressure form is nondegenerate at all generic, irreducible representations in $\mathcal{A}$.

6. The (first) simple root pressure metric

Bridgeman, Canary, Labourie and Sambarino [12] prove that if Γ is a closed surface group, then $\mathbb{P}^{\alpha_1}$ is non-degenerate on $\mathcal{H}_d(\Gamma)$. A key tool in their work is the fact, due to Potrie-Sambarino [40], that the topological entropy $h^{\alpha_1}(\rho) = 1$ if $\rho \in \mathcal{H}_d(\Gamma)$. Canary, Zhang and Zimmer generalize Potrie and Sambarino's result to the setting of torsion-free lattices which are not cocompact.

Theorem 6.1. (Potrie-Sambarino [40] and Canary-Zhang-Zimmer [16]) *If* $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ *is a torsion-free lattice, and* $\rho \in \mathcal{H}_d(\Gamma)$, *then* $h^{\alpha_1}(\rho) = 1$.

With this result in hand, we are ready to establish the non-degeneracy of the first simple root pressure metric.

Theorem 1.4. *If* $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ *is a torsion-free lattice, then the pressure form* $\mathbb{P}^{\alpha_1}$ *is non-degenerate, so it gives rise to a mapping class group invariant, analytic Riemannian metric on* $\mathcal{H}_d(\Gamma)$.

As in the previous section, Proposition 3.1 and Lemma 4.1 together imply that Theorem 1.4 follows from the following proposition.

Proposition 6.2. *If* $\vec{v} \in T_{[\eta]}\mathcal{H}_d(\Gamma)$ *and* $\mathbb{P}^{\alpha_1}(\vec{v},\vec{v}) = 0$*, then* $\mathrm{D}_{\vec{v}}\,\mathrm{Tr}_\beta = 0$ *for all* $\beta \in \Gamma$.

Here, we will only sketch the proof, since the proof proceeds exactly as in the proof of [12, Prop. 7.4], and the arguments will not need be referred to again in the next section.

Proof. We again abuse notation by identifying $[\rho]$ with $F([\rho]) \in \widehat{\mathcal{H}}_d(\Gamma)$. Since $h^{\alpha_1}(\rho) = 1$ for all $\rho \in \mathcal{H}_d(\Gamma)$, Proposition 3.2 implies that $\mathrm{D}_{\vec{v}}\ell^{\alpha_1}_\beta = 0$ for all $\beta \in \Gamma$.

If $\alpha \in \Gamma$ is parabolic, then Tr_α is constant on $\widehat{\mathcal{H}}_d(\Gamma)$, so $\mathrm{D}_{\vec{v}}\,\mathrm{Tr}_\alpha = 0$.

If β is hyperbolic, we may choose $\alpha \in \Gamma$, so that α is hyperbolic and α and β have non-intersecting axes.

We show that

$$\mathrm{D}_{\vec{v}}(\log \lambda_k(\rho(\beta))) = \mathrm{D}_{\vec{v}}(\log \lambda_1(\rho(\beta)))$$

for all k. Since $\lambda_1(\rho(\beta)) \cdots \lambda_d(\rho(\beta)) = 1$, this implies that $\mathrm{D}_{\vec{v}}\lambda_i(\rho(\gamma)) = 0$ for all i, and hence that $\mathrm{D}_{\vec{v}}\,\mathrm{Tr}_\beta = 0$.

We first notice that, since $\mathrm{D}_{\vec{v}}\ell^{\alpha_1}_\gamma = 0$,

$$\mathrm{D}_{\vec{v}}(\log \lambda_2(\rho(\beta))) = \mathrm{D}_{\vec{v}}(\log \lambda_1(\rho(\beta)))$$

for all $\beta \in \Gamma$. We proceed iteratively, assuming that $\mathrm{D}_{\vec{v}}(\log \lambda_i(\rho(\beta))) = \mathrm{D}_{\vec{v}}(\log \lambda_1(\rho(\beta)))$ for all $i < m$ and $\beta \in \Gamma$.

Consider the family of analytic functions $\{F_n : \widehat{\mathcal{H}}_d(\Gamma) \to \mathbb{R}\}_{n\in\mathbb{N}}$ defined by

$$F_n(\rho) = \frac{\left(\frac{\mathrm{Tr}(\mathbf{p}_{11}(\rho(\alpha))\mathsf{S}^2\rho(\beta^n))}{\mathrm{Tr}(\mathbf{q}_{12}(\rho(\alpha))\mathsf{E}^2\rho(\beta^n))}\right)}{\left(\frac{\mathsf{L}_1(\rho(\beta))}{\mathsf{L}_2(\rho(\beta))}\right)^n \left(\frac{\mathrm{Tr}\left(\mathbf{p}_{11}(\rho(\alpha))\mathbf{p}_{11}(\rho(\beta))\right)}{\mathrm{Tr}\left(\mathbf{q}_{12}(\rho(\alpha))\mathbf{q}_{12}(\rho(\beta))\right)}\right)}.$$

Lemma 4.7 implies that the numerator of F_n is an analytic function which is a limit of analytic functions which, by assumption, have derivative zero in the direction $\vec{v}$, so the numerator has derivative zero in direction $\vec{v}$. Similarly, Lemma 4.7 implies that the denominator of F_n is non-zero and has zero derivative in direction $\vec{v}$. Therefore, $\mathrm{D}_{\vec{v}}F_n = 0$ for all $n \in \mathbb{N}$.

Bridgeman, Canary, Labourie and Sambarino [12] then expand and regroup the terms of F_n and use another mysterious lemma [12, Lem. 7.5], to show that

$$\mathrm{D}_{\vec{v}}(\log \lambda_m(\rho(\beta))) = \mathrm{D}_{\vec{v}}(\log \lambda_1(\rho(\beta))).$$

The proof in [12, Prop. 7.4] goes through exactly, where one simply replaces their use of [12, Lem. 7.2] with our Lemma 4.6. This completes our proof. □

7. The Hilbert length pressure metric

Finally, we study the Hilbert length pressure form $\mathbb{P}^{\omega_H}$ on $\mathcal{H}_3(\Gamma)$. We hope that the Hilbert length pressure metric will be amenable to study via techniques from Hilbert geometry. Recall that if $[\rho] \in \mathcal{H}_3(\Gamma)$, then there exists a strictly convex domain $\Omega_\rho \subset \mathbb{RP}^2$ which is acted on freely and properly discontinously by $\rho(\Gamma)$. The domain Ω_ρ admits a Finsler metric, called the Hilbert metric, which is projectively invariant, so descends to a metric on the convex projective surface $X_\rho = \Omega_\rho/\rho(\Gamma)$. The Hilbert length $\ell^{\omega_H}(\rho(\gamma))$ is the translation length of $\rho(\gamma)$ in the Hilbert metric on Ω_ρ and hence the length of the geodesic on X_ρ in the homotopy class of γ. See the survey article by Marquis [34] for more details on projective structures and Hilbert geometry.

Let $C\colon \mathcal{H}_d(\Gamma) \to \mathcal{H}_d(\Gamma)$ be the contragredient involution, where $C(\rho)(\gamma) = \left(\rho(\gamma)^{-1}\right)^\top$ for all $\gamma \in \Gamma$. The *self-dual locus* $\mathcal{H}_d^{sd}(\Gamma)$ of $\mathcal{H}_d(\Gamma)$ is the fixed point set of C. When $d = 3$, the self-dual locus is just the Fuchsian locus. (See for instance [33, Thm. 6.17]). We say that a vector $\vec{v} \in T\mathcal{H}_d(\Gamma)$ is *anti-self-dual* if $\mathrm{D}C(\vec{v}) = -\vec{v}$. In particular, the basepoints of anti-self-dual vectors lie on the self-dual locus and if $[\rho]$ is in the self-dual locus, the involution $\mathrm{D}_\rho C$ induces a splitting

$$\mathsf{T}_{[\rho]}\mathcal{H}_d(\Gamma) = \mathsf{T}_{[\rho]}\mathcal{H}_d^{sd}(\Gamma) \oplus A_{[\rho]}$$

where $A_{[\rho]}$ is the space of anti-self-dual vectors based at $[\rho]$.

The key result in our analysis is that degenerate vectors are anti-self-dual when $d = 3$.

Proposition 7.1. *Suppose that $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is geometrically finite and torsion-free. If $\vec{v} \in \mathsf{T}\mathcal{H}_3(\Gamma)$ is non-zero and $\mathbb{P}^{\omega_H}(\vec{v},\vec{v}) = 0$, then $\vec{v}$ is anti-self-dual.*

Before establishing Proposition 7.1, we observe that it immediately gives Theorem 1.5

Theorem 1.5. *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is torsion-free and geometrically finite. If $\vec{v} \in \mathsf{T}\mathcal{H}_3(\Gamma)$ is non-zero and $\mathbb{P}^{\omega_H}(\vec{v},\vec{v}) = 0$, then $\vec{v}$ is anti-self-dual. Therefore, $\mathbb{P}^{\omega_H}$ gives rise to a mapping class group invariant path metric on $\mathcal{H}_3(\Gamma)$ which is an analytic Riemannian metric off of the Fuchsian locus.*

Proof. The first claim simply restates Proposition 7.1. Propositions 3.1 and 7.1 together imply that $\mathbb{P}^{\omega_H}$ is analytic, mapping class group-invariant and positive definite off of the Fuchsian locus. Moreover, its restriction to the tangent space of the Fuchsian locus is positive definite (see Kao [23]). Since the Fuchsian locus is a submanifold of positive codimension, it then follows from [11, Lemma 13.1] that $\mathbb{P}^{\omega_H}$ gives rise to a path metric on $\mathcal{H}_3(\Gamma)$. □

Proof of Proposition 7.1: Recall that, by the discussion at the beginning of section 4, $\mathcal{H}_3(\Gamma)$ coincides with a subspace $\widehat{\mathcal{H}}_3(\Gamma)$ of the character variety $X_3(\Gamma)$.

First suppose that $\vec{v} \in T_{[\eta]}\mathcal{H}_3(\Gamma)$, $[\eta]$ is in the Fuchsian locus and $\mathbb{P}^{\omega_H}(\vec{v},\vec{v}) = 0$. Recall that there are finitely many curves whose Hilbert lengths determine a point in the Fuchsian locus. Since length functions of curves are analytic, it follows that if $\vec{v}$ is tangent to the Fuchsian locus and $\mathrm{D}_{\vec{v}}\ell_\gamma^{\omega_H} = 0$ for all $\gamma \in \Gamma$, then $\vec{v}$ is trivial. Proposition 3.2 then implies that $\mathrm{D}_{\vec{v}}\ell_\gamma^{\omega_H} = 0$ for all $\gamma \in \Gamma$, so $\vec{v}$ is trivial. (Kao [23] uses this same argument in his proof that the pressure metric is non-degenerate on Teichmüller space.) We next check that if $\vec{v} \in A_{[\eta]}$, then $\mathbb{P}^{\omega_H}(\vec{v},\vec{v}) = 0$. To see this, observe that we can find a path $(\eta_t)_{t\in(-1,1)}$ such that $\dot{\eta}_0 = \vec{v}$ and $\eta_{-t} = C(\eta_t)$. Then the function $t \mapsto h_t \cdot \ell^{\omega_H}(\eta_t(\gamma))$ is even for all $\gamma \in \Gamma$, thus its derivative at $t = 0$ is zero and we conclude by applying Proposition 3.2. Since the tangent space at $[\eta]$ decomposes as $\mathsf{T}_{[\eta]}\mathcal{H}_d^{sd}(\Gamma) \oplus A_{[\eta]}$ this establishes our result for vectors based at points on the Fuchsian locus.

We may now assume that η does not lie in the Fuchsian locus. It then follows from work of Benoist [2, Theorem 1.3], Marquis [33, Theorem 6.17] or Sambarino [44] that η is

Zariski dense. We will need the following observation about Zariski dense representations into $\mathsf{SL}(3,\mathbb{R}) = \mathsf{PSL}(3,\mathbb{R})$.

Lemma 7.2. *If $\rho : \Gamma \to \mathsf{SL}(3,\mathbb{R})$ is a Zariski dense Hitchin representation, $\rho(\gamma)$ is diagonalizable and $\lambda_2(\rho(\gamma)) = 1$, then, there exists $\alpha \in \Gamma$ and $N \geqslant 0$ so that $\lambda_2(\rho(\gamma^n\alpha)) \neq 1$ for all $n \geqslant N$.*

Notice that this statement includes the case that γ is the identity.

Proof. Since $\rho(\gamma)$ is either trivial or loxodromic, we may assume that it is diagonal. If $\alpha \in \Gamma$ and $\lambda_2(\rho(\gamma^k\alpha)) = 1$ then

$$\mathrm{Tr}(\rho(\gamma^k\alpha)) = \mathrm{Tr}(\rho(\alpha^{-1}\gamma^{-k})) = \mathrm{Tr}(\rho(\gamma^{-k}\alpha^{-1})). \tag{2}$$

We will observe that if there exists $n < n+m < n+m+l$ with $n, m, l \in \mathbb{Z}_{>0}$ such that Equation (2) holds for $k = n, n+m, n+m+l$, then $\rho(\alpha)$ lies in a proper Zariski closed subset of $\mathsf{SL}(3,\mathbb{R})$. Then, since ρ is assumed to be Zariski dense, we must be able to find α which satisfies the hypotheses of the lemma.

Denote by d_1, d_2, d_3 the diagonal entries of $\rho(\alpha)$ and by D_1, D_2, D_3 the diagonal entries of $\rho(\alpha^{-1})$. Let $\mathsf{L} = \mathsf{L}_1(\rho(\gamma))$. Notice that $\mathsf{L} \geqslant 1$, since $\rho(\gamma)$ is a continuous deformation, through $\mathsf{SL}(3,\mathbb{R})$, of $\tau_3(\gamma)$, which has positive first eigenvalue.

Equation (2) can then be written as

$$\mathsf{L}^k d_1 + d_2 + \mathsf{L}^{-k} d_3 = \mathsf{L}^{-k} D_1 + D_2 + \mathsf{L}^k D_3.$$

If $\mathsf{L} = 1$, namely $\rho(\gamma) = \mathrm{id}$, then $\rho(\alpha)$ lies in the proper Zariski closed subset given by the polynomial equation $d_1 + d_2 + d_3 = D_1 + D_2 + D_3$ and we are done.

Assume $\mathsf{L} > 1$ and consider $n, m, l > 0$ such that

$$\begin{cases} \mathsf{L}^{2n}(d_1 - D_3) + \mathsf{L}^n(d_2 - D_2) + (d_3 - D_1) = 0 \\ \mathsf{L}^{2n+2m}(d_1 - D_3) + \mathsf{L}^{n+m}(d_2 - D_2) + (d_3 - D_1) = 0 \\ \mathsf{L}^{2n+2m+2l}(d_1 - D_3) + \mathsf{L}^{n+m+l}(d_2 - D_2) + (d_3 - D_1) = 0 \end{cases}$$

and observe that the Vandermonde's determinant

$$\det \begin{bmatrix} (\mathsf{L}^n)^2 & \mathsf{L}^n & 1 \\ (\mathsf{L}^{n+m})^2 & \mathsf{L}^{n+m} & 1 \\ (\mathsf{L}^{n+m+l})^2 & \mathsf{L}^{n+m+l} & 1 \end{bmatrix} = -(\mathsf{L}^n - \mathsf{L}^{n+m})(\mathsf{L}^n - \mathsf{L}^{n+m+l})(\mathsf{L}^{n+m} - \mathsf{L}^{n+m+l})$$

is nonzero since $n, m, l > 0$. Therefore, $\rho(\alpha)$ lies in the proper Zariski closed subset given by the polynomial equations $d_1 = D_3,\ d_2 = D_2$, and $d_3 = D_1$.

Therefore, there exists $\alpha \in \Gamma$ so that $\lambda_2(\rho(\gamma^n\alpha)) \neq 1$ for all large enough n. □

Now suppose that $\vec{v} \in T_{[\eta]}\mathcal{H}_3(\Gamma)$, $[\eta]$ is not in the Fuchsian locus and $\mathbb{P}^{\omega_H}(\vec{v}, \vec{v}) = 0$. We will again show that $\mathrm{D}_{\vec{v}}\,\mathrm{Tr}_\gamma = 0$ for all $\gamma \in \Gamma$. As usual, it suffices to consider hyperbolic elements. We proceed in two steps.

(1) First, we prove that if $\beta \in \Gamma$ is such that $\lambda_2(\eta(\beta)) \neq 1$, then $\mathrm{D}_{\vec{v}}\,\mathrm{Tr}_\beta = 0$. Observe that since η is Zariski dense, we can apply Lemma 7.2 with $\gamma = \mathrm{id}$ to show that there exists a $\beta \in \Gamma$ with $\lambda_2(\eta(\beta)) \neq 1$. This will allow us to conclude that $K = 0$, which is needed in the next step.

(2) If β is hyperbolic and $\lambda_2(\eta(\beta)) = 1$, we apply Lemma 7.2 to find a hyperbolic element α so that $\lambda_2(\eta(\beta^n\alpha)) \neq 1$ for all sufficiently large n. The previous step implies that $\mathrm{D}_{\vec{v}} \operatorname{Tr}_{\beta^n\alpha} = 0$ for all sufficiently large n, and we use this to show that $\mathrm{D}_{\vec{v}} \operatorname{Tr}_\beta = 0$ as well.

First suppose that $\lambda_2(\eta(\beta)) \neq 1$. We may choose a hyperbolic element $\alpha \in \Gamma$ so that α and β have non-intersecting axes. For all $n \in \mathbb{N}$, we consider the analytic function F_n given by

$$F_n(\rho) = \frac{\operatorname{Tr}(\mathbf{p}_1(\rho(\alpha))\rho(\beta)^n)}{\mathsf{L}_{1,\beta}(\rho)^n \cdot \operatorname{Tr}(\mathbf{p}_1(\rho(\alpha))\mathbf{p}_1(\rho(\beta)))} \cdot \frac{\mathsf{L}_{3,\beta}(\rho)^n \cdot \operatorname{Tr}(\mathbf{p}_3(\rho(\alpha))\rho(\beta)^{-n})}{\operatorname{Tr}(\mathbf{p}_3(\rho(\alpha))\mathbf{p}_3(\rho(\beta)))}.$$

Recall that $\mathsf{L}_{i,\beta} : \mathcal{H}_d(\Gamma) \to \mathbb{R}$ is the analytic function given by $\mathsf{L}_{i,\beta}(\rho) = \mathsf{L}_i(\rho(\beta))$. Notice that, by Proposition 3.2 and Lemma 4.5, F_n is non-zero and a product of functions which are log-type K at $\vec{v}$, so F_n itself is log-type K at $\vec{v}$. Recall that

$$\rho(\beta)^n = \sum_{i=1}^{3} \mathsf{L}_{i,\beta}(\rho)^n \mathbf{p}_i(\rho(\beta)) \qquad \text{and} \qquad \beta^{-n} = \sum_{i=1}^{3} \frac{1}{\mathsf{L}_{i,\beta}(\rho)^n} \mathbf{p}_i(\rho(\beta))$$

and set

$$a_2(\rho) = \frac{\operatorname{Tr}(\mathbf{p}_1(\rho(\alpha))\mathbf{p}_2(\rho(\beta)))}{\operatorname{Tr}(\mathbf{p}_1(\rho(\alpha))\mathbf{p}_1(\rho(\beta)))} \qquad b_2(\rho) = \frac{\operatorname{Tr}(\mathbf{p}_3(\rho(\alpha))\mathbf{p}_2(\rho(\beta)))}{\operatorname{Tr}(\mathbf{p}_3(\rho(\alpha))\mathbf{p}_3(\rho(\beta)))}$$

$$a_3(\rho) = \frac{\operatorname{Tr}(\mathbf{p}_1(\rho(\alpha))\mathbf{p}_3(\rho(\beta)))}{\operatorname{Tr}(\mathbf{p}_1(\rho(\alpha))\mathbf{p}_1(\rho(\beta)))} \qquad b_3(\rho) = \frac{\operatorname{Tr}(\mathbf{p}_3(\rho(\alpha))\mathbf{p}_1(\rho(\beta)))}{\operatorname{Tr}(\mathbf{p}_3(\rho(\alpha))\mathbf{p}_3(\rho(\beta)))}.$$

Notice that Lemma 4.5 implies that a_2, a_3, b_2 and b_3 are non-zero. Then we can expand

$$\begin{aligned} F_n &= \left(1 + a_2 \left(\frac{\mathsf{L}_{2,\beta}}{\mathsf{L}_{1,\beta}}\right)^n + a_3 \left(\frac{\mathsf{L}_{3,\beta}}{\mathsf{L}_{1,\beta}}\right)^n\right)\left(1 + b_2 \left(\frac{\mathsf{L}_{3,\beta}}{\mathsf{L}_{2,\beta}}\right)^n + b_3 \left(\frac{\mathsf{L}_{3,\beta}}{\mathsf{L}_{1,\beta}}\right)^n\right) \\ &= 1 + a_2 \left(\frac{\mathsf{L}_{2,\beta}}{\mathsf{L}_{1,\beta}}\right)^n + b_2 \left(\frac{\mathsf{L}_{3,\beta}}{\mathsf{L}_{2,\beta}}\right)^n + (a_3 + b_3 + a_2 b_2)\left(\frac{\mathsf{L}_{3,\beta}}{\mathsf{L}_{1,\beta}}\right)^n \\ &\quad + a_2 b_3 \left(\frac{\mathsf{L}_{2,\beta}}{\mathsf{L}_{1,\beta}}\right)^n \left(\frac{\mathsf{L}_{3,\beta}}{\mathsf{L}_{1,\beta}}\right)^n + a_3 b_2 \left(\frac{\mathsf{L}_{3,\beta}}{\mathsf{L}_{2,\beta}}\right)^n \left(\frac{\mathsf{L}_{3,\beta}}{\mathsf{L}_{1,\beta}}\right)^n + a_3 b_3 \left(\frac{\mathsf{L}_{3,\beta}}{\mathsf{L}_{1,\beta}}\right)^{2n}. \end{aligned}$$

We first assume that $\lambda_2(\eta(\beta)) > 1$, so $\frac{\lambda_2(\rho(\beta))}{\lambda_1(\rho(\beta))} > \frac{\lambda_3(\rho(\beta))}{\lambda_2(\rho(\beta))}$ in a neighborhood U of $[\eta]$. As in the proof of Proposition 5.1, we write

$$F_n(\rho) = 1 + \sum_{k=1}^{6} f_k(\rho) t_k(\rho)^n.$$

with $|t_1(\rho)| > |t_2(\rho)| > \cdots > |t_6(\rho)|$ if $\rho \in U$. More explicitly,

$$t_1 = \frac{\mathsf{L}_{2,\beta}}{\mathsf{L}_{1,\beta}},\ t_2 = \frac{\mathsf{L}_{3,\beta}}{\mathsf{L}_{2,\beta}},\ \ldots,\ t_6 = \left(\frac{\mathsf{L}_{3,\beta}}{\mathsf{L}_{1,\beta}}\right)^2$$

and

$$f_1 = a_2,\ f_2 = b_2, \ldots,\ f_6 = a_3 b_3.$$

Notice that if we set $u_k = t_k^2$, then

$$\mathrm{D}_{\vec{v}} F_{2n} = \sum_{k=1}^{6} n\left(\frac{2f_k(\eta)\dot{t}_k}{t_k(\eta)}\right) u_k(\eta)^n + \dot{f}_k u_k(\eta)^n$$

for all $n \in \mathbb{N}$, where $\dot{t}_k = \mathrm{D}_{\vec{v}} t_k$ and $\dot{f}_k = \mathrm{D}_{\vec{v}} f_k$. As in the proof of Theorem 1.3 we may write

$$F_{2n}(\eta)\log(F_{2n}(\eta)) = \left(1 + \sum_{k=1}^{6} f_k(\eta)t_k(\eta)^{2n}\right)\log\left(1 + \sum_{k=1}^{6} f_k(\eta)t_k(\eta)^{2n}\right) = \sum_{s=1}^{\infty} c_s w_s^n$$

for all $n \in \mathbb{N}$, where $\{w_s\}$ is a strictly decreasing sequence of positive terms. Then, since $\mathrm{D}_{\vec{v}} F_n = K F_n(\eta)\log(F_n(\eta))$ for all $n \in \mathbb{N}$, we see that

$$\sum_{k=1}^{6} n\left(\frac{2f_k(\eta)\dot{t}_k}{t_k(\eta)}\right) u_k(\eta)^n = \sum_{s=1}^{\infty} c_s w_s^n - \sum_{k=1}^{d} \dot{f}_k t_k(\eta)^{2n} = \sum_{s=1}^{\infty} b_s v_s^n$$

for all $n \in \mathbb{N}$, where $\{|v_s|\}$ is a strictly decreasing sequence of positive terms. Lemma 5.2 implies that $\frac{2f_k(\eta)\dot{t}_k}{t_k(\eta)} = 0$ for all k. Since $f_6 = a_3 b_3 \neq 0$ and $t_6 \neq 0$, this implies that $\dot{t}_6 = 0$, so

$$\mathrm{D}_{\vec{v}} \frac{\mathsf{L}_{3,\beta}}{\mathsf{L}_{1,\beta}} = 0.$$

It follows that $K = 0$, so

$$\mathrm{D}_{\vec{v}} \ell_\gamma^{\omega_H} = \mathrm{D}_{\vec{v}} \log\left|\frac{\mathsf{L}_{1,\gamma}}{\mathsf{L}_{3,\gamma}}\right| = 0$$

for all $\gamma \in \Gamma$.

In addition, Lemma 5.2 implies that

$$\frac{2f_1(\eta)\dot{t}_1}{t_1(\eta)} = \frac{2f_2(\eta)\dot{t}_2}{t_2(\eta)} = 0.$$

Then, since $f_1 = a_2 \neq 0$, $f_2 = b_2 \neq 0$ and $t_1, t_2 \neq 0$, we see that

$$\mathrm{D}_{\vec{v}}\left(\frac{\mathsf{L}_{2,\beta}}{\mathsf{L}_{1,\beta}}\right) = \mathrm{D}_{\vec{v}}\left(\frac{\mathsf{L}_{3,\beta}}{\mathsf{L}_{2,\beta}}\right) = 0.$$

By the quotient rule

$$\frac{\dot{\mathsf{L}}_{2,\beta}\mathsf{L}_{1,\beta}(\eta) - \mathsf{L}_{2,\beta}(\eta)\dot{\mathsf{L}}_{1,\beta}}{\mathsf{L}_{1,\beta}(\eta)^2} = 0 = \frac{\dot{\mathsf{L}}_{3,\beta}\mathsf{L}_{2,\beta}(\eta) - \mathsf{L}_{3,\beta}(\eta)\dot{\mathsf{L}}_{2,\beta}}{\mathsf{L}_{2,\beta}(\eta)^2}$$

and looking at the numerators, we get

$$\frac{\dot{\mathsf{L}}_{2,\beta}}{\mathsf{L}_{2,\beta}(\eta)} = \frac{\dot{\mathsf{L}}_{1,\beta}}{\mathsf{L}_{1,\beta}(\eta)} \qquad \text{and} \qquad \frac{\dot{\mathsf{L}}_{2,\beta}}{\mathsf{L}_{2,\beta}(\eta)} = \frac{\dot{\mathsf{L}}_{3,\beta}}{\mathsf{L}_{3,\beta}(\eta)} \tag{3}$$

where $\dot{\mathsf{L}}_{i,\beta} = \mathrm{D}_{\vec{v}}\mathsf{L}_{i,\beta}$. Thus

$$2\mathrm{D}_{\vec{v}} \log|\mathsf{L}_{2,\beta}| = 2\frac{\dot{\mathsf{L}}_{2,\beta}}{\mathsf{L}_{2,\beta}(\eta)} = \frac{\dot{\mathsf{L}}_{1,\beta}}{\mathsf{L}_{1,\beta}(\eta)} + \frac{\dot{\mathsf{L}}_{3,\beta}}{\mathsf{L}_{3,\beta}(\eta)} = \mathrm{D}_{\vec{v}} \log|\mathsf{L}_{1,\beta}\mathsf{L}_{3,\beta}| = -\mathrm{D}_{\vec{v}} \log|\mathsf{L}_{2,\beta}|.$$

It follows that $\mathrm{D}_{\vec{v}} \log|\mathsf{L}_{2,\beta}| = 0$, so $\dot{\mathsf{L}}_{2,\beta} = 0$. This also implies, by equations (3) that $\dot{\mathsf{L}}_{1,\beta} = \dot{\mathsf{L}}_{3,\beta} = 0$. Therefore, $\mathrm{D}_{\vec{v}} \operatorname{Tr}_\beta = 0$.

If $\lambda_2(\eta(\beta)) < 1$, then we argue that $K = 0$ and $\mathrm{D}_{\vec{v}} \operatorname{Tr}_\beta = 0$ just as above. The only differences are the forms of the f_k's and t_k's, e.g. $t_1 = \frac{\mathsf{L}_{3,\beta}}{\mathsf{L}_{2,\beta}}$, $t_2 = \frac{\mathsf{L}_{2,\beta}}{\mathsf{L}_{1,\beta}}$, $f_1 = b_2$ and $f_2 = a_2$.

We now assume that $\beta \in \Gamma$ is hyperbolic and $\lambda_2(\eta(\beta)) = 1$. Lemma 7.2 guarantees that there exists $\alpha \in \Gamma$ and $N > 0$, so that $\lambda_2(\eta(\beta^n\alpha)) \neq 1$ for all $n \geqslant N$. By the above argument, this implies that $\mathrm{D}_{\vec{v}} \operatorname{Tr}_{\beta^n\alpha} = 0$ for all $n \geqslant N$.

Combining the facts that

$$\mathrm{D}_{\vec{v}}\left(\frac{\mathsf{L}_{1,\beta}}{\mathsf{L}_{3,\beta}}\right) = \frac{\dot{\mathsf{L}}_{1,\beta}\mathsf{L}_{3,\beta}(\eta) - \mathsf{L}_{1,\beta}(\eta)\dot{\mathsf{L}}_{3,\beta}}{\mathsf{L}_{3,\beta}(\eta)^2} = 0$$

and $\mathsf{L}_{1,\beta}\mathsf{L}_{2,\beta}\mathsf{L}_{3,\beta} = 1$ we deduce that

$$\frac{\dot{\mathsf{L}}_{1,\beta}}{\mathsf{L}_{1,\beta}(\eta)} = \frac{\dot{\mathsf{L}}_{3,\beta}}{\mathsf{L}_{3,\beta}(\eta)} \qquad \text{and} \qquad \frac{\dot{\mathsf{L}}_{2,\beta}}{\mathsf{L}_{2,\beta}(\eta)} = -2\frac{\dot{\mathsf{L}}_{1,\beta}}{\mathsf{L}_{1,\beta}(\eta)}.$$

In particular, if $\dot{\mathsf{L}}_{i,\beta} = 0$ for some $i = 1, 2, 3$, then $\dot{\mathsf{L}}_{i,\beta} = 0$ for all $i = 1, 2, 3$.

We may again assume that $\eta(\beta)$ is diagonal and that $\vec{v} = \frac{d}{dt}\big|_{t=0}\eta_t$ where $\eta_t(\beta)$ is diagonal for all t. Let $a_{11}(t), a_{22}(t), a_{33}(t)$ be the diagonal entries of $\eta_t(\alpha)$ and observe that

$$\begin{aligned}
0 &= \frac{d}{dt}\Big|_{t=0}\Big(\mathsf{L}_{1,\beta}(\eta_t)^n a_{11} + \mathsf{L}_{2,\beta}(\eta_t)^n a_{22} + \mathsf{L}_{3,\beta}(\eta_t)^n a_{33}\Big) \\
&= n\mathsf{L}_{1,\beta}(\eta)^n \Bigg(a_{11}(0)\frac{\dot{\mathsf{L}}_{1,\beta}}{\mathsf{L}_{1,\beta}(\eta)} + a_{22}(0)\frac{\dot{\mathsf{L}}_{2,\beta}}{\mathsf{L}_{2,\beta}(\eta)}\frac{\mathsf{L}_{2,\beta}(\eta)^n}{\mathsf{L}_{1,\beta}(\eta)^n} + a_{33}(0)\frac{\dot{\mathsf{L}}_{3,\beta}}{\mathsf{L}_{3,\beta}(\eta)}\frac{\mathsf{L}_{3,\beta}(\eta)^n}{\mathsf{L}_{1,\beta}(\eta)^n} \\
&\qquad + \frac{\dot{a}_{11}}{n} + \frac{\dot{a}_{22}}{n}\frac{\mathsf{L}_{2,\beta}(\eta)^n}{\mathsf{L}_{1,\beta}(\eta)^n} + \frac{\dot{a}_{33}}{n}\frac{\mathsf{L}_{3,\beta}(\eta)^n}{\mathsf{L}_{1,\beta}(\eta)^n}\Bigg)
\end{aligned}$$

for all $n \geq N$, where $\dot{a}_{ii} = \frac{d}{dt}\big|_{t=0} a_{ii}(t)$ for all i. Dividing by $n\mathsf{L}_{1,\beta}(\eta)^n$ and letting n go to infinity, we see that $a_{11}(0)\frac{\dot{\mathsf{L}}_{1,\beta}}{\mathsf{L}_{1,\beta}(\eta)} = 0$. Thus, either $\dot{\mathsf{L}}_{1,\beta} = 0$, and we are done, or $a_{11}(0) = 0$. Assuming $a_{11}(0) = 0$, we see that

$$0 = \mathsf{L}_{1,\beta}(\eta)^n\left(na_{22}(0)\frac{\dot{\mathsf{L}}_{2,\beta}}{\mathsf{L}_{2,\beta}(\eta)}\frac{\mathsf{L}_{2,\beta}(\eta)^n}{\mathsf{L}_{1,\beta}(\eta)^n} + na_{33}(0)\frac{\dot{\mathsf{L}}_{3,\beta}}{\mathsf{L}_{3,\beta}(\eta)}\frac{\mathsf{L}_{3,\beta}(\eta)^n}{\mathsf{L}_{1,\beta}(\eta)^n} + \dot{a}_{11} + \dot{a}_{22}\frac{\mathsf{L}_{2,\beta}(\eta)^n}{\mathsf{L}_{1,\beta}(\eta)^n} + \dot{a}_{33}\frac{\mathsf{L}_{3,\beta}(\eta)^n}{\mathsf{L}_{1,\beta}(\eta)^n}\right)$$

and, since $\lim_{n\to\infty} n\frac{\mathsf{L}_{i,\beta}(\eta)^n}{\mathsf{L}_{1,\beta}(\eta)^n} = 0$ for $i = 2, 3$, we conclude that $\dot{a}_{11} = 0$. Therefore,

$$0 = n\mathsf{L}_{2,\beta}(\eta)^n\left(a_{22}(0)\frac{\dot{\mathsf{L}}_{2,\beta}}{\mathsf{L}_{2,\beta}(\eta)} + a_{33}(0)\frac{\dot{\mathsf{L}}_{3,\beta}}{\mathsf{L}_{3,\beta}(\eta)}\frac{\mathsf{L}_{3,\beta}(\eta)^n}{\mathsf{L}_{2,\beta}(\eta)^n} + \frac{\dot{a}_{22}}{n} + \frac{\dot{a}_{33}}{n}\frac{\mathsf{L}_{3,\beta}(\eta)^n}{\mathsf{L}_{2,\beta}(\eta)^n}\right)$$

and after dividing by $n\mathsf{L}_{2,\beta}(\eta)^n$ and letting n go to infinity, we conclude that $a_{22}(0)\dot{\mathsf{L}}_{2,\beta} = 0$. If $\dot{\mathsf{L}}_{2,\beta} = 0$, then we are done. Otherwise, we may assume $a_{22}(0) = 0$. We can again check that this implies that $\dot{a}_{22} = 0$ and we may repeat the argument to see that either $\dot{\mathsf{L}}_{3,\beta} = 0$ or $a_{33}(0) = 0$.

However, observe that $\mathrm{Tr}(\eta(\alpha)) = a_{11}(0) + a_{22}(0) + a_{33}(0)$ is nonzero. This is because $\eta(\alpha)$ can be continuously deformed, by Theorem 1.1, to $\rho_0(\alpha)$ for some Fuchsian representation $\rho_0 \in \mathcal{H}_3(\Gamma)$. It is a standard calcuation that $\rho_0(\alpha)$ has eigenvalues $\mu, 1, 1/\mu$ for some $\mu > 1$. Since the eigenvalues vary real analytically with the representation and they never vanish in $\mathcal{H}_3(\Gamma)$, we have that $\mathrm{Tr}(\eta(\alpha)) > 0$. Thus, necessarily $\dot{\mathsf{L}}_{3,\beta} = 0$. As we observed previously, this implies that $\dot{\mathsf{L}}_{j,\beta} = 0$ for all $j = 1, 2, 3$ and hence that $\mathrm{D}_{\vec{v}} \mathrm{Tr}_\beta = 0$.

We have shown that $\mathrm{D}_{\vec{v}} \mathrm{Tr}_\beta = 0$ if β is hyperbolic. Since Tr_β is constant if β is not hyperbolic, $\mathrm{D}_{\vec{v}} \mathrm{Tr}_\beta = 0$ if β for all $\beta \in \Gamma$. Lemma 4.1 then implies that $\vec{v} = 0$. □

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