1) a) TRUE If \( f \) is differentiable on \( \mathbb{R} \), then it is continuous on \( \mathbb{R} \), so it is continuous on \([-1, 1]\) and hence uniformly continuous on \([0, 1]\). But if a function is uniformly continuous on \([-1, 1]\) it is uniformly continuous on any subset of \([-1, 1]\) and hence uniformly continuous on \((-1, 1)\).

b) FALSE: Consider the function \( f(x) = x \) which is uniformly continuous on \( \mathbb{R} \), but not bounded on \( \mathbb{R} \).

c) FALSE: Consider the function \( f(x) = x^3 \), then \( f \) is strictly increasing and differentiable on \( \mathbb{R} \), but \( f'(0) = 0 \), so it is not the case that \( f'(x) > 0 \) for all \( x \in \mathbb{R} \).

d) TRUE: \( f(x) = |x| \) is continuous on \([-1, 1]\), hence uniformly continuous on \([-1, 1]\), but is not differentiable at 0.

e) TRUE: If \( f : \mathbb{R} \to \mathbb{R} \) is a function such that \( \lim_{x \to 2^+} f(x) = f(2) \) and \( \lim_{x \to 2^-} f(x) = f(2) \), then \( \lim_{x \to 2} f(x) = f(2) \), so \( f \) is continuous at 2.

2) Claim: The function \( f(x) = x^2 + x \) is continuous at \( x_0 = 1 \).

Proof: Given \( \epsilon > 0 \), let \( \delta = \min\{1, \frac{\epsilon}{4}\} \). If \( |x - 1| < \delta \), then \( |x - 1| < 1 \), so \( 0 < x < 2 \) which implies that \( 2 < x + 2 < 4 \), so \( |x + 2| < 4 \), and \( |x - 1| < \frac{\epsilon}{4} \). Therefore,

\[
|f(x) - f(1)| = |x^2 + x - 2| = |x + 2||x - 1| < 4 \left(\frac{\epsilon}{4}\right) = \epsilon.
\]

We have shown that for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( |x - 1| < \delta \), then \( |f(x) - f(1)| < \epsilon \), so \( f \) is continuous at 1.

3) Claim: If \( f : \mathbb{R} \to \mathbb{R} \) is defined so that \( f(x) = 2x \) if \( x \geq 0 \) and \( f(x) = 3x \) if \( x < 0 \), then \( f \) is not differentiable at 0.

Proof: Suppose that \( f \) is differentiable at 0. Then, there exists \( C \in \mathbb{R} \) so that \( \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = C \).

So, if \( (x_n) \) is a sequence in \( \mathbb{R} - \{0\} \) which converges to 0, then \( \lim_{n \to \infty} \frac{f(x_n) - f(0)}{x_n - 0} = C \).

Consider the sequence \( (x_n) = (\frac{1}{n}) \), then \( x_n = 0 \) and \( (x_n) \subset \mathbb{R} - \{0\} \). Then, \( \frac{f(x_n) - f(0)}{x_n - 0} = 2 \) for all \( n \), so \( \lim_{n \to \infty} \frac{f(x_n) - f(0)}{x_n - 0} = 2 \), which implies that \( C = 2 \).

Now consider the sequence \( (y_n) = (\frac{1}{n}) \), then \( y_n = 0 \) (as we have shown in class) and \( (y_n) \subset \mathbb{R} - \{0\} \). However, \( \frac{f(y_n) - f(0)}{y_n - 0} = 3 \) for all \( n \), so \( \lim_{n \to \infty} \frac{f(y_n) - f(0)}{y_n - 0} = 3 \), which implies that \( C = 3 \).

We have obtained a contradiction, so \( f \) must not be differentiable at 0.

4) Claim: There exists a real number \( x \) such that \( x^4 = x^3 + 1 \).

Proof: Consider the function \( g(x) = x^4 - x^3 - 1 \). Since \( g \) is a polynomial it is continuous on \( \mathbb{R} \). Notice that \( g(0) = -1 \) and \( g(-1) = 1 \). Since \( g \) is continuous on \([-1, 0]\) and 0 lies between \( g(-1) \) and \( g(0) \), the Intermediate Value Theorem implies that there exists \( x \in (-1, 0) \) such that \( g(x) = 0 \). Since \( g(x) = 0 \), \( x^4 - x^3 - 1 = 0 \) which implies that \( x^4 = x^3 + 1 \).

5) Claim: If \( f \) is continuous on \([0, 2]\) and differentiable on \((0, 2)\) and \( |f'(x)| \leq 2x \) for all \( x \in (0, 2) \), then

\[
|f(b) - f(a)| \leq 4|b - a|
\]

for all \( a, b \in [0, 2] \) such that \( a < b \).

Proof: Notice that since \( f \) is differentiable on \( \mathbb{R} \), it is continuous on \( \mathbb{R} \). If \( a < b \) and \( a, b \in [0, 2] \), then \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\), so the Mean Value Theorem implies that there exists \( y \in (a, b) \) so that \( f'(y) = \frac{f(b) - f(a)}{b - a} \), so \( |f(b) - f(a)| = |f'(y)| |b - a| \). Since \( y \in (a, b) \subset (0, 2) \), \( |f'(y)| \leq 2y \leq 4 \) which implies that \( |f(b) - f(a)| \leq 4|b - a| \) as desired.