New nonlinear hyperbolic groups

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Abstract

We construct nonlinear hyperbolic groups which are large, torsion-free, one-ended, and admit a finite $K(\pi, 1)$. Our examples are built from superrigid cocompact rank one lattices via amalgamated free products and HNN extensions.

1 Introduction

In this note, we construct new examples of nonlinear hyperbolic groups. For us, a group is "nonlinear" if it does not admit a faithful representation into $\mathrm{GL}_n(F)$ for F a local field. As with previous constructions, our groups are built from superrigid cocompact lattices in rank 1 Lie groups. Previous examples were quotients of such lattices, small cancellation theory was used to show that the quotients are hyperbolic, and superrigidity results were used to see that they are nonlinear (see M. Kapovich [K05, §8]). Our construction involves simple HNN extensions and free products with amalgamation, and one can prove that the resulting groups are hyperbolic using the Bestvina–Feighn Combination Theorem [BF]. Our examples are large (i.e., have finite index subgroups that surject a free group of rank two), torsion-free, one-ended, and admit a finite $K(\pi, 1)$.

Theorem 1.1. *For any $n \geq 0$, there exist large, torsion-free, one-ended, nonlinear hyperbolic groups that admit a finite $K(\pi, 1)$, have first betti number n, and surject a free group of rank n.*

We present two related constructions, both of which begin with a cocompact torsion-free lattice Γ in $\mathrm{Sp}(m, 1)$ (always with $m \geq 2$) or $\mathrm{F}_4^{(-20)}$. As in M. Kapovich [K05], our proofs rely crucially on Corlette's [Cor] and Gromov–Schoen's [GS] generalizations of the Margulis superrigidity theorem to lattices in these groups. In what follows, let G be $\mathrm{Sp}(m, 1)$ or $\mathrm{F}_4^{(-20)}$ and X be the associated rank one symmetric space, i.e., quaternionic hyperbolic m-space or the Cayley hyperbolic plane.

In our first construction, we choose elements γ_1 and γ_2 of Γ associated with primitive closed geodesics of different length in the locally symmetric space X/Γ. We consider the group Λ_1 obtained by taking the HNN extension of Γ such that the stable letter conjugates γ_1 to γ_2, i.e.,

$$\Lambda_1 = \langle \Gamma, t \mid t\gamma_1 t^{-1} = \gamma_2 \rangle.$$

We use superrigidity results to show that if Λ_1 is linear, then it admits a faithful representation ρ into $\mathrm{GL}_n(\mathbb{R})$ and there is a totally geodesic embedding of X into the symmetric space Y_n of $\mathrm{GL}_n(\mathbb{R})$ which is equivariant with respect to the restriction $\rho|_\Gamma$ of ρ to Γ. Since the translation lengths of $\rho(\gamma_1)$ and $\rho(\gamma_2)$ agree in Y_n and f is totally geodesic, the translation lengths of γ_1 and γ_2 on X agree, which gives a contradiction. It follows that Λ_1 is nonlinear. The Bestvina–Feighn combination theorem [BF] implies that Λ_1 is hyperbolic, and it is clear that Λ_1 has first betti number 1, has the same cohomological dimension as Γ, admits a finite $K(\pi,1)$, and is torsion-free. We will see that it is easy to iterate this construction to produce examples with arbitrarily large first betti number.

Our second construction involves amalgamated free products and produces examples with first betti number zero. Let $\Delta = \langle \alpha, \beta \rangle$ be a malnormal, infinite index subgroup of Γ freely generated by α and β that is contained in a malnormal Fuchsian subgroup of Δ. Let $\phi : \Delta \to \Delta$ be an isomorphism such that the ratio of the translation lengths of α and β is different than the ratio of the translation lengths of $\phi(\alpha)$ and $\phi(\beta)$. We then construct

$$\Lambda_0 = \Gamma *_\phi \Gamma$$

from two copies of Γ by identifying Δ in the first copy with Δ in the second copy via the isomorphism ϕ. We argue, as before, that if Λ_0 is linear, then there is a representation ρ of Λ_0 into $\mathrm{GL}_n(\mathbb{R})$ such that the restriction of ρ to each factor determines an equivariant totally geodesic embedding of X into Y_n. It follows that the ratio of the translation lengths of α and β agrees with the ratios of the translation lengths of $\phi(\alpha)$ and $\phi(\beta)$, which we have disallowed.

We regard the main advantage of our new constructions to be their relative simplicity and flexibility. For example, if one were given an explicit presentation of a superrigid lattice, one could easily write down an explicit presentation of a group of the form Λ_1.

The first published examples of nonlinear hyperbolic groups are due to M. Kapovich [K05]. Gromov [Gr] used small cancellation theory to show that suitable quotients of a lattice Γ as above are infinite hyperbolic groups (see also [Ch, D, Ol]), and then Kapovich used superrigidity results to show that any linear representation of these quotients has finite image. In particular, these examples have Property (T), since they are quotients of Property (T) groups. It follows that these groups are not large and hence are not abstractly commensurable with our examples.

The paper is organized as follows. In §2, we give the details of our constructions and show that our groups have the claimed group-theoretic properties. In §3 we recall the necessary consequences of superrigidity for lattices in $\mathrm{Sp}(m,1)$, $m \geq 2$, or $\mathrm{F}_4^{(-20)}$. The proofs of nonlinearity are given in §4.

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2 The constructions

In this section, we give the details of the constructions described in the introduction and establish the group-theoretic properties claimed there. Throughout this paper G will be either $\mathrm{Sp}(m,1)$ for $m \geq 2$ or $\mathrm{F}_4^{(-20)}$, so G acts by isometries on a rank one symmetric space X, which is quaternionic hyperbolic m-space or the Cayley hyperbolic plane, respectively. Then Γ will always denote a torsion-free cocompact lattice in G. In particular, Γ is hyperbolic, admits a finite $K(\pi, 1)$, $H^1(\Gamma, \mathbb{R}) = 0$, and the cohomological dimension of Γ is the dimension of X.

We first construct the examples with nontrivial first betti number. Let γ_1 and γ_2 be primitive elements of Γ with distinct translation length. The associated geodesics in X/Γ are distinct, so no nontrivial power of γ_1 is conjugate to a power of γ_2. Let Λ_1 be the HNN extension of Γ given by

$$\Lambda_1 = \langle \Gamma, t \mid t\gamma_1 t^{-1} = \gamma_2 \rangle.$$

We may iterate this construction by choosing $2n$ primitive elements $\{\gamma_1, \dots, \gamma_{2n}\}$ with distinct translation lengths and defining

$$\Lambda_n = \langle \Gamma, t_1, \dots, t_n \mid t_i \gamma_{2i-1} t_i^{-1} = \gamma_{2i} \rangle.$$

to be obtained by repeated HNN extensions.

We now construct the examples with trivial first betti number. Suppose that Γ contains a maximal cocompact Fuchsian group Σ. Examples of Γ containing such a Σ are well-known to exist. In fact, Meyer showed that all lattices in $\mathrm{Sp}(m, 1)$ contain such surface subgroups [Mey, Prop 8.]. Results of Long [Lo] and Bergeron [Ber, p. 113] on subgroup separability allow us to replace Γ with a finite index subgroup containing Σ so that $\mathbb{H}^2/\Sigma$ is embedded in X/Γ as a totally geodesic submanifold. It follows that we can assume Σ is malnormal in Γ, since if $\gamma \in \Gamma \smallsetminus \Sigma$ and $\gamma\Sigma\gamma^{-1} \cap \Sigma \neq \{1\}$, then the immersion of $\mathbb{H}^2/\Sigma$ into X/Γ cannot be an embedding.

Let $\Delta \subset \Sigma$ be an infinite index malnormal subgroup of Σ that is freely generated by α and β. For example, one may take Δ to be identified with the fundamental group of a proper, essential subsurface of $\mathbb{H}^2/\Sigma$ that is either an one-holed torus or a four-holed sphere. All finitely generated subgroups of a surface group are quasiconvex, so Δ is a quasiconvex subgroup of Σ. Then Σ is quasiconvex in Γ, since it is the fundamental group of a totally geodesic submanifold, and hence Δ is quasiconvex in Γ. Let $\phi : \Delta \to \Delta$ be an isomorphism such that the ratio of the translation lengths of α and β is different than the ratio of the translation lengths of $\phi(\alpha)$ and $\phi(\beta)$. Let

$$\Lambda_0 = \Gamma *_\phi \Gamma.$$

be obtained from two copies of Γ by identifying Δ in the first copy with Δ in the second copy via the isomorphism ϕ.

Proposition 2.1. *For all n, a group Λ_n constructed as above is hyperbolic, torsion-free, large, one-ended, has a finite $K(\pi, 1)$, has first betti number n, and its cohomological dimension is the dimension of X. Moreover, if $n \geq 1$, Λ_n admits a surjective homomorphism to the free group F_n of rank n.*

Proof. That Λ_n is torsion-free, one-ended, has a finite $K(\pi, 1)$, has first betti number n, and has cohomological dimension equal to the dimension of X follows from standard facts about graphs of groups (see, for example, Serre [Ser, Chap. 1] or Scott–Wall [SW]). If $n \geq 1$, then Λ_n clearly surjects onto the group freely generated by $\{t_1, \dots, t_n\}$. The fact that each Λ_n is hyperbolic is a special case of the Bestvina–Feighn combination theorem [BF], which is explicitly stated in I. Kapovich [K97, Ex. 1.3] as follows:

Theorem 2.2.

1. *If A and B are hyperbolic groups and C is a quasiconvex subgroup of both A and B that is malnormal in either A or B, then $A *_C B$ is hyperbolic.*

2. *If A is a hyperbolic group and a_1 and a_2 are elements of A so that no nontrivial power of a_1 is conjugate to a power of a_2, the the HNN extension*

$$\langle A, t \mid t a_1 t^{-1} = a_2 \rangle$$

is hyperbolic.

Part (1) immediately implies that Λ_0 is hyperbolic, while part (2) gives that Λ_n is hyperbolic if $n \geq 1$.

To complete the proof of the proposition, it remains to show that each Λ_n is large. If $n \geq 2$, then Λ_n clearly surjects onto the group freely generated by t_1 and t_2, so is large. Now suppose that $n = 1$. Since the cyclic subgroup generated by γ_i is separable in Γ (see Bergeron [Ber, p. 113]), there exist surjective homomorphisms $\pi_1 : \Gamma \to K_1$ where $\pi_1(\gamma_2)$ does not lie in the subgroup of the finite group K_1 generated by $\pi_1(\gamma_1)$ and $\pi_2 : \Gamma \to K_2$ where $\pi_2(\gamma_1)$ does not lie in the subgroup of the finite group K_2 generated by $\pi_2(\gamma_2)$. Consider

$$\pi = \pi_1 \times \pi_2 : \Gamma \to K_1 \oplus K_2,$$

and notice that there is a surjective homomorphism onto the HNN extension

$$\Lambda_1 \to H_1 = \langle K_1 \oplus K_2, t \mid t\pi(\gamma_1)t^{-1} = \pi(\gamma_2) \rangle.$$

However, since $\pi(\gamma_2)$ is not a power of $\pi(\gamma_1)$ and $\pi(\gamma_2)$ is not a power of $\pi(\gamma_1)$, H_1 contains a finite index subgroup isomorphic to F_r for some $r \geq 2$, so Λ_1 is large. In fact, the kernel of the obvious map from H_1 to $K_1 \oplus K_2$ is free of rank at least two (see [Ser, §I.4.3]).

We now consider Λ_0. Similarly, since Σ is separable in Γ [Ber, p. 113] there exists a surjective homomorphism $\pi_1 : \Gamma \to K_1$ so that $\pi_1(\Sigma)$ is a proper subgroup of the finite group K_1. Moreover, if we choose $\beta \in \mathrm{Ker}(\pi_1) \setminus \Sigma$, there exists a surjective homomorphism $\pi_2 : \Gamma \to K_2$ so that K_2 is finite and $\pi_2(\beta)$ does not lie in $\pi_2(\Sigma)$. Let $\pi_0 = \pi_1 \times \pi_2 : \Gamma \to K_1 \oplus K_2$ and consider the surjective map

$$\pi : \Lambda_0 \to H_0 = (K_1 \oplus K_2) *_{\pi_0(\Delta)} (K_1 \oplus K_2).$$

Since $\pi_0(\Delta)$ has index greater than two in $K_1 \oplus K_2$, H_0 contains a finite index subgroup isomorphic to F_r for some $r \geq 2$, so Λ_0 is large (again, see [Ser, §I.4.3]). □

Remarks: 1) I. Kapovich [K99] showed that every non-elementary hyperbolic group contains a malnormal quasiconvex subgroup which is free of rank two, so one can more generally construct nonlinear groups from any pair of superrigid rank one lattices by identifying such subgroups by an isomorphism which does not preserve ratios of translation lengths of generators. In our work, we only use the fact that Δ is contained in a Fuchsian subgroup in our proof that Λ_0 is large.

Kapovich [K99] further uses a malnormal quasiconvex free subgroup of a word hyperbolic group G to construct a group G^* which contains G as a non-quasiconvex subgroup. We note that G^* is a quotient of a group of the form Λ_2, obtained by identifying the two stable letters, so if G is a superrigid rank one lattice then G^* can be chosen to be nonlinear.

2) We expect that the techniques of Belegradek–Osin [BO], which also begin with quotients of superrigid lattices and employ more powerful small cancellation theoretic results, also produce large, one-ended, nonlinear hyperbolic groups (in particular, see [BO, Thm. 3.1]).

3) It is clear that one can construct infinitely many isomorphism classes of groups of the form Λ_n, for each n, even if one begins with a fixed superrigid lattice Γ. For example, if $n \geq 1$, it follows readily from the JSJ theory for hyperbolic groups, see Sela [Sel], that the isomorphism type of a group of the form Λ_1 is determined, up to finite ambiguity, by the conjugacy class of the pair $\{\gamma_1, \gamma_2\}$ in Γ.

3 Superrigidity

In this section, we record a version of the superrigidity theorem of Corlette [Cor] and Gromov–Schoen [GS] that is crafted for our purposes. In our statement Y_n will denote the symmetric space

$$Y_n = Z\,\mathrm{O}(n)\backslash\,\mathrm{GL}_n(\mathbb{R}) = \mathrm{PO}(n)\backslash\,\mathrm{PGL}_n(\mathbb{R})$$

associated with $\mathrm{GL}_n(\mathbb{R})$, where Z denotes the center of $\mathrm{GL}_n(\mathbb{R})$.

Theorem 3.1. *Suppose that Γ is a lattice in G, where G is either $Sp(m,1)$ or $\mathrm{F}_4^{(-20)}$, F is a local field of characteristic zero, and $\rho : \Gamma \to \mathrm{GL}_d(F)$ is a representation with infinite image.*

1. *There exists a faithful representation $\tau : \mathrm{GL}_d(F) \to \mathrm{GL}_n(\mathbb{R})$ for some n such that $\tau \circ \rho(\Gamma)$ has noncompact Zariski closure.*
2. *If $F = \mathbb{R}$ and $\rho(\Gamma) \subset \mathrm{GL}_n(\mathbb{R})$ has noncompact Zariski closure, then there exists a ρ-equivariant totally geodesic map*
$$f_\rho : X \to Y_n,$$
where $X = K\backslash G$ is the symmetric space associated with G.

Proof. First notice that every local field of characteristic zero is isomorphic to a subfield of $\mathbb{C}$. For example, the algebraic closure $\mathbb{C}_p$ of $\mathbb{Q}_p$ is isomorphic to $\mathbb{C}$. Moreover, $\mathrm{GL}_d(\mathbb{C})$ is a subgroup of $\mathrm{GL}_{2d}(\mathbb{R})$. It follows that there exists an injective representation $\eta : \mathrm{GL}_d(F) \to \mathrm{GL}_n(\mathbb{R})$ for some n, so we may assume that the original representation maps into $\mathrm{GL}_n(\mathbb{R})$.

Fisher and Hitchman [FH, Thm. 3.7] then observe that the existing results on superrigidity imply that one can factor ρ as two representations

$$\rho_i : \Gamma \to \mathrm{GL}_{n_i}(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$$

such that:

1. When ρ_1 is nontrivial, there is a group G' locally isomorphic to G, a continuous representation $\hat{\rho}_1 : G' \to \mathrm{GL}_{n_1}(\mathbb{R})$, and an embedding $\iota : \Gamma \hookrightarrow G'$ of Γ as a lattice in G' such that $\rho_1 = \hat{\rho}_1 \circ \iota$.
2. The image of ρ_2 is bounded, i.e., has compact Zariski closure.
3. The groups $\rho_1(\Gamma)$ and $\rho_2(\Gamma)$ commute and $\rho(\gamma) = \rho_1(\gamma)\rho_2(\gamma)$ for all $\gamma \in \Gamma$.

If ρ_1 is nontrivial, the continuous embedding $\hat{\rho}_1 : G' \to \mathrm{GL}_{n_1}(\mathbb{R})$ determines a totally geodesic embedding of X into Y_{n_1}. Since ρ_1 and ρ_2 commute, this is a ρ-equivariant map.

When ρ_1 is trivial, we follow arguments in the proof of [K05, Thm. 8.1]. Note that our use of [FH, Thm. 3.7] allows us to know beforehand that the solvable radical considered in [K05] is trivial. As in [K05], the fact that Γ has Property (T) allows us to conclude that we may conjugate ρ so that $\rho(\Gamma) \subseteq \mathrm{GL}_n(k)$ for some number field k. Given an element $\sigma \in \mathrm{Aut}(k/\mathbb{Q})$, we can choose an extension of σ to an element of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, which we continue to denote by σ. Applying σ to matrix entries induces an embedding $\tau_\sigma : \mathrm{GL}_n(F) \to \mathrm{GL}_n(\mathbb{C})$.

Following the adelic argument in [K05], if $\rho(\Gamma)$ were bounded for every valuation of k then $\rho(\Gamma)$ would be finite, which is a contradiction. Moreover, $\rho(\Gamma)$ must be bounded for every nonarchimedean valuation by nonarchimedean superrigidity [GS]. Consequently, there exists $\sigma \in \mathrm{Aut}(k/\mathbb{Q})$ such that $\tau_\sigma(\rho(\Gamma))$ has noncompact Zariski closure in $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_{2n}(\mathbb{R})$, according to whether $\sigma(k) \otimes_{\mathbb{Q}} \mathbb{R}$ is $\mathbb{R}$ or $\mathbb{C}$. Applying the previous argument to $\tau_\sigma \circ \rho$, there is a $(\tau_\sigma \circ \rho)$-equivariant totally geodesic embedding of X into Y_n or Y_{2n}, accordingly. This completes the sketch of the proof. □

M. Kapovich [K05] also points out that superrigidity rules out faithful representations of Γ into linear groups of local fields of nonzero characteristic. Briefly, one shows that the image of ρ lies in $\mathrm{GL}_n(k)$ for k a global field of characteristic p. Then, applying Gromov–Schoen superrigidity [GS] to each local completion of k, one sees that $\rho(\Gamma)$ is bounded in each local field associated with a valuation on k, as all valuations are nonarchimedean. It follows that $\rho(\Gamma)$ is bounded and hence finite. Thus we have:

Proposition 3.2. *If Γ is a lattice in either $Sp(m,1)$ or $\mathrm{F}_4^{(-20)}$ and F is a local field of characteristic $p > 0$, then there does not exist a faithful representation of Γ into $\mathrm{GL}_n(F)$ for any n.*

4 Proofs of nonlinearity

To complete the proof of Theorem 1.1 it remains to prove:

Theorem 4.1. *Groups of the form Λ_n constructed in Section 2 are nonlinear.*

Proof. We begin with a group of the form

$$\Lambda_1 = \langle \Gamma, t \mid t\gamma_1 t^{-1} = \gamma_2 \rangle$$

constructed in Section 2, where Γ is a cocompact lattice in G and G is either $Sp(m,1)$ or $\mathrm{F}_4^{(-20)}$. Recall that X is the symmetric space associated with G and that γ_1 and γ_2 are assumed to have different translation lengths on X.

Suppose that F is a local field and $\eta : \Lambda_1 \to \mathrm{GL}_d(F)$ is a faithful representation. Applying Proposition 3.2 to the restriction $\rho = \eta|_\Gamma$ of η to Γ, we conclude that F has characteristic zero. Theorem 3.1 implies that there exists a faithful representation $\tau_\sigma : \mathrm{GL}_d(F) \to \mathrm{GL}_n(\mathbb{R})$, for some n and a $(\tau_\sigma \circ \rho)$-equivariant embedding f of X into Y_n, where Y_n is the symmetric space associated with $\mathrm{GL}_n(\mathbb{R})$.

Since $\tau_\sigma(\rho(\gamma_1))$ is conjugate to $\tau_\sigma(\rho(\gamma_2))$ in $\tau_\sigma(\eta(\Lambda_1))$, and hence in $\mathrm{GL}_n(\mathbb{R})$, they have the same translation length on Y_n. However, since f is a ρ-equivariant totally geodesic embedding, this implies that γ_1 and γ_2 have the same translation length in X, which is a contradiction, hence Λ_1 is nonlinear. Notice that if $n \geq 2$, then any group of the form Λ_n constructed in Section 2 contains a subgroup of the form Λ_1, so Λ_n is also nonlinear.

Now suppose we have a group of the form

$$\Lambda_0 = \langle \Gamma_1, \Gamma_2 \mid \alpha_1 = \phi(\alpha)_2,\ \beta_1 = \phi(\beta)_2 \rangle$$

where each Γ_i is a copy of Γ, $\Delta = \langle \alpha, \beta \rangle$ is a subgroup of Γ freely generated by α and β, Δ_i is the copy of Δ in Γ_i and if $\delta \in \Delta$, then δ_i is the copy of δ in Δ_i. Moreover, ϕ is an automorphism of Δ so that the ratio of the translation lengths of α and β on X differs from the ratio of translation lengths of $\phi(\alpha)$ and $\phi(\beta)$ on X.

Suppose that F is a local field and $\eta : \Lambda_0 \to \mathrm{GL}_d(F)$ is a faithful representation. We again apply Proposition 3.2 to conclude that F has characteristic zero, Theorem 3.1 implies that there exists a faithful representation $\tau_\sigma : \mathrm{GL}_d(F) \to \mathrm{GL}_n(\mathbb{R})$, for some n and a $(\tau_\sigma \circ \rho_1)$-equivariant embedding f of X into Y_n, where Y_n is the symmetric space associated with $\mathrm{GL}_n(\mathbb{R})$. Let $\rho_2 = \eta|_{\Gamma_2}$. Since $\tau_\sigma(\rho_1(\Delta_1)) = \tau_\sigma(\rho_2(\Delta_2))$

has noncompact Zariski closure, Theorem 3.1 implies that there exists a $(\tau_\sigma \circ \rho_2)$-equivariant embedding g of X into Y_n. Notice that $\tau_\sigma(\rho_1(\alpha_1)) = \tau_\sigma(\rho_2(\phi(\alpha)_2))$ and that $\tau_\sigma(\rho_1(\beta_1)) = \tau_\sigma(\rho_2(\phi(\beta)_2))$.

Since f and g are equivariant totally geodesic embeddings, there exist positive constants c_1 and c_2 so that if $\gamma \in \Gamma$, then the ratio of the translation length of $\tau_\sigma(\rho_i(\gamma_i))$ on Y_n and the translation length of γ on X is c_i. Indeed, the metrics on $f(X)$ and $g(X)$ differ by a scalar multiple. It follows that the ratio of the translation lengths of α and β on X agrees with the ratio of the translation lengths of $\phi(\alpha)$ and $\phi(\beta)$ on X. However, this contradicts our assumptions, so Λ_0 is nonlinear. □

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