THE PRESSURE METRIC FOR ANOSOV REPRESENTATIONS

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Abstract. Using the thermodynamic formalism, we introduce a notion of intersection for projective Anosov representations, show analyticity results for the intersection and the entropy, and rigidity results for the intersection. We use the renormalized intersection to produce an $\mathrm{Out}(\Gamma)$-invariant Riemannian metric on the smooth points of the deformation space of irreducible, generic, projective Anosov representations of a word hyperbolic group Γ into $\mathsf{SL}_m(\mathbb{R})$. In particular, we produce mapping class group invariant Riemannian metrics on Hitchin components which restrict to the Weil–Petersson metric on the Fuchsian loci. Moreover, we produce $\mathrm{Out}(\Gamma)$-invariant metrics on deformation spaces of convex cocompact representations into $\mathsf{PSL}_2(\mathbb{C})$ and show that the Hausdorff dimension of the limit set varies analytically over analytic families of convex cocompact representations into any rank 1 semi-simple Lie group.

1. Introduction

In this paper we produce a mapping class group invariant Riemannian metric on a Hitchin component of the character variety of representations of a closed surface group into $\mathsf{SL}_m(\mathbb{R})$ whose restriction to the Fuchsian locus is a multiple of the Weil-Petersson metric. More generally, we produce a $\mathrm{Out}(\Gamma)$-invariant Riemannian metric on the smooth generic points of the deformation space of irreducible, projective Anosov representations of a word hyperbolic group Γ into $\mathsf{SL}_m(\mathbb{R})$. We use Plücker representations to produce metrics on deformation spaces of convex cocompact representations into $\mathsf{PSL}_2(\mathbb{C})$ and on the smooth points of deformation spaces of Zariski dense Anosov representations into an arbitrary semi-simple Lie group.

Our metric is produced using the thermodynamic formalism developed by Bowen [12, 13], Parry–Pollicott [58], Ruelle [64] and others. It generalizes earlier work done in the Fuchsian and quasifuchsian cases by McMullen [56] and Bridgeman [9]. In order to use the thermodynamic formalism, we associate a natural flow $\mathsf{U}_\rho\Gamma$ to any projective Anosov representation ρ, and show that it is a topologically transitive metric Anosov flow and is a Hölder reparameterization of the geodesic flow $\mathsf{U}_0\Gamma$ of Γ as defined by Gromov. We then see that entropy varies analytically over any smooth analytic family of projective Anosov homomorphisms of Γ into $\mathsf{SL}_m(\mathbb{R})$. As a consequence, again using the Plücker embedding, we see that the Hausdorff

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dimension of the limit set varies analytically over analytic families of convex cocompact representations into a rank one semi-simple Lie group. We also introduce a renormalized intersection $\mathbf{J}$ on the space of projective Anosov representations. Our metric is given by the Hessian of this renormalized intersection $\mathbf{J}$.

We now introduce the notation necessary to give more careful statements of our results. Let Γ be a word hyperbolic group with Gromov boundary $\partial_\infty\Gamma$. Loosely speaking, a representation $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ is *projective Anosov* if it has transverse projective limit maps, the image of every infinite order element is proximal, and the proximality "spreads uniformly" (see Section 2.1 for a careful definition). An element $A \in \mathsf{SL}_m(\mathbb{R})$ is *proximal* if its action on $\mathbb{RP}(m)$ has an attracting fixed point. A representation $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ is said to have *transverse projective limit maps* if there exist continuous ρ-equivariant maps $\xi : \partial_\infty\Gamma \to \mathbb{RP}(m)$ and $\theta : \partial_\infty\Gamma \to \mathbb{RP}(m)^*$ such that if x and y are distinct points in $\partial_\infty\Gamma$, then

$$\xi(x) \oplus \theta(y) = \mathbb{R}^m$$

(where we identify $\mathbb{RP}(m)^*$ with the Grassmanian of $(m-1)$-dimensional vector subspaces of $\mathbb{R}^m$). If $\gamma \in \Gamma$ has infinite order, ρ is projective Anosov and γ^+ is the attracting fixed point of the action of γ on $\partial\Gamma$, then $\xi(\gamma^+)$ is the attracting fixed point for the action of $\rho(\gamma)$ on $\mathbb{RP}(m)$. Moreover, Guichard and Wienhard [29, Proposition 4.10] proved that every irreducible representation $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ with transverse projective limit maps is projective Anosov.

If ρ is a projective Anosov representation, we can associate to every conjugacy class $[\gamma]$ of $\gamma \in \Gamma$ its *spectral radius* $\Lambda(\gamma)(\rho)$. The collection of these radii form the *radius spectrum* of ρ. For every positive real number T we define

$$R_T(\rho) = \{[\gamma] \mid \log(\Lambda(\gamma)(\rho)) \leqslant T\}.$$

We will see that $R_T(\rho)$ is finite (Proposition 2.8). We also define the *entropy* of a representation by

$$h(\rho) = \lim_{T\to\infty} \frac{1}{T} \log \sharp(R_T(\rho)).$$

If ρ_1 and ρ_2 are two projective Anosov representations, we define their *intersection* by

$$\mathbf{I}(\rho_1,\rho_2) = \lim_{T\to\infty} \left(\frac{1}{\sharp(R_T(\rho_1))} \sum_{[\gamma]\in R_T(\rho_0)} \frac{\log(\Lambda(\gamma)(\rho_2))}{\log(\Lambda(\gamma)(\rho_1))} \right).$$

We also define the *renormalized intersection* by

$$\mathbf{J}(\rho_1,\rho_2) = \frac{h(\rho_2)}{h(\rho_1)}\, \mathbf{I}(\rho_1,\rho_2).$$

We prove, see Theorem 1.3, that all these quantities are well defined and obtain the following inequality and rigidity result for the renormalized intersection. Let $\pi_m : \mathsf{SL}_m(\mathbb{R}) \to \mathsf{PSL}_m(\mathbb{R})$ be the projection map. If $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ is a representation, let G_ρ be the Zariski closure of $\rho(\Gamma)$.

Theorem 1.1. [Intersection] *If Γ is a word hyperbolic group and $\rho_1 : \Gamma \to \mathsf{SL}_{m_1}(\mathbb{R})$ and $\rho_2 : \Gamma \to \mathsf{SL}_{m_2}(\mathbb{R})$ are projective Anosov representations, then*

$$\mathbf{J}(\rho_1,\rho_2) \geqslant 1.$$

Moreover, if ρ_1 and ρ_2 are irreducible, G_{ρ_1} and G_{ρ_2} are connected and $\mathbf{J}(\rho_1,\rho_2)=1$, then there exists an isomorphism $\phi:\pi_{m_1}(\mathsf{G}_{\rho_1})\to\pi_{m_2}(\mathsf{G}_{\rho_2})$ such that

$$\phi\circ\pi_{m_1}\circ\rho_1=\pi_{m_2}\circ\rho_2.$$

We also establish a spectral rigidity result. If $\rho:\Gamma\to\mathsf{SL}_m(\mathbb{R})$ is projective Anosov and $\gamma\in\Gamma$, then let $\mathsf{L}(\gamma)(\rho)$ denote the eigenvalue of maximal absolute value of $\rho(\gamma)$, so

$$\Lambda(\gamma)(\rho)=|\mathsf{L}(\gamma)(\rho)|.$$

Theorem 1.2. [Spectral rigidity] *Let Γ be a word hyperbolic group and let $\rho_1:\Gamma\to\mathsf{SL}_m(\mathbb{R})$ and $\rho_2:\Gamma\to\mathsf{SL}_m(\mathbb{R})$ be projective Anosov representations with limit maps ξ_1 and ξ_2 such that*

$$\mathsf{L}(\gamma)(\rho_1)=\mathsf{L}(\gamma)(\rho_2)$$

for every γ in Γ. Then there exists $g\in\mathsf{GL}_m(\mathbb{R})$ such that $g\xi_1=\xi_2$.

Moreover, if ρ_1 is irreducible, then $g\rho_1g^{-1}=\rho_2$.

We now introduce the deformation spaces which occur in our work. In section 7, we will see that each of these deformation spaces is a real analytic manifold. Let us introduce some terminology. If G is a reductive subgroup of $\mathsf{SL}_m(\mathbb{R})$, we say that an element of G is *generic* if its centralizer is a maximal torus in G. For example, an element of $\mathsf{SL}_m(\mathbb{R})$ is generic if and only if it is diagonalizable over $\mathbb{C}$ with distinct eigenvalues. We say that a representation $\rho:\Gamma\to\mathsf{G}$ is G-*generic* if the Zariski closure of $\rho(\Gamma)$ contains a generic element of G. Finally, we say that $\rho\in\mathrm{Hom}(\Gamma,\mathsf{G})$ is *regular* if it is a smooth point of the algebraic variety $\mathrm{Hom}(\Gamma,\mathsf{G})$.

- Let $\mathcal{C}(\Gamma,m)$ denote the space of (conjugacy classes of) regular, irreducible, projective Anosov representations of Γ into $\mathsf{SL}_m(\mathbb{R})$.
- Let $\mathcal{C}_g(\Gamma,\mathsf{G})$ denote the space of (conjugacy classes of) G-generic, regular, irreducible, projective Anosov representations.

We show that the entropy and the renormalized intersection vary analytically over our deformation spaces. Moreover, we obtain analyticity on analytic families of projective Anosov homomorphisms. An analytic family of projective Anosov homomorphisms is a continuous map $\beta:M\to\mathrm{Hom}(\Gamma,\mathsf{SL}_m(\mathbb{R}))$ such that M is an analytic manifold, $\beta_m=\beta(m)$ is projective Anosov for all $m\in M$, and $m\to\beta_m(\gamma)$ is an analytic map of M into $\mathsf{SL}_m(\mathbb{R})$ for all $\gamma\in\Gamma$.

Theorem 1.3. [Analyticity] *If Γ is a word hyperbolic group, then the entropy h and the renormalized intersection $\mathbf{J}$ are well-defined positive,* $\mathrm{Out}(\Gamma)$*-invariant analytic functions on the spaces $\mathcal{C}(\Gamma,m)$ and $\mathcal{C}(\Gamma,m)\times\mathcal{C}(\Gamma,m)$ respectively. More generally, they are analytic functions on any analytic family of projective Anosov homomorphisms.*

Moreover, let $\gamma:(-1,1)\to\mathcal{C}(\Gamma,m)$ be any analytic path with values in the deformation space, let $\mathbf{J}_\gamma(t)=\mathbf{J}(\gamma(0),\gamma(t))$ then

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\mathbf{J}_\gamma=0\ \text{and}\ \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0}\mathbf{J}_\gamma\geqslant 0. \tag{1}$$

Theorem 1.3 allows us to define a non-negative analytic 2-tensor on $\mathcal{C}_g(\Gamma,\mathsf{G})$. The pressure form is defined to be the Hessian of the restriction of the renormalized intersection $\mathbf{J}$. Our main result is the following.

Theorem 1.4. [Pressure metric] *Let* Γ *be a word hyperbolic group and let* G *be a reductive subgroup of* $\mathsf{SL}_m(\mathbb{R})$. *The pressure form is an analytic* $\mathrm{Out}(\Gamma)$*-invariant Riemannian metric on* $\mathcal{C}_g(\Gamma,\mathsf{G})$.

If S is a closed, connected, orientable, hyperbolic surface, Hitchin [33] exhibited a component $\mathcal{H}_m(S)$ of $\mathrm{Hom}(\pi_1(S),\mathsf{PSL}_m(\mathbb{R}))/\mathsf{PGL}_m(\mathbb{R})$ now called the *Hitchin component*, which is an analytic manifold diffeomorphic to a ball. Each Hitchin component contains a Fuchsian locus which consists of representations obtained by composing Fuchsian representations of $\pi_1(S)$ into $\mathsf{PSL}_2(\mathbb{R})$ with the irreducible representation $\tau_m : \mathsf{PSL}_2(\mathbb{R}) \to \mathsf{PSL}_m(\mathbb{R})$. The representations in a Hitchin component are called *Hitchin representations* and can be lifted to representations into $\mathsf{SL}_m(\mathbb{R})$. Labourie [44] showed that lifts of Hitchin representations are projective Anosov, irreducible and $\mathsf{SL}_m(\mathbb{R})$-generic. In particular, if $\rho_i : \pi_1(S) \to \mathsf{PSL}_m(\mathbb{R})$ are Hitchin representations, then one can define $h(\rho_i)$, $\mathbf{I}(\rho_1,\rho_2)$ and $\mathbf{J}(\rho_1,\rho_2)$ just as for projective Anosov representations. Guichard has recently announced a classification of the possible Zariski closures of Hitchin representations, see Section 11.3 for a statement. As a corollary of Theorem 1.1 and Guichard's work we obtain a stronger rigidity result for Hitchin representations.

Corollary 1.5. [Hitchin rigidity] *Let* S *be a closed, orientable surface and let* $\rho_1 \in \mathcal{H}_{m_1}(S)$ *and* $\rho_2 \in \mathcal{H}_{m_2}(S)$ *be two Hitchin representations such that*

$$\mathbf{J}(\rho_1,\rho_2) = 1.$$

Then, either

- $m_1 = m_2$ *and* $\rho_1 = \rho_2$ *in* $\mathcal{H}_{m_1}(S)$, *or*
- *there exists an element* ρ *of the Teichmüller space* $\mathcal{T}(S)$ *so that* $\rho_1 = \tau_{m_1}(\rho)$ *and* $\rho_2 = \tau_{m_2}(\rho)$.

In section 11.4 we use work of Benoist [5, 6] to obtain a similar rigidity result for representations which arise as monodromies of strictly convex projective structures on compact manifolds with word hyperbolic fundamental group. We will call such representations Benoist representations.

Each Hitchin component lifts to a component of $\mathcal{C}_g(\pi_1(S),\mathsf{SL}_m(\mathbb{R}))$. As a corollary of Theorem 1.4 and work of Wolpert [71] we obtain:

Corollary 1.6. [Hitchin component] *The pressure form on the Hitchin component is an analytic Riemannian metric which is invariant under the mapping class group and restricts to the Weil-Petersson metric on the Fuchsian locus.*

The same naturally holds for Hitchin components of representations into $\mathsf{PSp}(n,\mathbb{R})$, $\mathsf{SO}(n,n+1)$ and $\mathsf{G}_{2,0}$, since they embed in Hitchin components of representations into $\mathsf{PSL}(n,\mathbb{R})$. Labourie and Wentworth [49] have announced an explicit formula (in term of the Hitchin parametrisation) for the pressure metric along the Fuchsian locus.

Li [51] used the work of Loftin [53] and Labourie [46] to exhibit a metric on $\mathcal{H}_3(S)$, which she calls the Loftin metric, which is invariant with respect to the mapping class group, restricts to a multiple of the Weil-Petersson metric on the Fuchsian locus and such that the Fuchsian locus is totally geodesic. She further shows that a metric on $\mathcal{H}_3(S)$ constructed earlier by Darvishzadeh and Goldman [26] restricts to a multiple of the Weil-Petersson metric on the Fuchsian locus. Kim and Zhang [42] introduced a mapping class group invariant Kähler metric on the Hitchin component $H_3(S)$ for $\mathsf{SL}(3,R)$, which Labourie [48] generalized to the Hitchin components associated to all real split simple Lie groups of rank 2.

If Γ is a word hyperbolic group, we let $\mathcal{C}_c(\Gamma,\mathsf{PSL}_2(\mathbb{C}))$ denote the space of (conjugacy classes of) convex cocompact representations of Γ into $\mathsf{PSL}_2(\mathbb{C})$. In Section 2.3 we produce a representation, called the Plücker representation, $\alpha:\mathsf{PSL}_2(\mathbb{C})\to\mathsf{SL}_m(\mathbb{R})$ (for some m), so that if $\rho\in\mathcal{C}_c(\Gamma,\mathsf{PSL}_2(\mathbb{C}))$, then $\alpha\circ\rho$ is projective Anosov. The deformation space $\mathcal{C}_c(\Gamma,\mathsf{PSL}_2(\mathbb{C}))$ is an analytic manifold and we may define a renormalized intersection $\mathbf{J}$ and thus a pressure form on $\mathcal{C}_c(\Gamma,\mathsf{PSL}_2(\mathbb{C}))$. The following corollary is a direct generalization of Bridgeman's pressure metric on quasifuchsian space (see [9]).

Corollary 1.7. [Kleinian groups] *Let Γ be a torsion-free word hyperbolic group. The pressure form gives rise to a $\mathrm{Out}(\Gamma)$-invariant metric on the analytic manifold $\mathcal{C}_c(\Gamma,\mathsf{PSL}_2(\mathbb{C}))$ which is Riemannian on the open subset consisting of Zariski dense representations. Moreover,*

1. *If Γ does not have a finite index subgroup which is either a free group or a surface group, then the metric is Riemannian at all points in $\mathcal{C}_c(\Gamma,\mathsf{PSL}_2(\mathbb{C}))$.*
2. *If Γ is the fundamental group of a closed, connected, orientable surface, then the metric is Riemannian off of the Fuchsian locus in $\mathcal{C}_c(\Gamma,\mathsf{PSL}_2(\mathbb{C}))$ and restricts to a multiple of the Weil-Petersson metric on the Fuchsian locus.*

If G is a rank one semi-simple Lie group, then work of Patterson [59], Sullivan [69], Yue [72] and Corlette-Iozzi [22] shows that the entropy of a convex cocompact representation $\rho:\Gamma\to\mathsf{G}$ agrees with the Hausdorff dimension of the limit set of $\rho(\Gamma)$. We may then apply Theorem 1.3 and the Plücker representation to conclude that that the Hausdorff dimension of the limit set varies analytically over analytic families of convex cocompact representations into rank one semi-simple Lie groups.

Corollary 1.8. [Analyticity of Hausdorff Dimension] *If Γ is a finitely generated group and G is a rank one semi-simple Lie group, then the Hausdorff dimension of the limit set varies analytically on any analytic family of convex cocompact representations of Γ into G. In particular, the Hausdorff dimension varies analytically over $\mathcal{C}_c(\Gamma,\mathsf{PSL}_2(\mathbb{C}))$*

One may further generalize our construction into the setting of virtually Zariski dense Anosov representations into an arbitrary semi-simple Lie group G. A representation $\rho:\Gamma\to\mathsf{G}$ is *virtually Zariski dense* if the Zariski closure of $\rho(\Gamma)$ is a finite index subgroup of G. If Γ is a word hyperbolic group, G is a semi-simple Lie group with finite center and P is a non-degenerate parabolic subgroup, then we let $\mathcal{Z}(\Gamma;\mathsf{G},\mathsf{P})$ denote the space of (conjugacy classes of) regular virtually Zariski dense (G,P)-Anosov representations of Γ into G. The space $\mathcal{Z}(\Gamma;\mathsf{G},\mathsf{P})$ is an analytic orbifold, see Proposition 7.3, and we can again use a Plücker representation to define a pressure metric on $\mathcal{Z}(\Gamma;\mathsf{G},\mathsf{P})$. If G is connected, then $\mathcal{Z}(\Gamma;\mathsf{G},\mathsf{P})$ is an analytic manifold.

Corollary 1.9. [Anosov representations] *Suppose that Γ is a word hyperbolic group, G is a semi-simple Lie group with finite center and P is a non-degenerate parabolic subgroup of G. Then there exists an $\mathrm{Out}(\Gamma)$-invariant analytic Riemannian metric on the orbifold $\mathcal{Z}(\Gamma;\mathsf{G},\mathsf{P})$.*

There are $2g-3$ components of the space of representations of the fundamental group of a closed orientable surface of genus g into $\mathsf{Sp}(4,\mathbb{R})$ which are (non-simply connected) analytic manifolds consisting entirely of Zariski dense, maximal representations (see Guichard-Wienhard [28, Theorem 15] and Bradlow-Garca-Prada-Gothen [16, Theorem 1.2]). Since maximal representations into $\mathsf{Sp}(4,\mathbb{R})$ are $(\mathsf{Sp}(4,\mathbb{R}),\mathsf{GL}(2,\mathbb{R}))$-Anosov (see Burger-Iozzi-Labourie-Wienhard [18, Theorem 6.1]), Corollary 1.9 implies that such components admit mapping class group invariant, analytic Riemannian metrics.

A key tool in our proof is the introduction of a flow $\mathsf{U}_\rho\Gamma$ associated to a projective Anosov representation ρ. Let $\rho:\Gamma\to\mathsf{SL}_m(\mathbb{R})$ be a projective Anosov representation with limit maps ξ and θ. Let F be the total space of the principal $\mathbb{R}$-bundle over $\mathbb{RP}(m)\times\mathbb{RP}(m)^*$ whose fiber at the point (x,y) is the space of norms on the line $\xi(x)$. There is a natural $\mathbb{R}$-action on F which takes a norm u on x to the norm $e^{-t}u$. Let F_ρ be $\mathbb{R}$-principal bundle over

$$\partial_\infty\Gamma^{(2)}=\partial_\infty\Gamma\times\partial_\infty\Gamma\setminus\{(x,x)\mid x\in\partial_\infty\Gamma\}.$$

which is the pull back of F by (ξ,θ). The $\mathbb{R}$-action on F gives rise to a flow on F_ρ. (An analogue of this flow was first introduced by Sambarino [66, 65] in the setting of projective Anosov irreducible representations of fundamental groups of closed negatively curved manifolds.)

We then show that this flow is metric Anosov and is a Hölder reparameterization of the *Gromov geodesic flow* $\mathsf{U}_0\Gamma$ of Γ. Moreover, this flow encodes the spectral radii of elements of $\rho(\Gamma)$, i.e. the period of the flow associated to (the conjugacy class of) an element $\gamma\in\Gamma$ is $\log\Lambda(\gamma)(\rho)$. (Metric Anosov flows are a natural generalization of Anosov flows in the setting of compact metric spaces and were studied by Pollicott [60].)

Theorem 1.10. [GEODESIC FLOW] *The action of Γ on F_ρ is proper and cocompact. Moreover, the $\mathbb{R}$ action on $\mathsf{U}_\rho\Gamma=F_\rho/\Gamma$ is a topologically transitive metric Anosov flow which is Hölder orbit equivalent to the geodesic flow $\mathsf{U}_0\Gamma$.*

Theorem 1.10 allows us to make use of the thermodynamic formalism. We show that if f_ρ is the Hölder function regulating the change of speed of $\mathsf{U}_\rho\Gamma$ and $\mathsf{U}_0\Gamma$, then $\Phi_\rho=-h(\rho)f_\rho$ is a pressure zero function on $\mathsf{U}_0\Gamma$. Therefore, we get a mapping

$$\mathfrak{T}:\mathcal{C}(\Gamma,m)\to\mathcal{H}(\mathsf{U}_0\Gamma),$$

called the *thermodynamic mapping*, from $\mathcal{C}(\Gamma,m)$ into the space $\mathcal{H}(\mathsf{U}_0\Gamma)$ of Livšic cohomology classes of pressure zero Hölder functions on $\mathsf{U}_0\Gamma$. Given any $[\rho]\in\mathcal{C}(\Gamma,m)$, there exists an open neighborhood U of $[\rho]$ and a lift of $\mathfrak{T}|_U$ to an analytic map of U into the space $\mathcal{P}(\mathsf{U}_0\Gamma)$ of pressure zero Hölder functions on $\mathsf{U}_0\Gamma$. Our pressure form is obtained as a pullback of the pressure 2-tensor on $\mathcal{P}(\mathsf{U}_0\Gamma)$ with respect to this lift.

Remarks and references: Anosov representations were introduced by Labourie [44] in his study of Hitchin representations, and their theory was further developed by Guichard and Wienhard [29]. Benoist [5, 6, 7] studied holonomy maps of strictly convex projective structures on closed manifolds which he showed were irreducible representations with transverse projective limit maps, hence projective Anosov. Sambarino [65, 66, 67] introduced a flow, closely related to our flow, associated to a representation with transverse projective limit maps and used it to prove the continuity of the associated entropy on a Hitchin component. Pollicott and Sharp

[61] applied the thermodynamic formalism and work of Dreyer [25] to show that a closely related entropy gives rise to an analytic function on any Hitchin component.

Our metric generalizes Thurston's Riemannian metric on Teichmüller space which he defined to be the Hessian of the length of a random geodesic. Wolpert [71] proved that Thurston's Riemannian metric was a multiple of the more classical Weil-Petersson metric. Bonahon [11] gave an interpretation of Thurston's metric in terms of the Hessian of an intersection function. Burger [17] previously studied the intersection number for convex cocompact subgroups of rank 1 simple Lie groups and proved a strong version of Theorem 1.1 in this setting (see also Kim [40]). The study of geometric properties of surfaces using the thermodynamic formalism originated in Bowen [14]. Using a Bowen-Series coding and building on work of Bridgeman and Taylor [10], McMullen [56] gave a pressure metric formulation of the Weil–Petersson metric on Teichmüller space. Bridgeman [9] developed a pressure metric on quasifuchsian space which restricts to the Weil–Petersson metric on the Fuchsian locus. Our Theorem 1.4 is a natural generalization of Bridgeman's work into the setting of projective Anosov representations, while Corollary 1.7 is a generalization into the setting of general deformation spaces of convex cocompact representations into $\mathsf{PSL}_2(\mathbb{C})$.

Corollary 1.8 was established by Ruelle [63] for quasifuchsian representations, *i.e.* when $\Gamma = \pi_1(S)$ and $\mathsf{G} = \mathsf{PSL}_2(\mathbb{C})$, and by Anderson and Rocha [2] for function groups, *i.e.* when Γ is a free product of surface groups and free groups and $\mathsf{G} = \mathsf{PSL}_2(\mathbb{C})$. Previous work of Tapie [70] implies that the Hausdorff dimension of the limit set is a C^1 function on C^1-families of convex cocompact representations of Γ into a rank one Lie group G. Tapie's work was inspired by work of Katok, Knieper, Pollicott and Weiss [38, 39] who established analytic variation of the entropy for analytically varying families of Anosov flows on closed Riemannian manifolds. Our Theorem 1.2 is related to the marked length spectrum rigidity theorem of Dal'Bo-Kim [23].

Coornaert–Papadopoulos [21] showed that if Γ is word hyperbolic, then there is a symbolic coding of its geodesic flow $\mathsf{U}_0\Gamma$. However, this coding is not necessarily one-to-one on a large enough set to apply the thermodynamic formalism. Therefore, word hyperbolic groups admitting projective Anosov representations represent an interesting class of groups from the point of view of symbolic dynamics.

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Contents

2. Anosov representations

In this section, we recall the theory of Anosov representations. We begin by defining projective Anosov representations and developing their basic properties. In section 2.3, we will see that any Anosov representation can be transformed, via post-composition with a Plücker representation, into a projective Anosov representation, while in section 2.4 we will study properties of irreducible projective Anosov representations.

2.1. Projective Anosov representations.

A representation $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ is projective Anosov if it has transverse projective limit maps and the associated flat bundle over its Gromov geodesic flow has a contraction property we will define carefully below.

Definition 2.1. *Let Γ be a word hyperbolic group and ρ be a representation of Γ in $\mathsf{SL}_m(\mathbb{R})$. We say ρ has* transverse projective limit maps *if there exist ρ-equivariant continuous maps $\xi : \partial_\infty\Gamma \to \mathbb{RP}(m)$ and $\theta : \partial_\infty\Gamma \to \mathbb{RP}(m)^*$ such that if $x \neq y$, then*

$$\xi(x) \oplus \theta(y) = \mathbb{R}^m.$$

Conventions: Denote by $\mathbb{RP}(m)$ the projective space of $\mathbb{R}^m$. We will often identify $\mathbb{RP}(m)^*$ with the Grassmannian $\mathrm{Gr}_{m-1}(\mathbb{R}^m)$ of $(m-1)$-dimensional subspaces of $\mathbb{R}^m$, via $\varphi \mapsto \ker\varphi$. The action of $\mathsf{SL}_m(\mathbb{R})$ on $\mathbb{RP}(m)^*$ consistent wth this identification is

$$g \cdot \varphi = \varphi \circ g^{-1}.$$

We will also assume throughout this paper that our word hyperbolic group does not have a finite index cyclic subgroup. Since all the word hyperbolic groups we study are linear, Selberg's Lemma implies that they contain finite index torsion-free subgroups.

Gromov [27] defined a geodesic flow $\mathsf{U}_0\Gamma$ for a word hyperbolic group – that we shall call the *Gromov geodesic flow* – (see Champetier [19] and Mineyev [57] for details). He defines a proper cocompact action of Γ on $\partial_\infty\Gamma^{(2)} \times \mathbb{R}$ which commutes with the action of $\mathbb{R}$ by translation on the final factor. The action of Γ restricted to $\partial_\infty\Gamma^{(2)}$ is the diagonal action arising from the standard action of Γ on $\partial_\infty\Gamma$. There is a metric on $\partial_\infty\Gamma^{(2)} \times \mathbb{R}$, well-defined up to Hölder equivalence, so that Γ acts by isometries, every orbit of the $\mathbb{R}$ action gives a quasi-isometric embedding and the geodesic flow acts by Lipschitz homeomorphisms. The flow on

$$\widetilde{\mathsf{U}_0\Gamma} = \partial_\infty\Gamma^{(2)} \times \mathbb{R}$$

descends to a flow on the quotient

$$\mathsf{U}_0\Gamma = \partial_\infty\Gamma^{(2)} \times \mathbb{R}/\Gamma.$$

In the case that M is a closed negatively curved manifold and $\Gamma = \pi_1(M)$, $\mathsf{U}_0\Gamma$ may be identified with T^1M in such a way that the flow on $\mathsf{U}_0\Gamma$ is identified with the geodesic flow on T^1M. Since the action of Γ on $\partial_\infty\Gamma^2$ is topologically transitive, the Gromov geodesic flow is topologically transitive.

If $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ is a representation, we let E_ρ be the associated flat bundle over the geodesic flow of the word hyperbolic group $\mathsf{U}_0\Gamma$. Recall that

$$E_\rho = \widetilde{\mathsf{U}_0\Gamma} \times \mathbb{R}^m/\Gamma$$

where the action of $\gamma \in \Gamma$ on $\mathbb{R}^m$ is given by $\rho(\gamma)$. If ρ has transverse projective limit maps ξ and θ, there is an induced splitting of E_ρ as

$$E_\rho = \Xi \oplus \Theta$$

where Ξ and Θ are sub-bundles, parallel along the geodesic flow, of rank 1 and $m-1$ respectively. Explicitly, if we lift Ξ and Θ to sub-bundles $\tilde{\Xi}$ and $\tilde{\Theta}$ of the bundle $\widetilde{\mathsf{U}_0\Gamma} \times \mathbb{R}^m$ over $\widetilde{\mathsf{U}_0\Gamma}$, then the fiber of $\tilde{\Xi}$ above (x, y, t) is simply $\xi(x)$ and the fiber of $\tilde{\Theta}$ is $\theta(y)$.

The $\mathbb{R}$-action on $\widetilde{\mathsf{U}_0\Gamma}$ extends to a flow $\{\tilde{\psi}_t\}_{t\in\mathbb{R}}$ on $\widetilde{\mathsf{U}_0\Gamma} \times \mathbb{R}^m$ (which acts trivially on the $\mathbb{R}^m$ factor). The flow $\{\tilde{\psi}_t\}_{t\in\mathbb{R}}$ descends to a flow $\{\psi_t\}_{t\in\mathbb{R}}$ on E_ρ which is a lift of the geodesic flow on $\mathsf{U}_0\Gamma$. In particular, the flow respects the splitting $E_\rho = \Xi \oplus \Theta$.

In general, we say that a vector bundle E over a compact topological space whose total space is equipped with a flow $\{\phi_t\}_{t\in\mathbb{R}}$ of bundle automorphisms is *contracted* by the flow if for any metric $\|.\|$ on E, there exists $t_0 > 0$ such that if $v \in E$, then

$$\|\phi_{t_0}(v)\| \leqslant \frac{1}{2}\|v\|.$$

Observe that if bundle is contracted by a flow, its dual is contracted by the inverse flow. Moreover, if the flow is contracting, it is also *uniformly contracting*, i.e. given any metric, there exists positive constants A and c such that

$$\|\phi_t(v)\| \leqslant Ae^{-ct}\|v\|$$

for any $v \in E$.

Definition 2.2. *A representation $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ with transverse projective limit maps is* projective Anosov *if the bundle* $\mathrm{Hom}(\Theta, \Xi)$ *is contracted by the flow* $\{\psi_t\}_{t\in\mathbb{R}}$.

In the sequel, we will use the notation $\Theta^* = \mathrm{Hom}(\Theta, \mathbb{R})$. The following alternative description will be useful.

Proposition 2.3. *A representation $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ with transverse projective limit maps ξ and θ is projective Anosov if and only if there exists $t_0 > 0$ such that for all $Z \in \mathsf{U}_0\Gamma$, $v \in \Xi_Z \setminus \{0\}$ and $w \in \Theta_Z \setminus \{0\}$,*

$$\frac{\|\psi_{t_0}(v)\|}{\|\psi_{t_0}(w)\|} \leq \frac{1}{2}\frac{\|v\|}{\|w\|}. \tag{2}$$

Notation: If E is a bundle over a base space X and $Z \in X$, then E_Z will denote the fibre of E over the point Z.

Proof. Given a projective Anosov representation $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ and a metric $\|.\|$ on E_ρ, let $t_0 > 0$ be chosen so that

$$\|\psi_{t_0}(\eta)\| \leqslant \frac{1}{2}\|\eta\|.$$

for all $\eta \in \Xi \otimes \Theta^*$. If $Z \in \mathsf{U}_0\Gamma$, $v \in \Xi_Z \setminus \{0\}$ and $w \in \Theta_Z \setminus \{0\}$, then there exists $\eta \in \mathrm{Hom}(\Theta_Z, \Xi_Z) = (\Xi \otimes \Theta^*)_Z$ such that $\eta(w) = v$ and $\|\eta\| = \|v\|/\|w\|$. Then,

$$\frac{\|\psi_{t_0}(v)\|}{\|\psi_{t_0}(w)\|} \leq \|\psi_{t_0}(\eta)\| \leqslant \frac{1}{2}\|\eta\| = \frac{\|v\|}{\|w\|}.$$

The converse is immediate. ∎

Furthermore, projective Anosov representations are contracting on Ξ.

Lemma 2.4. *If $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ is projective Anosov, then $\{\psi_t\}_{t\in\mathbb{R}}$ is contracting on Ξ.*

Proof. Since the bundle $\Xi \otimes \Theta^*$ is contracted, so is

$$\Omega = \det(\Xi \otimes \Theta^*) = \Xi^{\otimes(m-1)} \otimes \det(\Theta^*).$$

One may define an isomorphism from Ξ to $\det(\Theta)^*$ by taking the vector u to the map $\alpha \to \mathrm{Vol}(u \wedge \alpha)$. Since $\det(\Theta)^*$ is isomorphic to $\det(\Theta^*)$, it follows that Ω is isomorphic to $\Xi^{\otimes m}$. Thus Ξ is contracted. □

It follows from standard techniques in hyperbolic dynamics that our limit maps are Hölder. We will give a proof of a more general statement in Section 6 (see [44, Proposition 3.2] for a proof in a special case).

Lemma 2.5. *Let ρ be a projective Anosov representation, then the limit maps ξ and θ are Hölder.*

If γ is an infinite order element of Γ, then there is a periodic orbit of $\mathsf{U}_0\Gamma$ associated to γ. If γ^+ is the attracting fixed point of γ on $\partial_\infty\Gamma$ and γ^- is its other fixed point, then this periodic orbit is the image of $(\gamma^+, \gamma^-) \times \mathbb{R}$. Inequality (2) and Lemma 2.4 applied to the periodic orbit of $\mathsf{U}_0\Gamma$ associated to γ imply that $\rho(\gamma)$ is proximal and that $\xi(\gamma^+)$ is the eigenspace associated to the largest modulus eigenvalue of $\rho(\gamma)$. Similarly, $\xi(\gamma^-)$ is the repelling hyperplane of $\rho(\gamma)$. It follows that the limit maps ξ and θ are uniquely determined by ρ (see also [29, Lemmas 3.1 and 3.3]).

Let $\mathsf{L}(\gamma)(\rho)$ denote the eigenvalue of $\rho(\gamma)$ of maximal absolute value and let $\Lambda(\gamma)(\rho)$ denote the spectral radius of $\rho(\gamma)$, so $\Lambda(\gamma)(\rho) = |\mathsf{L}(\gamma)(\rho)|$. If S is a fixed generating set for Γ and $\gamma \in \Gamma$, then we let $l(\gamma)$ denote the translation length of the action of γ on the Cayley graph of Γ with respect to S; more explicitly, $l(\gamma)$ is the minimal word length of any element conjugate to γ. Since the contraction is uniform and the length of the periodic orbit of $\mathsf{U}_0\Gamma$ associated to γ is comparable to $l(\gamma)$, we obtain the following uniform estimates:

Proposition 2.6. *If $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ is a projective Anosov representation, then there exists $\delta \in (0,1)$ such that if $\gamma \in \Gamma$ has infinite order, then $\mathsf{L}(\gamma)(\rho)$ and $(\mathsf{L}(\gamma^{-1})(\rho))^{-1}$ are both eigenvalues of $\rho(\gamma)$ of multiplicity one and*

$$\rho(\gamma) = \mathsf{L}(\gamma)(\rho)\mathsf{p}_\gamma + \mathsf{m}_\gamma + \frac{1}{\mathsf{L}(\gamma^{-1})(\rho)}\mathsf{q}_\gamma$$

where

- p_γ *is the projection on $\xi(\gamma^+)$ parallel to $\theta(\gamma^-)$,*
- $\mathsf{q}_\gamma = \mathsf{p}_{\gamma^{-1}}$,
- $\mathsf{m}_\gamma = A \circ (1 - \mathsf{q}_\gamma - \mathsf{p}_\gamma)$ *and A is an endomorphism of $\theta(\gamma^-) \cap \theta(\gamma+)$ whose spectral radius is less than*

$$\delta^{\ell(\gamma)}\Lambda(\gamma)(\rho).$$

Moreover, we see that ρ is well-displacing in the following sense:

Proposition 2.7. [Displacing property] *If $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ is a projective Anosov representation, then there exists constants $K > 0$ and $C > 0$, and a neighborhood U of ρ_0 in $\mathrm{Hom}(\Gamma, \mathsf{SL}_m(\mathbb{R}))$ such that that for every $\gamma \in \Gamma$ and $\rho \in U$ we have*

$$\frac{1}{K}\ell(\gamma) - C \leqslant \quad \log(\Lambda(\gamma)(\rho)) \quad \leqslant K\ell(\gamma) + C, \tag{3}$$

Proposition 2.7 immmediately implies:

Proposition 2.8. *For every real number T, the set*

$$R_T(\rho) = \{[\gamma] \mid \log(\Lambda(\gamma)(\rho)) \leqslant T\}$$

is finite.

Remark: Proposition 2.6 is a generalization of results of Labourie [44, Proposition 3.4], Sambarino [66, Lemma 5.1] and Guichard-Wienhard [29, Lemma 3.1]. Proposition 2.7 is a generalization of a result of Labourie [47, Theorem 1.0.1] and a special case of a result of Guichard-Wienhard [29, Theorem 5.14]. See [24] for a discussion of well-displacing representations and their relationship with quasi-isometric embeddings.

2.2. **Anosov representations.** We now recall the general definition of an Anosov representation and note that projective Anosov representations are examples of Anosov representations.

We first recall some notation and definitions. Let G be a semi-simple Lie group with finite center and Lie algebra $\mathfrak{g}$. Let K be a maximal compact subgroup of G and let τ be the Cartan involution on $\mathfrak{g}$ whose fixed point set is the Lie algebra of K. Let $\mathfrak{a} = \mathfrak{a}_{\mathsf{G}}$ be a maximal abelian subspace contained in $\{v \in \mathfrak{g} : \tau v = -v\}$.

For $a \in \mathfrak{a}$, let M be the connected component of the centralizer of $\exp a$ which contains the identity, and let $\mathfrak{m}$ denote its Lie algebra. Let E_λ be the eigenspace of the action of a on $\mathfrak{g}$ with eigenvalue λ and consider

$$\begin{aligned} \mathfrak{n}^+ &= \bigoplus_{\lambda>0} E_\lambda, \\ \mathfrak{n}^- &= \bigoplus_{\lambda<0} E_\lambda, \end{aligned}$$

so that

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-. \tag{4}$$

Then $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are Lie algebras normalized by M. Let $\mathsf{P}^\pm$ be the Lie subgroups of G which normalize the Lie algebras $\mathfrak{p}^\pm = \mathfrak{m} \oplus \mathfrak{n}^\pm$. Then P^+ and P^- are *opposite parabolic subgroups* and their Lie algebras are $\mathfrak{p}^+$ and $\mathfrak{p}^-$ respectively. We will say that P^+ is *non-degenerate* if $\mathfrak{p}^+$ does not contain a simple factor of $\mathfrak{g}$.

We may identify a point $([X],[Y])$ in $\mathsf{G}/\mathsf{P}^+ \times \mathsf{G}/\mathsf{P}^-$ with the pair $(\mathrm{Ad}(X)\mathsf{P}^+, \mathrm{Ad}(Y)\mathsf{P}^-)$ of parabolic subgroups. The pair $(\mathrm{Ad}(X)\mathsf{P}^+, \mathrm{Ad}(Y)\mathsf{P}^-)$ is *transverse* if their intersection $\mathrm{Ad}(X)\mathsf{P}^+ \cap \mathrm{Ad}(Y)\mathsf{P}^-$ is conjugate to M.

We now suppose that $\rho : \Gamma \to \mathsf{G}$ is a representation of a word hyperbolic group Γ and $\xi^+ : \partial_\infty\Gamma \to \mathsf{G}/\mathsf{P}^+$ and $\xi^- : \Gamma \to \mathsf{G}/\mathsf{P}^-$ are continuous ρ-equivariant maps. We say that ξ^+ and ξ^- are *transverse* if given any two distinct points $x, y \in \partial_\infty\Gamma$, $\xi^+(x)$ and $\xi^-(y)$ are transverse. The G-invariant splitting described by Equation

(4) then gives rise to bundles over $\mathsf{U}_0\Gamma$. Let $\widetilde{\mathcal{N}}_\rho^+$ and $\widetilde{\mathcal{N}}_\rho^-$ be the bundles over $\widetilde{\mathsf{U}_0\Gamma}$ whose fibers over the point (x,y,t) are

$$\mathrm{Ad}(\xi^-(y))\mathfrak{n}^+ \quad \text{and} \quad \mathrm{Ad}(\xi^+(x))\mathfrak{n}^-.$$

There is a natural action of Γ on $\widetilde{\mathcal{N}}_\rho^+$ and $\widetilde{\mathcal{N}}_\rho^-$, where the action on the fiber is given by $\rho(\Gamma)$, and we denote the quotient bundles over $\mathsf{U}_0\Gamma$ by $\mathcal{N}_\rho^+$ and $\mathcal{N}_\rho^-$. We may lift the geodesic flow to a flow on the bundles $\mathcal{N}_\rho^+$ and $\mathcal{N}_\rho^-$ which acts trivially on the fibers.

Definition 2.9. *Suppose that G is a semi-simple Lie group with finite center, P^+ is a parabolic subgroup of G and Γ is a word hyperbolic group. A representation $\rho : \Gamma \to \mathsf{G}$ is* $(\mathsf{G},\mathsf{P}^+)$-Anosov *if there exist transverse ρ-equivariant maps*

$$\xi^+ : \partial_\infty\Gamma \to \mathsf{G}/\mathsf{P}^+ \text{ and } \xi^- : \partial_\infty\Gamma \to \mathsf{G}/\mathsf{P}^-$$

so that the geodesic flow is contracting on the associated bundle $\mathcal{N}_\rho^+$ and the inverse flow is contracting on the bundle $\mathcal{N}_\rho^-$.

We now recall some basic properties of Anosov representations which were established by Labourie, [44, Proposition 3.4] and [47, Theorem 6.1.3], and Guichard-Wienhard [29, Theorem 5.3 and Lemma 3.1]. We recall that an element $g \in \mathsf{G}$ is *proximal* relative to P^+ if g has fixed points $x^+ \in \mathsf{G}/\mathsf{P}^+$ and $x^- \in \mathsf{G}/\mathsf{P}^-$ so that x^+ is transverse to x^- and if $x \in \mathsf{G}/\mathsf{P}^+$ is transverse to x^- then $\lim_{n\to\infty} g^n(x) = x^+$.

Theorem 2.10. *Let G be a semi-simple Lie group, P^+ a parabolic subgroup, Γ a word hyperbolic group and $\rho : \Gamma \to \mathsf{G}$ a $(\mathsf{G},\mathsf{P}^+)$-Anosov representation.*

1. *ρ has finite kernel, so Γ is virtually torsion-free.*
2. *ρ is well-displacing, so $\rho(\Gamma)$ is discrete.*
3. *If $\gamma \in \Gamma$ has infinite order, then $\rho(\gamma)$ is proximal relative to P^+*

In this language, projective Anosov representations are exactly the same as $(\mathsf{SL}_m(\mathbb{R}),\mathsf{P}^+)$-Anosov representations where P^+ is the stabilizer of a line in $\mathbb{R}^m$.

Proposition 2.11. *Let P^+ be the stabilizer of a line in $\mathbb{R}^m$. A representation $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ is projective Anosov if and only if it is $(\mathsf{SL}_m(\mathbb{R}),\mathsf{P}^+)$-Anosov. Moreover, the limit maps ξ and θ in the definition of projective Anosov representation agree with the limit maps ξ^+ and ξ^- in the definition of a $(\mathsf{SL}_m(\mathbb{R}),\mathsf{P}^+)$-Anosov representation.*

Proof. If ρ is projective Anosov with limit maps ξ and θ, one may identify $\mathsf{SL}_m(\mathbb{R})/P^+$ with $\mathbb{RP}(m)$ and $\mathsf{SL}_m(\mathbb{R})/P^-$ with $\mathbb{RP}(m)^*$ so that, after letting $\xi^+ = \xi$ and $\xi^- = \theta$, N_ρ^+ is identified with $\mathrm{Hom}(\Theta,\Xi)$ and N_ρ^- is identified with $\mathrm{Hom}(\Xi,\Theta)$.

The same identification holds if ρ is $(\mathsf{SL}_m(\mathbb{R}),\mathsf{P}^+)$-Anosov with limit maps ξ^+ and ξ^-. □

2.3. **Plücker representations.** Guichard and Wienhard [29] showed how to obtain a projective Anosov representation from any Anosov representation by post-composing with a Plücker representation. We first recall the following general result.

Theorem 2.12. [Guichard-Wienhard [29, Prop. 4.3]] *Let $\phi : \mathsf{G} \to \mathsf{SL}(V)$ be a finite dimensional irreducible representation. Let $x \in \mathbb{P}(V)$ and assume that*

$$\mathsf{P} = \{g \in \mathsf{G} : \phi(g)(x) = x\}$$

is a parabolic subgroup of G *with opposite parabolic* Q. *If* Γ *is a word hyperbolic group, then a representation* $\rho : \Gamma \to \mathsf{G}$ *is* (G, P)*-Anosov if and only if* $\phi \circ \rho$ *is projective Anosov.*

Furthermore, if ρ *is* (G, P)*-Anosov with limit maps* ξ^+ *and* ξ^-*, then the limit maps of* $\phi \circ \rho$ *are given by* $\xi = \beta \circ \xi^+$ *and* $\theta = \beta^* \circ \xi^-$ *where* $\beta : \mathsf{G}/\mathsf{P} \to \mathbb{P}(V)$ *and* $\beta^* : \mathsf{G}/\mathsf{Q} \to \mathbb{P}(V^*)$ *are the maps induced by* ϕ.

The following corollary is observed by Guichard-Wienhard [29, Remark 4.12]. We provide a proof here for the reader's convenience. The representation given in the proof will be called the *Plücker representation* of G with respect to P.

Corollary 2.13. [Guichard-Wienhard] *For any parabolic subgroup* P *of a semi-simple Lie group* G *with finite center, there exists a finite dimensional irreducible representation* $\alpha : \mathsf{G} \to \mathsf{SL}(V)$ *such that if* Γ *is a word hyperbolic group and* $\rho : \Gamma \to \mathsf{G}$ *is a* (G, P)*-Anosov representation, then* $\alpha \circ \rho$ *is projective Anosov.*

Moreover, if P *is non-degenerate, then* $\ker(\alpha) = Z(\mathsf{G})$ *and* α *is an immersion.*

Proof. In view of Theorem 2.12 it suffices to find a finite dimensional irreducible representation $\alpha : \mathsf{G} \to \mathsf{SL}(V)$ such that $\alpha(\mathsf{P})$ is the stabilizer (in $\alpha(\mathsf{G})$) of a line in V.

Let $\Lambda^k W$ denote the k-th exterior power of the vector space W. Let $n = \dim \mathfrak{n}^+ = \dim \mathfrak{n}^-$ and consider $\alpha : \mathsf{G} \to \mathsf{SL}(\Lambda^n \mathfrak{g})$ given by

$$\alpha(g) = \Lambda^n \operatorname{Ad}(g).$$

One may readily check that the restriction of α to $V = \langle \alpha(\mathsf{G}) \cdot \Lambda^n \mathfrak{n}^+ \rangle$ works.

If P is non-degenerate, then $\ker(\alpha|_V)$ is a normal subgroup of G which is contained in P, so $\ker(\alpha|_V)$ is contained in $Z(\mathsf{G})$ (see [62]). Since $Z(\mathsf{G})$ is in the kernel of the adjoint representation, we see that $\ker(\alpha|_V) = Z(\mathsf{G})$. Since $\alpha|_V$ is algebraic and $Z(\mathsf{G})$ is finite, it follows that $\alpha|_V$ is an immersion. ☐

If G has rank one, then it contains a unique conjugacy class of parabolic subgroups. A representation $\rho : \Gamma \to \mathsf{G}$ is Anosov if and only if it is convex cocompact (see [29, Theorem 5.15]). We then get the following.

We recall that the topological entropy of a convex cocompact representation $\rho : \Gamma \to \mathsf{G}$ of a word hyperbolic group into a rank one semi-simple Lie group is given by

$$h(\rho) = \lim_{T \to \infty} \frac{1}{T} \log \left(\sharp \{ [\gamma] \; | \mathrm{d}(\rho(\gamma)) \leqslant T \} \right),$$

where $\mathrm{d}(\rho(\gamma))$ denotes the translation length of $\rho(\gamma)$. We obtain the following immediate corollary.

Corollary 2.14. *Let* G *be a rank one semi-simple Lie group, let* Γ *be a word hyperbolic group and let* $\alpha : \mathsf{G} \to \mathsf{SL}(V)$ *be the Plücker representation. There exists* $K > 0$*, such that if* $\rho : \Gamma \to \mathsf{G}$ *is convex cocompact, then* $\alpha \circ \rho$ *is projective Anosov and*

$$h(\alpha \circ \rho) = \frac{h(\rho)}{K}.$$

Proof. Let $\lambda_{\mathsf{G}} : \mathsf{G} \to \mathfrak{a}_{\mathsf{G}}$ be the Jordan projection of G. Since $\mathfrak{a}_{\mathsf{G}}$ is one dimensional, we can identify it with $\mathbb{R}$ by setting $\lambda_{\mathsf{G}}(g) = \mathrm{d}(g)$.

Denote by $\chi_\alpha \in \mathfrak{a}_{\mathsf{G}}$ the highest (restricted) weight of the representation α (see, for example, Humphreys [35]). By definition, one has $\log \Lambda(\alpha(g)) = \chi_\alpha(\mathrm{d}(g))$, for every $g \in \mathsf{G}$. Hence, since $\mathfrak{a}_{\mathsf{G}}$ is one dimensional, one has

$$\log \Lambda(\alpha(\rho(\gamma))) = K\mathrm{d}(\rho(\gamma)) \tag{5}$$

for every $\gamma \in \Gamma$.

Since

$$h(\alpha \circ \rho) = \lim_{T\to\infty} \frac{1}{T} \log \sharp\left(\{[\gamma] \mid \log \Lambda(\alpha(\rho(\gamma)) \leq T \}\right),$$

it follows immediately that

$$h(\alpha \circ \rho) = \frac{h(\rho)}{K}.$$

□

2.4. **Irreducible representations.** Guichard and Wienhard [29, Proposition 4.10] proved that irreducible representations with transverse projective limit maps are projective Anosov (see also [44] for hyperconvex representations).

Proposition 2.15. [Guichard–Wienhard] *If Γ is a word hyperbolic group, then every irreducible representation $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ with transverse projective limit maps is projective Anosov.*

It will be useful to note that if $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ is projective Anosov and irreducible, then $\xi(\partial_\infty\Gamma)$ contains a projective frame for $\mathbb{RP}(m)$. We recall that a collection of $m+1$ elements in $\mathbb{RP}(m)$ is a *projective frame* if every subset containing m elements spans $\mathbb{R}^m$. We first prove the following lemma.

Lemma 2.16. *Let $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ be a representation with a continuous ρ-equivariant map $\xi : \partial_\infty\Gamma \to \mathbb{RP}(m)$, then the preimage $\xi^{-1}(V)$ of a vector subspace $V \subset \mathbb{R}^m$ is either $\partial_\infty\Gamma$ or has empty interior on $\partial_\infty\Gamma$.*

Proof. Choose $\{x_1, \ldots, x_p\} \subset \partial_\infty\Gamma$ so that $\{\xi(x_1), \ldots, \xi(x_p)\}$ spans the vector subspace $\langle\xi(\partial_\infty\Gamma)\rangle$ spanned by $\xi(\partial_\infty\Gamma)$.

Suppose that $\xi^{-1}(V) = \{x \in \partial_\infty\Gamma : \xi(x) \in V\}$ has non-empty interior in $\partial_\infty\Gamma$. Choose $\gamma \in \Gamma$ so that $\gamma^- \notin \{x_1, \ldots, x_p\}$ and γ^+ belongs to the interior of $\xi^{-1}(V)$.

Since $\gamma^n(x_i) \to \gamma^+$ for every $i \in \{1, \ldots, p\}$, if we choose n large enough, then $\gamma^n(x_i)$ is contained in the interior of $\xi^{-1}(V)$, so $\xi(\gamma^n x_i) \in V$. Since $\{\xi(\gamma^n(x_1)), \ldots, \xi(\gamma^n(x_p))\}$ still spans $\langle\xi(\partial_\infty\Gamma)\rangle$, we see that $\langle\xi(\partial_\infty\Gamma)\rangle \subset V$, in which case $\xi^{-1}(V) = \partial_\infty\Gamma$. □

The following generalization of the fact that every irreducible projective Anosov representation admits a projective frame will be useful in Section 11.

Lemma 2.17. *Let $\rho_1 : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ and $\rho_2 : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ be representations with continuous equivariant limit maps ξ_1 and ξ_2 such that $\dim\langle\xi_1(\partial_\infty\Gamma)\rangle = \dim\langle\xi_2(\partial_\infty\Gamma)\rangle = p$. Then there exist $p+1$ distinct points $\{x_0, \ldots, x_p\}$ in $\partial_\infty\Gamma$ such that*

$$\{\xi_1(x_0), \ldots, \xi_1(x_p)\} \text{ and } \{\xi_2(x_0), \ldots, \xi_2(x_p)\}$$

are projective frames of $\langle\xi_1(\partial_\infty\Gamma)\rangle$ and $\langle\xi_2(\partial_\infty\Gamma)\rangle$ respectively.

Proof. We first proceed by iteration to produce $\{x_1, \ldots, x_p\}$ so that $\{\xi_1(x_1), \ldots, \xi_1(x_p)\}$ and $\{\xi_2(x_1), \ldots, \xi_2(x_p)\}$ generate

$$V = \langle\xi_1(\partial_\infty\Gamma)\rangle \quad \text{and} \quad W = \langle\xi_2(\partial_\infty\Gamma)\rangle.$$

Assume we have found $\{x_1, \ldots, x_k\}$ so that $\{\xi_1(x_1), \ldots, \xi_1(x_k)\}$ and $\{\xi_2(x_1), \ldots, \xi_2(x_k)\}$ are both linearly independent. Define

$$V_k = \langle\{\xi_1(x_1), \ldots, \xi_1(x_k)\}\rangle \text{ and } W_k = \langle\{\xi_2(x_1), \ldots, \xi_2(x_k)\}\rangle.$$

By the previous lemma, if $k < p$, then $\xi_1^{-1}(V_k)$ and $\xi_2^{-1}(W_k)$ have empty interior, so their complements must intersect. Pick

$$x_{k+1} \in \xi_1^{-1}(V_k)^c \cap \xi_2^{-1}(W_k)^c.$$

This process is complete when $k = p$.

It remains to find x_0. For each $i = 1, \ldots, p$, let

$$U_i^1 = \langle \{\xi_1(x_1), \ldots, \xi_1(x_p)\} \setminus \{\xi_1(x_i)\} \rangle$$

and

$$U_i^2 = \langle \{\xi_2(x_1), \ldots, \xi_2(x_p)\} \setminus \{\xi_2(x_i)\} \rangle .$$

Then, choose

$$x_0 \in \bigcap_i \xi_1^{-1}(U_i^1)^c \cap \xi_2^{-1}(U_i^2)^c.$$

One easily sees that $\{x_0, \ldots, x_p\}$ has the claimed properties. □

If $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ is projective Anosov and irreducible, then $\langle \xi(\partial_\infty \Gamma) \rangle = \mathbb{R}^m$ (since $\langle \xi(\partial_\infty \Gamma) \rangle$ is $\rho(\Gamma)$-invariant), so Lemma 2.17 immediately gives:

Lemma 2.18. *If $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ is an irreducible projective Anosov representation with limit maps ξ and θ, then then there exist $\{x_0, \ldots, x_m\} \subset \partial_\infty \Gamma$ so that $\{\xi(x_0), \ldots, \xi(x_m)\}$ is a projective frame for $\mathbb{RP}(m)$.*

We will also need the following lemma which was explained to us by J.-F. Quint.

Lemma 2.19. [Quint] *If Δ is an irreducible subgroup of $\mathsf{SL}_m(\mathbb{R})$ that contains a proximal element, then the Zariski closure G of Δ is a semi-simple Lie group without compact factors whose center $Z(\mathsf{G}) \subset \{\pm I\}$.*

Proof. Since G acts irreducibly on $\mathbb{R}^m$, it is a reductive group. Moreover, since G contains a proximal matrix, one easily sees that attracting lines of proximal matrices in G span $\mathbb{R}^m$, and that each attracting line of a proximal matrix in G is invariant under $Z(\mathsf{G})$. Therefore, $Z(\mathsf{G}) \subset \{\pm I\}$, so G is a semi-simple Lie group.

Let K be the maximal normal connected compact subgroup of G, and let H be the product of the non-compact Zariski connected, simple factors of G. Then H and K commute and HK has finite index in G.

Consider now a proximal element $g \in \mathsf{G}$. Replacing g by a large enough power, we can assume that $g = hk$ for some $h \in \mathsf{H}$ and $k \in \mathsf{K}$. Since all eigenvalues of k have modulus 1 and k and h commute, we conclude that h is proximal. So we can assume that $g \in \mathsf{H}$.

Since g and K commute, the attracting line of g is fixed by K, and, since K is connected, each vector of this attracting line is fixed by K. Let W be the vector space of K-fixed vectors on $\mathbb{R}^m$, then W is G-invariant , since K is normal in G, and nonzero. Since G is irreducible, $W = \mathbb{R}^m$ and so $\mathsf{K} = \{I\}$. □

Proposition 2.6 and Lemma 2.19 together have the following immediate consequence.

Corollary 2.20. *Let $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ be an irreducible projective Anosov representation, then the Zariski closure G_ρ of $\rho(\Gamma)$ is a semi-simple Lie group without compact factors such that $Z(\mathsf{G}_\rho) \subset \{\pm I\}$.*

2.5. **G-generic representations.** Let G be a reductive subgroup of $\mathsf{SL}_m(\mathbb{R})$. We recall that an element in G is *generic* if its centralizer is a maximal torus in G. We say that a representation $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ of Γ is G-*generic* if $\rho(\Gamma) \subset \mathsf{G}$ and the Zariski closure $\overline{\rho(\Gamma)}^Z$ of $\rho(\Gamma)$ contains a G-generic element.

We will need the following observation.

Lemma 2.21. *If G is a reductive subgroup of $\mathsf{SL}_m(\mathbb{R})$ and $\rho : \Gamma \to \mathsf{G}$ is a G-generic representation, then there exists $\gamma \in \Gamma$ such that $\rho(\gamma)$ is a generic element of G.*

Proof. We first note that the set of non-generic elements of G is Zariski closed in G, so the set of generic elements is Zariski open in G. Therefore, if the Zariski closure of $\rho(\Gamma)$ contains generic elements of G, then $\rho(\Gamma)$ must itself contain generic elements of G. □

3. Thermodynamic formalism

In this section, we recall facts from the thermodynamic formalism, as developed by developed by Bowen [12, 13], Parry–Pollicott [58], Ruelle [64] and others, which we will need in our work. In section 3.5, we will describe a variation of a construction of McMullen [56], which produces a pressure form on the space of pressure zero functions on a flow space. Our pressure metric will be a pull-back of this form.

3.1. **Hölder flows on compact spaces.** Let X be a compact metric space with a Hölder continuous flow $\phi = \{\phi_t\}_{t\in\mathbb{R}}$ without fixed points.

3.1.1. *Flows and parametrisations.* Let $f : X \to \mathbb{R}$ be a positive Hölder continuous function. Then, since X is compact, f has a positive minimum and for every $x \in X$, the function $\kappa_f : X \times \mathbb{R} \to \mathbb{R}$, defined by $\kappa_f(x,t) = \int_0^t f(\phi_s x)ds$, is an increasing homeomorphism of $\mathbb{R}$. We then have a map $\alpha_f : X \times \mathbb{R} \to \mathbb{R}$ that verifies

$$\alpha_f(x, \kappa_f(x,t)) = \kappa_f(x, \alpha_f(x,t)) = t, \tag{6}$$

for every $(x,t) \in X \times \mathbb{R}$.

The *reparametrization* of ϕ by f, is the flow $\phi^f = \{\phi_t^f\}_{t\in\mathbb{R}}$ on X, defined by $\phi_t^f(x) = \phi_{\alpha_f(x,t)}(x)$, for all $t \in \mathbb{R}$ and $x \in X$.

3.1.2. *Livšic-cohomology classes.* Two Hölder functions $f, g : X \to \mathbb{R}$ are *Livšic-cohomologous* if there exists $V : X \to \mathbb{R}$ of class C^1 in the flow's direction such that

$$f(x) - g(x) = \left.\frac{\partial}{\partial t}\right|_{t=0} V(\phi_t(x)).$$

Then one easily notices that:

(1) If f and g are Livšic cohomologous then they have the same integral over any ϕ-invariant measure, and
(2) If f and g are both positive and Livšic cohomologous, then the flows ϕ^f and ϕ^g are Hölder conjugate.

3.1.3. *Periods and measures.* Let O be the set of periodic orbits of ϕ. If $a \in O$ then its *period* as a $\{\phi_t^f\}$ periodic orbit is

$$\int_0^{p(a)} f(\phi_s(x))\mathrm{d}s$$

where $p(a)$ is the period of a for ϕ and $x \in a$. In particular, if $\widehat{\delta}_a$ is the probability measure invariant by the flow and supported by the orbit a, and if

$$\widehat{\delta}_a = \frac{\delta_a}{\langle \delta_a | 1 \rangle},$$

then

$$\langle \delta_a | f \rangle = \int_0^{p(a)} f(\phi_s(x))\mathrm{d}s \quad \text{and} \quad p(a) = \langle \delta_a | 1 \rangle\,.$$

In general, if μ is a ϕ-invariant measure on X and $f : X \to \mathbb{R}$ is a Hölder function, we will use the notation

$$\langle \mu | f \rangle = \int_X f \mathrm{d}\mu.$$

Let μ be a ϕ-invariant probability measure on X and let ϕ^f be the reparametrization of ϕ by f. We define $\widehat{f.\mu}$ by

$$\widehat{f.\mu} = \frac{1}{\langle \mu | f \rangle} f.\mu.$$

The map $\mu \mapsto \widehat{f.\mu}$ induces a bijection between ϕ-invariant probability measures and ϕ^f-invariant probability measures. If $\widehat{\delta}_a^f$ is the unique ϕ_f invariant probability measure supported by a, then $\widehat{\delta}_a^f = \widehat{f.\delta_a}$. In particular, we have

$$\langle \widehat{\delta}_a^f | g \rangle = \frac{\langle \delta_a | f.g \rangle}{\langle \delta_a | f \rangle} \tag{7}$$

3.1.4. *Entropy, pressure and equilibrium states.* If μ is a ϕ-invariant probability measure on X, then we denote by $h(\phi, \mu)$, its metric entropy. The Abramov formula [1] relates the metric entropies of a flow and its reparameterization:

$$h(\phi^f, \widehat{f.\mu}) = \frac{1}{\int f \,\mathrm{d}\mu} h(\phi, \mu). \tag{8}$$

Let $\mathcal{M}^\phi$ denote the set of ϕ-invariant probability measures. The *pressure* of a function $f : X \to \mathbb{R}$ is defined by

$$\mathbf{P}(\phi, f) = \sup_{m \in \mathcal{M}^\phi} \left(h(\phi, m) + \int_X f \,\mathrm{d}m \right). \tag{9}$$

In particular,

$$h_{\mathrm{top}}(\phi) = \mathbf{P}(\phi, 0)$$

is the *topological entropy* of the flow ϕ.

A measure $m \in \mathcal{M}^\phi$ on X such that

$$\mathbf{P}(\phi, f) = h(\phi, m) + \int_X f \mathrm{d}m,$$

is called an *equilibrium state* of f.

An equilibrium state for the function $f \equiv 0$ is called a *measure of maximal entropy.*

Remark: The pressure $\mathbf{P}(\phi, f)$ only depends on the Livšic cohomology class of f.

The following lemma from Sambarino [66] is a consequence of the definition and the Abramov formula.

Lemma 3.1. (Sambarino [66, Lemma 2.4]) *If ϕ is a Hölder continuous flow on a compact metric space X and $f : X \to \mathbb{R}$ is a positive Hölder continuous function, then*

$$P(\phi, -hf) = 0$$

if and only if $h = h_{\mathrm{top}}(\phi^f)$.

Moreover, if $h = h_{\mathrm{top}}(\phi^f)$ and m is an equilibrium state of $-hf$, then $\widehat{f.m}$ is a measure of maximal entropy for the reparameterized flow ϕ^f.

3.2. **Metric Anosov flows.** We shall assume from now on that the flow $\{\phi_t\}_{t\in\mathbb{R}}$ is a topologically transitive metric Anosov flow on X.

We recall that a flow $\{\phi\}_{t\in\mathbb{R}}$ on a metric space X is *topologically transitive* if given any two open sets U and V in X, there exists $t \in \mathbb{R}$ so that $\phi_t(U) \cap V$ is non-empty.

Let X be metric space. Let $\mathcal{L}$ be an equivalence relation on X. We denote by $\mathcal{L}_x$ the equivalence class of x and call it the *leaf* through x, so that we have a partition of X into leaves

$$X = \bigsqcup_{y\in Y} \mathcal{L}_y,$$

where Y is the collection of equivalence classes of $\mathcal{L}$. Such a partition is a *lamination* if we can find, for every x in X, an open neighbourhood O_x of x, two topological spaces U and K, a homeomorphism $\nu_x = (\nu_x^1, \nu_x^2)$ called a *chart* from O_x to $U \times K$ satisfying the following conditions

- for all $z, w \in O_x \cap O_y$,
$$\nu_x^1(w) = \nu_x^1(z) \iff \nu_y^1(w) = \nu_y^1(z),$$
- we have that $w \; \mathcal{L} \; z$ if and only if there exists a sequence w_i, $i \in \{1, \dots n\}$ with $w_1 = w$ and $w_n = z$, such that $w_{i+1} \in O_{w_i}$ and $\nu_{w_i}^1(w_i) = \nu_{w_i}^1(w_{i+1})$.

A *plaque open set* in the chart corresponding to ν is a set of the form $\nu\,(O \times \{z_0\})$ where $x = \nu(y_0, z_0)$ and O is an open set in U containing y_0. The *plaque topology* on $\mathcal{L}_x$ is the topology generated by the plaque open sets. A *plaque neighborhood* of x is a neighborhood for the plaque topology on $\mathcal{L}_x$.

We say that two laminations $\mathcal{L}$ and $\mathcal{L}'$ define a *local product structure*, if for any point x in X there exist plaque neighborhoods U and U' of x in $\mathcal{L}$ and $\mathcal{L}'$ respectively, and a map $\nu : U \times U' \to X$, which is an homeomorphism onto an open set of X, such that ν is both a chart for $\mathcal{L}$ and for $\mathcal{L}'$.

Assume now we have a flow $\{\phi_t\}_{t\in\mathbb{R}}$ on X. If $\mathcal{L}$ is a lamination invariant by $\{\phi_t\}$, we say that $\mathcal{L}$ is *transverse to the flow*, if for every x in X, there exists a plaque neighborhood U of x in $\mathcal{L}_x$, a topological space K, $\epsilon > 0$, and a chart

$$\nu : U \times K \times (-\epsilon, \epsilon) \to X,$$

such that

$$\phi_t(\nu(u, k, s)) = \nu(u, k, s + t).$$

If $\mathcal{L}$ is tranverse to the flow, we define a new lamination, called the *central lamination* with respect to $\mathcal{L}$, denoted by $\mathcal{L}^c$, by letting $x\,\mathcal{L}^c\,y$ if and only if there exists s such that $\phi_s(x)\,\mathcal{L}\,y$.

Finally, a $\{\phi_t\}$ invariant lamination $\mathcal{L}$ is *contracted by the flow*, if there exists $t_0 > 0$ such that for all $x \in X$, there exists a chart $\nu_x : U \times K \to V$ of an open neighborhood V of x, such that if

$$z = \nu_x(u, k), \quad \text{and} \quad y = \nu_x(v, k),$$

then for all $t > t_0$

$$d(\phi_t(z), \phi_t(y)) < \frac{1}{2} d(z, y).$$

Definition 3.2. [Metric Anosov flow] *A flow $\{\phi_t\}_{t\in\mathbb{R}}$ on a compact metric space X is* metric Anosov, *if there exist two laminations, $\mathcal{L}^+$ and $\mathcal{L}^-$, transverse to the flow, such that*

1. *$(\mathcal{L}^+, \mathcal{L}^{-,c})$ defines a local product structure,*
2. *$(\mathcal{L}^-, \mathcal{L}^{+,c})$ defines a local product structure,*
3. *$\mathcal{L}^+$ is contracted by the flow, and*
4. *$\mathcal{L}^-$ is contracted by the inverse flow.*

Then $\mathcal{L}^+$, $\mathcal{L}^-$, $\mathcal{L}^{+,c}$, $\mathcal{L}^{-,c}$ are respectively called the stable, unstable, central stable *and* central unstable *laminations.*

Remark: In the language of Pollicott [60], a metric Anosov flow is a Smale flow: the local product structure of $(\mathcal{L}^+, \mathcal{L}^{-,c})$ is what he calls the map

$$\langle \cdot, \cdot \rangle : \{(x, y) \in X \times X : d(x, y) < \varepsilon\} \to X.$$

3.2.1. *Livšic's Theorem.* Livšic [52] shows that the Livšic cohomology class of a Hölder function $f : X \to \mathbb{R}$ is determined by its periods:

Theorem 3.3. *Let $f : X \to \mathbb{R}$ be a Hölder continuous function, then $\langle \delta_a | f \rangle = 0$ for every $a \in O$ if and only if f is Livšic cohomologous to zero.*

3.2.2. *Coding.* We shall say that the triple (Σ, π, r) is a *Markov coding* for ϕ if Σ is an irreducible two-sided subshift of finite type, the maps $\pi : \Sigma \to X$ and $r : \Sigma \to \mathbb{R}_+^*$ are Hölder-continuous and verify the following conditions: Let $\sigma : \Sigma \to \Sigma$ be the shift, and let $\hat{r} : \Sigma \times \mathbb{R} \to \Sigma \times \mathbb{R}$ be the homeomorphism defined by

$$\hat{r}(x, t) = (\sigma x, t - r(x)),$$

then

- i) the map $\Pi : \Sigma \times \mathbb{R} \to X$ defined by $\Pi(x, t) = \phi_t(\pi(x))$ is surjective and $\hat{r}$-invariant,
- ii) consider the suspension flow $\sigma^r = \{\sigma_t^r\}_{t\in\mathbb{R}}$ on $(\Sigma \times \mathbb{R})/\hat{r}$, then the induced map $\Pi : (\Sigma \times \mathbb{R})/\hat{r} \to X$ is bounded-to-one and, injective on a residual set which is of full measure for every ergodic invariant measure of total support of σ^r.

Remark: If a flow ϕ admits a Markov coding, then every reparametrization ϕ^f of ϕ also admits a Markov coding, simply by changing the roof function r.

We recall, see Remark 3.2, that a metric Anosov flow is a Smale flow. One then has the following theorem of Bowen [12, 13] and Pollicott [60].

Theorem 3.4. *A topologically transitive metric Anosov flow on a compact metric space admits a Markov coding.*

3.3. **Entropy and pressure for Anosov flows.** The thermodynamic formalism of suspensions of subshifts of finite type extends thus to topologically transitive metric Anosov flows. For a positive Hölder function $f : X \to \mathbb{R}_+$ and $T \in \mathbb{R}$, we define

$$R_T(f) = \{a \in O \mid \langle \delta_a | f \rangle \leqslant T\}.$$

Observe that $R_T(f)$ only depends on the cohomology class of f.

3.3.1. *Entropy.* For a topologically transitive metric Anosov flow Bowen [12] (see also Pollicott [60]) showed:

Proposition 3.5. *The topological entropy of a topologically transitive metric Anosov flow $\phi = \{\phi_t\}_{t\in\mathbb{R}}$ on a compact metric space X is finite and positive. Moreover,*

$$h_{\text{top}}(\phi) = \lim_{T\to\infty} \frac{1}{T} \log \sharp\ \{a \in O \mid p(a) \leqslant T\}.$$

In particular, for a nowhere vanishing Hölder continuous function f,

$$h_f = \lim_{T\to\infty} \frac{1}{T} \log \sharp\,(R_T(f)) = h_{\text{top}}(\phi^f)$$

is finite and positive.

3.3.2. *Pressure.* The Markov coding may be used to show the pressure of a Hölder function $g : X \to \mathbb{R}$ is finite and that there is a unique equilibrium state of g. We shall denote this equilibrium state as m_g.

Theorem 3.6. [Bowen–Ruelle [15], Pollicott [60]] *Let $\phi = \{\phi_t\}_{t\in\mathbb{R}}$ be a topologically transitive metric Anosov flow on a compact metric space X and let $g : X \to \mathbb{R}$ be a Hölder function, then there exists a unique equilibrium state m_g for g. Moreover, if $f : X \to \mathbb{R}$ is a Hölder function such that $m_f = m_g$, then $f - g$ is Livšic cohomologous to a constant.*

The pressure function has the following alternative formulation in this setting (see Bowen–Ruelle [15]):

$$\mathbf{P}(\phi, g) = \lim_{T\to\infty} \frac{1}{T} \log \left(\sum_{a\in R_T(1)} e^{\langle \delta_a | g \rangle} \right). \tag{10}$$

3.3.3. *Measure of maximal entropy.* We have the following equidistribution result of Bowen [12] (see also Pollicott [60]).

Theorem 3.7. *A topologically transitive metric Anosov flow $\phi = \{\phi_t\}_{t\in\mathbb{R}}$ on a compact metric space X has a unique probability measure μ_ϕ of maximal entropy. Moreover,*

$$\mu_\phi = \lim_{T\to\infty} \left(\frac{1}{\sharp R_T(1)} \sum_{a\in R_T(1)} \widehat{\delta}_a \right). \tag{11}$$

The probability measure of maximal entropy for ϕ is called the *Bowen–Margulis measure* of ϕ.

3.4. **Intersection and renormalized intersection.**

3.4.1. *Intersection.* Let $\phi = \{\phi_t\}_{t\in\mathbb{R}}$ be a topologically transitive metric Anosov flow on a compact metric space X. Consider a positive Hölder function $f : X \to \mathbb{R}_+$ and a continuous function $g : X \to \mathbb{R}$. We define the *intersection of f and g as*

$$\mathbf{I}(f,g) = \int \frac{g}{f} \mathrm{d}\mu_{\phi^f},$$

where μ_{ϕ^f} is the Bowen–Margulis measure of the flow ϕ^f. We also have the following two alternative ways to define the intersection

$$\mathbf{I}(f,g) = \lim_{T\to\infty} \left(\frac{1}{\sharp R_T(f)} \sum_{a\in R_T(f)} \frac{\langle \delta_a | g\rangle}{\langle \delta_a | f\rangle} \right) \tag{12}$$

$$\mathbf{I}(f,g) = \frac{\int g \,\mathrm{d}m_{-h_f f}}{\int f \,\mathrm{d}m_{-h_f f}} \tag{13}$$

where h_f is the topological entropy of ϕ^f, and $m_{-h_f.f}$ is the equilibrium state of $-h_f.f$. The first equality follows from Theorem 3.7 and Equation (7), the second equality follows from the second part of Lemma 3.1.

Since $\langle \delta_a | f\rangle$ depends only on the Livšic cohomology class of f and $\langle \delta_a | g\rangle$ depends only on the Livšic cohomology class of g, the intersection $\mathbf{I}(f,g)$ depends only on the Livšic cohomology classes of f and g.

3.4.2. *A lower bound on the renormalized intersection.* For two positive Hölder functions $f, g : X \to \mathbb{R}_+$ define the *renormalized intersection* as

$$\mathbf{J}(f,g) = \frac{h_g}{h_f} \mathbf{I}(f,g),$$

where h_f and h_g are the topological entropies of ϕ^f and ϕ^g. Uniqueness of equilibrium states together with the definition of the pressure imply the following proposition.

Proposition 3.8. *If $\phi = \{\phi_t\}_{t\in\mathbb{R}}$ is a topologically transitive metric Anosov flow on a compact metric space X, and $f : X \to \mathbb{R}_+$ and $g : X \to \mathbb{R}_+$ are positive Hölder functions, then*

$$\mathbf{J}(f,g) \geqslant 1.$$

Moreover, $\mathbf{J}(f,g) = 1$ if and only if $h_f f$ and $h_g g$ are Livšic cohomologous.

Proof. Since $\mathbf{P}(\phi, -h_g g) = 0$,

$$h_g \int g \,\mathrm{d}m \geqslant h(\phi, m)$$

for all $m \in \mathcal{M}^\phi$ and, by Theorem 3.6, equality holds only for $m = m_{-h_g g}$, the equilibrium state of $-h_g g$. Applying the analogous inequality for $m_{-h_f f}$, together with Abramov's formula (8) and Lemma 3.1, one sees that

$$h_g \int g \,\mathrm{d}m_{-h_f.f} \geqslant h(\phi, m_{-h_f.f}) = h_f \int f \,\mathrm{d}m_{-h_f.f},$$

which implies that $\mathbf{J}(f,g) \geqslant 1$.

If $\mathbf{J}(f,g)=1$, then $m_{-h_g g}=m_{-h_f f}$ and thus, applying theorem 3.6, one sees that $h_g g - h_f f$ is Livšic cohomologous to a constant c. Thus,

$$0=\mathbf{P}(\phi,-h_g g)=\mathbf{P}(\phi,-h_f f-c)=\mathbf{P}(\phi,-h_f f)-c=-c.$$

Therefore, $h_g g$ and $h_f f$ are Livšic cohomologous. □

3.5. **Variation of the pressure and the pressure form.** McMullen [56] introduced a pressure metric on the space of Livšic cohomology classes of pressure zero Hölder functions on a shift space Σ. In this section, we use his construction to produce a pressure form, and associated semi-norm, on the space of pressure zero Hölder functions on our flow space X.

3.5.1. *First and second derivatives.* For g a Hölder continuous function with mean zero (i.e. $\int g\, dm_f = 0$), we define the *variance of* g with respect to f as

$$\mathrm{Var}(g,m_f)=\lim_{T\to\infty}\frac{1}{T}\int\left(\int_0^T g(\phi_s(x))\,\mathrm{d}s\right)^2 \mathrm{d}m_f(x),$$

where m_f is the equilibrium state of f. We will see that the variance is well defined. Similarly, for two mean zero Hölder continuous functions g and h, we define the *covariance* of g and h with respect to f as

$$\mathrm{Cov}(g,h,m_f)=\lim_{T\to\infty}\frac{1}{T}\int\left(\int_0^T g(\phi_u(x))\,\mathrm{d}u\right)\left(\int_0^T h(\phi_s(x))\,\mathrm{d}s\right)\mathrm{d}m_f(x).$$

We shall omit the background flow in the notation of the pressure function and simply write

$$\mathbf{P}(\cdot)=\mathbf{P}(\phi,\cdot).$$

Proposition 3.9. (Parry-Pollicott [58, Prop. 4.10,4.11], Ruelle [64]) *Suppose that $\phi=\{\phi_t\}_{t\in\mathbb{R}}$ is a topologically transitive metric Anosov flow on a compact metric space X, and $f:X\to\mathbb{R}$ and $g:X\to\mathbb{R}$ are Hölder functions. If m_f is the equilibrium state of f, then*

1. *The function $t\mapsto\mathbf{P}(f+tg)$ is analytic,*
2. *The first derivative is given by*

$$\left.\frac{\partial\mathbf{P}(f+tg)}{\partial t}\right|_{t=0}=\int g\,\mathrm{d}m_f,$$

3. *If $\int g\,\mathrm{d}m_f=0$ then*

$$\left.\frac{\partial^2\mathbf{P}(f+tg)}{\partial t^2}\right|_{t=0}=\mathrm{Var}(g,m_f),$$

4. *If $\mathrm{Var}(g,m_f)=0$ then g is Livšic cohomologous to zero.*

As a corollary, the variance is well defined.

3.5.2. *The pressure form.* Let $C^h(X)$ be the set of real valued Hölder continuous functions on X. Define $\mathcal{P}(X)$ to be the set of pressure zero Hölder functions on X, i.e.

$$\mathcal{P}(X) = \left\{\Phi \in C^h(X) : \mathbf{P}(\Phi) = 0\right\}.$$

The tangent space of $\mathcal{P}(X)$ at Φ is the set

$$\mathsf{T}_\Phi \mathcal{P}(X) = \ker \mathrm{d}_\Phi \mathbf{P} = \left\{ g \in C^h(X) \mid \int g \, \mathrm{d}m_\Phi = 0 \right\}$$

where m_Φ is the equilibrium state of Φ. Define the *pressure semi-norm* of $g \in \mathsf{T}_\Phi \mathcal{P}(X)$ as

$$\|g\|^2_{\mathbf{P}} = -\frac{\mathrm{Var}(g, m_\Phi)}{\int \Phi \, \mathrm{d}m_\Phi}.$$

One has the following computation.

Lemma 3.10. *Let $\phi = \{\phi_t\}_{t\in\mathbb{R}}$ be a topologically transitive metric Anosov flow on a compact metric space X. If $\{\Phi_t\}_{t\in(-1,1)}$ is a smooth one parameter family contained in $\mathcal{P}(X)$, then*

$$\|\dot\Phi_0\|^2_{\mathbf{P}} = \frac{\int \ddot\Phi_0 \, \mathrm{d}m_{\Phi_0}}{\int \Phi_0 \, \mathrm{d}m_{\Phi_0}}.$$

Proof. As $\mathbf{P}(\Phi_t) = 0$ by differentiating twice we get the equation

$$\mathrm{D}^2\mathbf{P}(\Phi_0)(\dot\Phi_0, \dot\Phi_0) + \mathrm{D}\mathbf{P}(\Phi_0)(\ddot\Phi_0) = 0 = \mathrm{Var}(\dot\Phi_0, m_{\Phi_0}) + \int \ddot\Phi_0 \mathrm{d}m_{\Phi_0}.$$

Thus

$$\|\dot\Phi_0\|^2_{\mathbf{P}} = -\frac{\mathrm{Var}(\dot\Phi_0, m_\Phi)}{\int \Phi_0 \, \mathrm{d}m_{\Phi_0}} = \frac{\int \ddot\Phi_0 \, \mathrm{d}m_{\Phi_0}}{\int \Phi_0 \, \mathrm{d}m_{\Phi_0}}.$$

□

We then have the following relation, generalizing Bonahon [11], between the renormalized intersection and the pressure metric.

Proposition 3.11. *Let $\phi = \{\phi_t\}_{t\in\mathbb{R}}$ be a topologically transitive metric Anosov flow on a compact metric space X. If $\{f_t : X \to \mathbb{R}_+\}_{t\in(-1,1)}$ is a one-parameter family of positive Hölder functions and $\Phi_t = -h_{f_t} f_t$ for all $t \in (-1,1)$, then*

$$\left.\frac{\partial^2}{\partial t^2}\right|_{t=0} \mathbf{J}(f_0, f_t) = \|\dot\Phi_0\|^2_{\mathbf{P}}.$$

Proof. By Equation (13) and the definition of the renormalized intersection, we see that

$$\mathbf{J}(f_0, f_t) = \frac{\int \Phi_t \, \mathrm{d}m_{\Phi_0}}{\int \Phi_0 \, \mathrm{d}m_{\Phi_0}}.$$

Differentiating twice and applying the previous lemma, one obtains

$$\left.\frac{\partial^2}{\partial t^2}\right|_{t=0} \mathbf{J}(f_0, f_t) = \frac{\int \ddot\Phi_0 \, \mathrm{d}m_{\Phi_0}}{\int \Phi_0 \, \mathrm{d}m_{\Phi_0}} = \|\dot\Phi_0\|^2_{\mathbf{P}}$$

which completes the proof. □

So, the pressure semi-norm arises naturally from the *pressure form* p which is the symmetric 2-tensor on $\mathsf{T}_\Phi\mathcal{P}(X)$ given by the Hessian of $\mathbf{J}_\Phi = \mathbf{J}(\Phi,\cdot)$. One may compute that if $f,g \in \mathsf{T}_\Phi\mathcal{P}(X)$, then

$$p(f,g) = -\frac{\mathrm{Cov}(f,g,m_\Phi)}{\int \Phi \,\mathrm{d}m_\Phi}.$$

3.6. **Analyticity of entropy, pressure and intersection.** We now show that pressure, entropy and intersection vary analytically for analytic families of positive Hölder functions.

Proposition 3.12. *Let $\phi = \{\phi_t\}_{t\in\mathbb{R}}$ be a topologically transitive metric Anosov flow on a compact metric space X. Let $\{f_u : X \to \mathbb{R}\}_{u\in D}$ and $\{g_v : X \to \mathbb{R}\}_{v\in D}$ be two analytic families of Hölder functions. Then the function*

$$u \mapsto \mathbf{P}(f_u)$$

is analytic. Moreover, if the family $\{f_u\}_{u\in D}$ consists of positive functions then the functions

$$u \mapsto h_u = h_{f_u}, \tag{14}$$
$$(u,v) \mapsto \mathbf{I}(f_u, g_v). \tag{15}$$

are both analytic.

Proof. Since the pressure function is analytic on the space of Hölder functions (see Parry-Pollicott [58, Prop. 4.7] or Ruelle [64, Cor. 5.27]) the function $u \mapsto \mathbf{P}(f_u)$ is analytic.

Since the family $\{f_u\}_{u\in D}$ consists of positive functions, Proposition 3.9 implies that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=h_u} \mathbf{P}(-tf_u) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=h_u} \mathbf{P}(-h_uf_u - (t-h_u)f_u) = -\int f_u \mathrm{d}m_{-h_uf_u} < 0.$$

Thus an application of the Implicit Function Theorem yields that $u \mapsto h_u$ is analytic.

We also get that

$$(u,v,t) \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbf{P}(-h_uf_u + tg_v),$$

is analytic. But, applying Proposition 3.9 again,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbf{P}(-h_uf_u + tg_v) = \int g_v \,\mathrm{d}m_{-h_uf_u}.$$

Thus the function $(u,v) \mapsto \int g_v \,\mathrm{d}m_{-h_uf_u}$ is analytic. Similarly (taking $g_v = f_u$), the function $u \mapsto \int f_u \,\mathrm{d}m_{-h_uf_u}$ is analytic. Thus, we get, by Equation (13) that

$$(u,v) \mapsto \mathbf{I}(f_u,g_v) = \frac{\int g_v \mathrm{d}m_{-h_uf_u}}{\int f_u \mathrm{d}m_{-h_uf_u}},$$

is analytic.

□

4. The geodesic flow of a projective Anosov representation

In this section, we define a flow $(\mathsf{U}_\rho\Gamma, \{\phi_t\}_{t\in\mathbb{R}})$ associated to a projective Anosov representation $\rho:\Gamma\to\mathsf{SL}_m(\mathbb{R})$. We will show that $\mathsf{U}_\rho\Gamma$ is a Hölder reparameterization of the geodesic flow $\mathsf{U}_0\Gamma$ of the domain group Γ, so it will make sense to refer to $\mathsf{U}_\rho\Gamma$ as the *geodesic flow* of the representation.

Let F be the total space of the bundle over

$$\mathbb{RP}(m)^{(2)} = \mathbb{RP}(m)\times\mathbb{RP}(m)^* \setminus \{(U,V) \mid U\not\subset V\},$$

whose fiber at the point (U,V) is the space

$$\mathsf{M}(U,V) = \{(u,v) \mid u\in U,\ v\in V,\ \langle v|u\rangle = 1\}/\sim,$$

where $(u,v)\sim(-u,-v)$ and $\mathbb{RP}(m)^*$ is identified with the projective space of the dual space $(\mathbb{R}^m)^*$. Notice that u determines v, so that F is an $\mathbb{R}$-bundle. One may also identify $\mathsf{M}(U,V)$ with the space of norms on U.

Then F is equipped with a natural $\mathbb{R}$-action, given by

$$\phi_t(U,V,(u,v)) = (U,V,(e^tu,e^{-t}v)).$$

If $\rho:\Gamma\to\mathsf{SL}_m(\mathbb{R})$ is a projective Anosov representation and ξ and θ are the associated limit maps, we consider the associated pullback bundle

$$F_\rho = (\xi,\theta)^*F$$

over $\partial_\infty\Gamma^{(2)}$ which inherits an $\mathbb{R}$ action from the action on F. The action of Γ on $\partial_\infty\Gamma^{(2)}$ extends to an action on F_ρ. If we let

$$\mathsf{U}_\rho\Gamma = F_\rho/\Gamma,$$

then the $\mathbb{R}$-action on F_ρ descends to a flow $\{\phi_t\}_{t\in\mathbb{R}}$ on $\mathsf{U}_\rho\Gamma$, which we call the *geodesic flow* of the representation.

Proposition 4.1. [The geodesic flow] *If $\rho:\Gamma\to\mathsf{SL}_m(\mathbb{R})$ is a projective Anosov representation, then the action of Γ on F_ρ is proper and cocompact. Moreover, the flow $\{\phi_t\}_{t\in\mathbb{R}}$ on $\mathsf{U}_\rho\Gamma$ is Hölder conjugate to a Hölder reparameterization of the Gromov geodesic flow on $\mathsf{U}_0\Gamma$ and the orbit associated to $[\gamma]$, for any infinite order primitive element $\gamma\in\Gamma$, has period $\Lambda(\rho)(\gamma)$.*

We produce a Γ-invariant Hölder orbit equivalence between $\widetilde{\mathsf{U}_0\Gamma}$ and F_ρ which is a homeomorphism. Recall that $\widetilde{\mathsf{U}_0\Gamma} = \partial_\infty\Gamma^{(2)}\times\mathbb{R}$ and that $\widetilde{\mathsf{U}_0\Gamma}/\Gamma = \mathsf{U}_0\Gamma$. Since the action of Γ on $\widetilde{\mathsf{U}_0\Gamma}$ is proper and cocompact, it follows immediately that $\mathsf{U}_\rho\Gamma$ is Hölder conjugate to a Hölder reparameterization of the Gromov geodesic flow on $\mathsf{U}_0\Gamma$.

Proposition 4.2. *If $\rho:\Gamma\to\mathsf{SL}_m(\mathbb{R})$ is a projective Anosov representation, there exists a Γ-equivariant Hölder orbit equivalence*

$$\tilde\nu:\widetilde{\mathsf{U}_0\Gamma}\to F_\rho$$

which is a homeomorphism.

Let E_ρ be the flat bundle associated to ρ on $\mathsf{U}_0\Gamma$. Recall that E_ρ splits as

$$E_\rho = \Xi\oplus\Theta.$$

Let $\{\psi_t\}_{t\in\mathbb{R}}$ be the lift of the geodesic flow on $\mathsf{U}_0\Gamma$ to a flow on E_ρ. We first observe that we may produce a Hölder metric on the bundle Ξ which is contracting on all scales.

Lemma 4.3. *There exists a Hölder metric* τ^0 *on the bundle* Ξ *and* $\beta > 0$ *such that for all* $t > 0$ *we have,*

$$\psi_t^*(\tau^0) < e^{-\beta t}\tau^0.$$

Proof. Let τ be any Hölder metric on Ξ. Since ρ is projective Anosov, Lemma 2.4 implies that there exists $t_0 > 0$ such that

$$\psi_{t_0}^*(\tau) \leqslant \frac{1}{4}\tau.$$

Choose $\beta > 0$ so that $2 < e^{\beta t_0} < 4$ and, for all s, let $\tau_s = \psi_s^*(\tau)$. Let

$$\tau^0 = \int_0^{t_0} e^{\beta s}\tau_s \, \mathrm{d}s.$$

Notice that τ^0 has the same regularity as τ. If $t > 0$, then

$$\begin{aligned} \psi_t^*(\tau^0) &= \int_0^{t_0} e^{\beta s}\tau_{t+s} \, \mathrm{d}s \\ &= e^{-\beta t}\int_t^{t+t_0} e^{\beta u}\tau_u \, \mathrm{d}u. \end{aligned} \tag{16}$$

Now observe that

$$\begin{aligned} \int_t^{t+t_0} e^{\beta u}\tau_u \, \mathrm{d}u &= \tau^0 + \int_{t_0}^{t+t_0} e^{\beta u}\tau_u \mathrm{d}u - \int_0^t e^{\beta u}\tau_u \, \mathrm{d}u \\ &= \tau^0 + \int_0^t e^{\beta u}\psi_u^*\left(e^{\beta t_0}\psi_{t_0}^*(\tau) - \tau\right) \mathrm{d}u. \end{aligned} \tag{17}$$

But

$$e^{\beta t_0}\psi_{t_0}^*(\tau) \leqslant \frac{e^{\beta t_0}}{4}\tau < \tau.$$

Thus

$$\int_t^{t+t_0} e^{\beta u}\tau_u \, \mathrm{d}u < \tau^0.$$

and the result follows from Inequality (16). □

Proof of Proposition 4.2 Let τ^0 be the metric provided by Lemma 4.3 and let β be the associated positive number. Let $\tilde{\Xi}$ denote the line bundle over $\partial_\infty\Gamma^{(2)} \times \mathbb{R}$ which is the lift of Ξ. Notice that τ^0 lifts to a Hölder metric $\tilde{\tau}^0$ on $\tilde{\Xi}$. Our Hölder orbit equivalence

$$\tilde{\nu} : \partial_\infty\Gamma^{(2)} \times \mathbb{R} \to F_\rho$$

will be given by

$$\tilde{\nu}(x, y, t) = (x, y, (u(x, y, t), v(x, y, t))),$$

where $\tilde{\tau}^0_{(x,y,t)}(u(x, y, t)) = 1$ and $\tilde{\tau}^0_{(x,y,t)}$ is the metric on the line $\xi(x)$ induced by the metric $\tilde{G}^0$ by regarding $\xi(x)$ as the fiber of $\tilde{\Xi}$ over the point (x, y, t). The fact that $\psi_t^*\tau^0 < \tau^0$ for all $t > 0$ implies that $\tilde{\nu}$ is injective. Since $\tilde{\tau}^0$ is Hölder and Γ-equivariant, $\tilde{\nu}$ is also Hölder and Γ-equivariant.

It remains to prove that $\tilde{\nu}$ is proper. We will argue by contradiction. If $\tilde{\nu}$ is not proper, then there exists a sequence $\{(x_n, y_n, t_n)\}_{n\in\mathbb{N}}$ leaving every compact subset of $\partial_\infty\Gamma^{(2)} \times \mathbb{R}$, such that $\{\tilde{\nu}(x_n, y_n, t_n)\}_{n\in\mathbb{N}}$ converges to $(x, y, (u, v))$ in F_ρ. Letting $\tilde{\nu}(x_n, y_n, t_n) = (x_n, y_n, (u_n, v_n))$, we see immediately that

$$\lim_{n\to\infty} x_n = x, \ \lim_{n\to\infty} y_n = y, \ \text{and} \ \lim_{n\to\infty} (u_n, v_n) = (u, v).$$

Writing $\tilde{\nu}(x_n, y_n, 0) = (x_n, y_n, (\hat{u}_n, \hat{v}_n))$ and $\tilde{\nu}(x, y, 0) = (x, y, (\hat{u}, \hat{v}))$, we obtain, by the continuity of the map $\tilde{\nu}$,

$$\lim_{n\to\infty} (\hat{u}_n, \hat{v}_n) = (\hat{u}, \hat{v}).$$

If $t > 0$, then

$$\frac{\tilde{\tau}^0_{(x,y,t)}}{\tilde{\tau}^0_{(x,y,0)}} = \frac{\psi^*_t\left(\tilde{\tau}^0_{(x,y,0)}\right)}{\tilde{\tau}^0_{(x,y,0)}} < e^{-\beta t}.$$

In particular,

$$\left|\frac{\langle v \mid u_n\rangle}{\langle v \mid \hat{u}_n\rangle}\right| < e^{-\beta t_n}. \tag{18}$$

Without loss of generality, either $t_n \to \infty$ or $t_n \to -\infty$. If $t_n \to \infty$, then by Inequality (18),

$$0 = \lim_{n\to\infty} \frac{\langle v \mid u_n\rangle}{\langle v \mid \hat{u}_n\rangle},$$

on the other hand,

$$\lim_{t\to\infty} \frac{\langle v \mid u_n\rangle}{\langle v \mid \hat{u}_n\rangle} = \frac{\langle v \mid u\rangle}{\langle v \mid \hat{u}\rangle} \neq 0.$$

We have thus obtained a contradiction. Symmetrically, if $t_n \to -\infty$, then

$$0 = \lim_{n\to\infty} \frac{\langle v \mid \hat{u}_n\rangle}{\langle v \mid u_n\rangle} = \frac{\langle v \mid \hat{u}\rangle}{\langle v \mid u\rangle} \neq 0,$$

which is again a contradiction.

The restriction of $\tilde{\nu}$ to each orbit $\{(x, y)\} \times \mathbb{R}$ is a proper, continuous, injection into the fiber of F_ρ over (x, y) (which is also homeomorphic to $\mathbb{R}$). It follows that the restriction of $\tilde{\nu}$ to each orbit is a homeomorphism onto the image fiber. We conclude that $\tilde{\nu}$ is surjective and hence a proper, continuous, bijection. Therefore, $\tilde{\nu}$ is a homeomorphism. This completes the proof of Proposition 4.2.

In order to complete the proof of Proposition 4.1, it only remains to compute the period of the orbit associated to $[\gamma]$ for an infinite order primitive element $\gamma \in \Gamma$. Since ρ is projective Anosov, Proposition 2.6 implies that $\rho(\gamma)$ is proximal, $\xi(\gamma^+)$ is the attracting line and $\theta(\gamma^-)$ is the repelling hyperplane. If $u \in \xi(\gamma^+)$ and $v \in \theta(\gamma^-)$ one sees that

$$\rho(\gamma)(u) = \mathsf{L}(\gamma)(\rho)\ u \text{ and } \rho(\gamma)(v) = \frac{1}{\mathsf{L}(\gamma)(\rho)}\ v.$$

Thus, $(\gamma^+, \gamma^-, (u, v))$ and

$$(\gamma^+, \gamma^-, \mathsf{L}(\gamma)(\rho)u, \frac{1}{\mathsf{L}(\gamma)(\rho)}v) = \phi_{\log(\Lambda(\gamma)(\rho))}(\gamma^+, \gamma^-, (u, v))$$

project to the same point on $\mathsf{U}_\rho\Gamma$. (Recall that

$$(\mathsf{L}(\gamma)(\rho)u, \frac{1}{\mathsf{L}(\gamma)(\rho)}v) \sim (-\mathsf{L}(\gamma)(\rho)u, \frac{-1}{\mathsf{L}(\gamma)(\rho)}v)$$

in $M(\xi(\gamma^+), \theta(\gamma^-))$.) Since γ is primitive, this finishes the proof. $\square$

5. The geodesic flow is a metric Anosov flow

In this section, we prove that the geodesic flow of a projective Anosov representation is a metric Anosov flow:

Proposition 5.1. [Anosov] *If $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ is a projective Anosov representation, then the geodesic flow $(\mathsf{U}_\rho\Gamma, \{\phi_t\}_{t\in\mathbb{R}})$ is a topologically transitive metric Anosov flow.*

The reader with a background in hyperbolic dynamics may be convinced by the following heuristic argument: essentially the splitting of an Anosov representation yields a section of some (product of) flag manifolds and the graph of this section should be thought as a Smale locally maximal hyperbolic set; then the result follows from the "fact" that the restriction of the flow on such a set is a metric Anosov flow. However, the above idea does not exactly work, and moreover it is not easy to extricate it from the existing literature in the present framework. Therefore, we give a detailed and *ad-hoc* construction, although the result should be true in a rather general setting.

The topological transitivity of $(\mathsf{U}_\rho\Gamma, \{\phi_t\}_{t\in\mathbb{R}})$ follows immediately from the topological transitivity of the action of Γ on $\partial_\infty\Gamma^2$. We define a metric on the geodesic flow in Section 5.1, introduce the stable and unstable leaves in Section 5.2, explain how to control the metric along the unstable leaves in Section 5.3 and finally proceed to the proof in Section 5.4. A more precise version of Proposition 5.1 is given by Proposition 5.7.

5.1. The geodesic flow as a metric space.

Recall that F is the total space of an $\mathbb{R}$-bundle over $\mathbb{RP}(m)^{(2)}$ whose fiber at the point (U, V) is the space

$$\mathsf{M}(U, V) = \{(u, v) \mid u \in U,\ v \in V,\ \langle v|u\rangle = 1\}/\sim .$$

Since $\mathbb{RP}(m)^{(2)} \subset \mathbb{RP}(m) \times \mathbb{RP}(m)^*$, any Euclidean metric on $\mathbb{R}^m$ gives rise to a metric on F which is a subset of

$$\mathbb{RP}(m) \times \mathbb{RP}(m)^* \times \left(\left(\mathbb{R}^m \times (\mathbb{R}^m)^*\right) / \pm 1\right).$$

The metric on F pulls back to a metric on F_ρ. A metric on F_ρ obtained by this procedure is called a *linear metric*. Any two linear metrics are bilipschitz equivalent.

The following lemma allows us to use a linear metric to study F_ρ.

Lemma 5.2. *There exists a Γ-invariant metric d_0 on F_ρ which is locally bilipschitz equivalent to any linear metric.*

The Γ-invariant metric d_0 descends to a metric on $\mathsf{U}_\rho\Gamma$ which we will also call d_0 and is defined for every x and y in F_ρ by

$$d_0(\pi(x), \pi(y)) = \inf_{\gamma\in\Gamma}(x, \gamma(y)),$$

where π is the projection $F_\rho \to \mathsf{U}_\rho\Gamma$.

Proof. We first notice that all linear metrics on F_ρ are bilipschitz to one another, so that it suffices to construct a metric which is locally bilipschitz to a fixed linear metric d.

Let V be an open subset of F_ρ with compact closure which contains a closed fundamental domain for the action of Γ on F_ρ. Since the action of Γ on F_ρ is proper, $\{V_\gamma = \gamma(V)\}_{\gamma\in\Gamma}$ is a locally finite cover of F_ρ. Let $\{d_\gamma = \gamma^* d\}_{\gamma\in\Gamma}$ be the associated family of metrics on F_ρ. Since each element of Γ acts as a bilipschitz automorphism with respect to any linear metric, any two metrics in the family $\{d_\gamma = \gamma^* d\}_{\gamma\in\Gamma}$ are bilipschitz equivalent.

We will use this cover and the associated family of metrics to construct a Γ-invariant metric on F_ρ. A *path* joining two points x and y in F_ρ is a pair of tuples

$$\mathcal{P} = ((z_0,\ldots,z_n),(\gamma_0,\ldots,\gamma_n)),$$

where $(z_0,\ldots,z_n)$ is an n-tuple of points in F_ρ and $(\gamma_0,\ldots,\gamma_n)$ is an n-tuple of elements of Γ such that

- $x = z_0 \in V_{\gamma_0}$ and $y = z_n \in V_{\gamma_n}$,
- for all $n > i > 0$, $z_i \in V_{\gamma_{i-1}} \cap V_{\gamma_i}$.

The *length* of a path is given by

$$\ell(\mathcal{P}) = \frac{1}{2}\left(\sum_{i=0}^{n-1} d_{\gamma_i}(z_i,z_{i+1}) + d_{\gamma_{i+1}}(z_i,z_{i+1})\right)$$

We then define

$$d_0(x,y) = \inf\{\ell(\mathcal{P}) \mid \mathcal{P} \text{ joins } x \text{ and } y\}.$$

It is clear that d_0 is a Γ-invariant pseudo metric. It remains to show that d_0 is a metric which is locally bilipschitz to d.

Let z be a point in F_ρ. Then there exists a neighborhood Z of z so that

$$A = \{\gamma \mid V_\gamma \cap Z \neq \emptyset\},$$

is a finite set. Choose $\alpha > 0$ so that

$$\bigcup_{\gamma\in A} \{x \mid d_\gamma(z,x) \leqslant \alpha\} \subset Z.$$

Let K be chosen so that if $\alpha,\beta \in A$, then d_α and d_β are K-bilipschitz. Finally, let

$$W = \bigcap_{\gamma\in A} \left\{x \mid d_\gamma(z,x) \leqslant \frac{\alpha}{10K}\right\}.$$

By construction, if x and y belong to W, then for all $\gamma \in A$,

$$d_\gamma(x,y) \leqslant \frac{\alpha}{5K}. \tag{19}$$

Let x be a point in W. Let $\mathcal{P} = ((z_0,\ldots,z_n),(\gamma_0,\ldots,\gamma_n))$ be a path joining x to a point y.

If there exists j such that $\gamma_j \notin A$, then

$$\begin{aligned}
\ell(\mathcal{P}) &\geqslant \frac{1}{2}\sum_{i=0}^{i=j-1} d_{\gamma_i}(z_{i-1},z_i) \\
&\geqslant \frac{1}{2K}\left(\sum_{i=0}^{i=j-1} d_{\gamma_{j-1}}(z_i,z_{i+1})\right) \\
&\geqslant \frac{1}{2K} d_{\gamma_{j-1}}(z_0,z_j) \geqslant \frac{1}{2K}\left(d_{\gamma_{j-1}}(z,z_j)) - d_{\gamma_{j-1}}(z_0,z)\right) \\
&\geqslant \frac{1}{2K}\left(\alpha - \frac{\alpha}{10K}\right) \geqslant \frac{\alpha}{5K}.
\end{aligned} \tag{20}$$

If $\gamma_j \in A$ for all j, then the triangle inequality and the definition of K immediately imply that for all $\gamma \in A$,

$$\ell(\mathcal{P}) \geqslant \frac{1}{K} d_\gamma(x,y). \tag{21}$$

Inequalities (20) and (21) imply that

$$d_0(x,y) \geqslant \frac{1}{K} \inf\left(\frac{\alpha}{5}, d_\gamma(x,y)\right) > 0, \tag{22}$$

hence d_0 is a metric. Moreover, if $x, y \in W$, then by inequalities (22) and (19),

$$d_0(x,y) \geqslant \frac{1}{K} d_\gamma(x,y). \tag{23}$$

By construction, and taking the path $\mathcal{P}_0 = ((x,y),(\gamma,\gamma))$ with γ in A, we also get

$$d_0(x,y) \leqslant \ell(\mathcal{P}_0) = d_\gamma(x,y). \tag{24}$$

As consequence of inequalities (23) and (24), d_0 is bilipschitz on W to any d_γ with $\gamma \in A$.

Since d is bilipschitz to d_γ for any $\gamma \in A$, we see that d_0 is bilipschitz to d on W.

Since z was arbitrary, it follows that d_0 is locally bilipschitz to d. □

5.2. **Stable and unstable leaves.** In this section, we define the stable and unstable laminations of the geodesic flow F_ρ. Let

$$Z = (x_0, y_0, (u_0, v_0))$$

be a point in F_ρ.

(1) The *unstable leaf through* Z is

$$\mathcal{L}_Z^- = \{(x, y_0, (u, v_0)) \mid x \in \partial_\infty\Gamma,\ u \in \xi(x),\ \langle v_0 | u\rangle = 1\}.$$

The *central unstable leaf through* Z is

$$\begin{aligned}\mathcal{L}_Z^{-,c} &= \{(x, y_0, (u,v)) \mid x \in \partial_\infty\Gamma, (u,v) \in \mathsf{M}(\xi(x), \theta(y_0))\} \\ &= \bigcup_{t\in\mathbb{R}} \phi_t(\mathcal{L}_Z+).\end{aligned}$$

(2) The *stable leaf through* Z is

$$\mathcal{L}_Z^+ = \{(x_0, y, (u_0, v)) \mid y \in \partial_\infty\Gamma,\ v \in \theta(y),\ \langle v | u_0\rangle = 1\}.$$

The *central stable leaf through* Z is

$$\begin{aligned}\mathcal{L}_Z^{+,c} &= \{(x_0, y, (u,v)) \mid y \in \partial_\infty\Gamma, (u,v) \in \mathsf{M}(\xi(x_0), \theta(y))\} \\ &= \bigcup_{t\in\mathbb{R}} \phi_t(\mathcal{L}_Z^-).\end{aligned}$$

Observe that $\mathcal{L}_Z^+$ is homeomorphic to $\partial_\infty\Gamma \setminus \{x_0\}$ and $\mathcal{L}_Z^-$ is homeomorphic to $\partial_\infty\Gamma \setminus \{y_0\}$.

The following two propositions are immediate from our construction.

Proposition 5.3. [INVARIANCE] *If* $\gamma \in \Gamma$ *and* $t \in \mathbb{R}$, *then*

$$\mathcal{L}^\pm_{\gamma(Z)} = \gamma\left(\mathcal{L}^\pm_Z\right) \quad \text{and} \quad \mathcal{L}^\pm_{\phi_t(Z)} = \phi_t\left(\mathcal{L}^\pm_Z\right).$$

Proposition 5.4. [PRODUCT STRUCTURE] *The (two) pairs of lamination* $(\mathcal{L}^\pm, \mathcal{L}^{\mp,c})$ *define a local product structure on* F_ρ, *and hence on* $\mathsf{U}_\rho\Gamma$.

Remark: Throughout this section, we abuse notation by allowing $\{\phi_t\}_{t\in\mathbb{R}}$ to denote both the flow on $\mathsf{U}_\rho\Gamma$ and the flow on F_ρ which covers it and letting $\mathcal{L}^\pm$ denote both the lamination on F_ρ and the induced lamination on $\mathsf{U}_\rho\Gamma$.

5.3. **The leaf lift and the distance.** In this section we introduce the *leaf lift* and show that it helps in controlling distances in F_ρ.

We first define the leaf lift for points in the bundle F. Let $A = (U, V, (u_0, v_0))$ be a point in F. We observe that there exists a unique continuous map, called the *leaf lift* from

$$O_A = \{w \in \mathbb{RP}(m)^* \mid U \cap \ker(w) = \{0\}\}.$$

to $((\mathbb{R}^m)^* \setminus \{0\}) / \pm 1$ such that w is taken to $\Omega_{w,A}$ such that

$$\Omega_{w,A} \in w, \qquad \langle \Omega_{w,A} | u_0 \rangle = 1. \tag{25}$$

In particular, $\Omega_{v_0,A} = v_0$. Observe that at this stage the leaf lift coincides with the classical notion of an affine chart.

The following lemma records immediate properties of the leaf lift .

Lemma 5.5. *Let $\|.\|_1$ be a Euclidean norm on $\mathbb{R}^n$ and d_1 the associated metric on $\mathbb{RP}(m)^*$. If $A = (x, y, (u, v)) \in F$, then there exist constants $K_1 > 0$ and $\alpha_1 > 0$ such that for $w_0, w_1 \in \mathbb{RP}(m)^*$*

- *If $d_1(w_i, y) \leqslant \alpha_1$, for $i = 0, 1$, then*
$$\|\Omega_{w_0,A} - \Omega_{w_1,A}\|_1 \leqslant K_1 d_1(w_0, w_1),$$
- *If $\|\Omega_{w_i,A} - \Omega_{y,A}\|_1 \leqslant \alpha_1$ for $i = 0, 1$, then*
$$d_1(w_0, w_1) \leqslant K_1 \|\Omega_{w_0,A} - \Omega_{w_1,A}\|_1.$$

If $Z = (x, y, (u_0, v_0)) \in F_\rho$ and $W = (x, w, (u_0, v)) \in \mathcal{L}^+_Z$, then we define the *leaf lift*

$$\omega_{W,Z} = \Omega_{\xi^*(w),(\xi(x),\xi^*(y),(u_0,v_0))} = v.$$

The following result allows us to use the leaf lift to bound distances in F_ρ

Proposition 5.6. *Let d_0 be a Γ-invariant metric on F_ρ which is locally bilipschitz equivalent to a linear metric and let $Z \to \|.\|_Z$ be a Γ-invariant map from F_ρ into the space of Euclidean metrics on $\mathbb{R}^m$. There exist positive constants K and α such that for any $Z \in F_\rho$ and any $W \in \mathcal{L}^-_Z$,*

- *if $d_0(W, Z) \leqslant \alpha$, then*
$$\left\|\omega_{W,Z} - \omega_{Z,Z}\right\|_Z \leqslant K d_0(W, Z), \tag{26}$$
- *if $\left\|\omega_{W,Z} - \omega_{Z,Z}\right\|_Z \leqslant \alpha$ then*
$$d_0(W, Z) \leqslant K \left\|\omega_{W,Z} - \omega_{Z,Z}\right\|_Z. \tag{27}$$

Proof. Since Γ acts cocompactly on F_ρ and both d_0 and the section $\|.\|$ are Γ-invariant, it suffices to prove the previous assertion for Z in a compact subset R of F_ρ. Observe first that d_0 is uniformly C-bilipschitz on R to any of the linear metrics d_Z coming from $\|.\|_Z$ for Z in R for some constant C.

Lemma 5.5 implies that, for all $Z \in R$, there exist positive constants K_Z and α_Z such that if $W_0, W_1 \in \mathcal{L}^-_Z \cap O$, then

- If $d_0(W_i, Z) \leqslant \alpha_Z$ for $i = 0, 1$, then
$$\left\|\omega_{W_0,Z} - \omega_{W_1,Z}\right\|_Z \leqslant K_Z d_0(W_0, W_1),$$
- If $\left\|\omega_{W_i,Z} - \omega_{Z,Z}\right\|_Z \leqslant \alpha_Z$ for $i = 0, 1$, then
$$d_0(W_0, W_1) \leqslant K_Z \left\|\omega_{W_0,Z} - \omega_{W_1,Z}\right\|_Z.$$

Since R is compact, one may apply the classical argument which establishes that continuous functions are uniformly continuous on compact sets, to show that there are positive constants K and α which work for all $Z \in R$. □

5.4. **The geodesic flow is Anosov.** The following result completes the proof of Proposition 5.1

Proposition 5.7. [ANOSOV PROPERTY] *Let $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ be a projective Anosov representation, and let $\mathcal{L}^{\pm}$ be the laminations on $\mathsf{U}_\rho\Gamma$ defined above. Then there exists a metric on $\mathsf{U}_\rho\Gamma$, Hölder equivalent to the Hölder structure on $\mathsf{U}_\rho\Gamma$, such that*

1. *$\mathcal{L}^+$ is contracted by the flow,*
2. *$\mathcal{L}^-$ is contracted by the inverse flow,*

We first show that the leaf lift is contracted by the flow.

Lemma 5.8. *There exists a Γ-invariant map $Z \mapsto \|.\|_Z$ from F_ρ into the space of Euclidean metrics on $\mathbb{R}^m$, such that for every positive integer n, there exists $t_0 > 0$ such that if $t > t_0$, $Z \in F_\rho$, and $W \in \mathcal{L}^+_Z$ then*

$$\left\|\omega_{\phi_t(W),\phi_t(Z)} - \omega_{\phi_t(Z),\phi_t(Z)}\right\|_{\phi_t(Z)} \leqslant \frac{1}{2^n}\left\|\omega_{W,Z} - \omega_{Z,Z}\right\|_Z. \tag{28}$$

The following notation will be used in the proof.

- For a vector space A and a subspace $B \subset A$, let
$$B^\perp = \{\omega \in A^* \mid B \subset \ker(\omega)\}.$$
- We consider the Γ-invariant splitting of the trivial $\mathbb{R}^m$-bundle
$$F_\rho \times \mathbb{R}^m = \hat{\Xi} \oplus \hat{\Theta}$$
  - where $\hat{\Xi}$ is the line bundle over F_ρ such that the fiber above $(x, y, (u, v))$ is given by $\xi(x)$ and
  - $\hat{\Theta}$ is a hyperplane bundle over F_ρ with fiber $\theta(y)$ above the point $(x, y, (u, v)) \in F_\rho$.

Proof. Suppose that $Z = (x, y, (u_0, v_0))$ and $W = (x, w, (u_0, v)) \in \mathcal{L}^+_Z$, then by the definition of the leaf lift

$$(\omega_{W,Z} - \omega_{Z,Z})(u_0) = 0,$$

and thus

$$\omega_{W,Z} = \alpha_{W,Z} + \omega_{Z,Z},$$

where $\alpha_{W,Z} \in \xi(x)^\perp$. Then

$$\phi_t(\omega_{W,Z}) = \phi_t(\alpha_{W,Z}) + \phi_t(\omega_{Z,Z}).$$

We choose a Γ-invariant map from F_ρ into the space of Euclidean metrics on $\mathbb{R}^m$ so that for all $Y \in F_\rho$

$$\left\|\omega_{Y,Y}\right\|_Y = 1.$$

Then

$$\omega_{\phi_t(Z),\phi_t(Z)} = \frac{1}{\left\|\phi_t(\omega_{Z,Z})\right\|_{\phi_t(Z)}}\phi_t(\omega_{Z,Z}),$$

hence

$$\omega_{\phi_t(W),\phi_t(Z)} = \frac{\phi_t(\alpha_{W,Z})}{\left\|\phi_t(\omega_{Z,Z})\right\|_{\phi_t(Z)}} + \omega_{\phi_t(Z),\phi_t(Z)}.$$

It follows that

$$\left\|\omega_{\phi_t(W),\phi_t(Z)} - \omega_{\phi_t(Z),\phi_t(Z)}\right\|_{\phi_t(Z)} = \frac{\left\|\phi_t(\alpha_{W,Z})\right\|_{\phi_t(Z)}}{\left\|\phi_t(\omega_{Z,Z})\right\|_{\phi_t(Z)}}$$

Since ρ is projective Anosov, and $(\mathsf{U}_\rho\Gamma, \{\phi_t\}_{t\in\mathbb{R}})$ is a Hölder reparameterization of $(\mathsf{U}_0\Gamma, \{\psi_t\}_{t\in\mathbb{R}})$, there exists $t_1 > 0$ so that for all $Z \in F_\rho$ and for all $t > t_1$, if $v \in \hat{\Xi}_Z^\perp$ and $w \in \hat{\Theta}_Z^\perp$, then

$$\frac{\left\|\phi_t(v)\right\|_{\phi_t(Z)}}{\left\|\phi_t(w)\right\|_{\phi_t(Z)}} \leqslant \frac{1}{2}\frac{\left\|v\right\|_Z}{\left\|w\right\|_Z}.$$

Thus, since $\alpha_{W,Z} \in \hat{\Xi}_Z^\perp$ and $\omega_{Z,Z} \in \hat{\Theta}_Z^\perp$, for all $n \in \mathbb{N}$ and $t > nt_1$, we have

$$\left\|\omega_{\phi_t(W),\phi_t(Z)} - \omega_{\phi_t(Z),\phi_t(Z)}\right\|_{\phi_t(Z)} \leqslant \frac{1}{2^n}\frac{\left\|\alpha_{W,Z}\right\|_Z}{\left\|\omega_{Z,Z}\right\|_Z}.$$

Since $\alpha_{W,Z} = \omega_{W,Z} - \omega_{Z,Z}$ and $\left\|\omega_{Z,Z}\right\|_Z = 1$, the previous assertion yields the result with $t_0 = nt_1$. □

We are now ready to establish Proposition 5.7.

Proof of Proposition 5.7: Let K and α be as in Proposition 5.6. Choose $n \in \mathbb{N}$ so that

$$\frac{K}{2^n} \leqslant 1 \quad \text{and} \quad \frac{K^2}{2^n} \leqslant \frac{1}{2}. \tag{29}$$

Let t_0 be the constant from Lemma 5.8 with our choice of n.

Suppose that $Z \in F_\rho$, $W \in \mathcal{L}_Z^+$, $t > t_0$ and $d_0(W, Z) \leqslant \alpha$. Then, by Inequality (26),

$$\left\|\omega_{W,Z} - \omega_{Z,Z}\right\| \leqslant K d_0(W, Z). \tag{30}$$

By Lemma 5.8,

$$\left\|\omega_{\phi_t(W),\phi_t(Z)} - \omega_{\phi_t(Z),\phi_t(Z)}\right\|_{\phi_t(Z)} \leqslant \frac{1}{2^n}\left\|\omega_{W,Z} - \omega_{Z,Z}\right\|_Z. \tag{31}$$

In particular, combining Equations (30), (31) and (29),

$$\left\|\omega_{\phi_t(W),\phi_t(Z)} - \omega_{\phi_t(Z),\phi_t(Z)}\right\|_{\phi_t(Z)} \leqslant \frac{1}{2^n}K\alpha \leqslant \alpha. \tag{32}$$

Thus, using Inequality (27),

$$d_0(\phi_t(W), \phi_t(Z)) \leqslant K\left\|\omega_{\phi_t(W),\phi_t(Z)} - \omega_{\phi_t(Z),\phi_t(Z)}\right\|_{\phi_t(Z)}. \tag{33}$$

Combining finally Equations (30), (31), (33) and (29), we get that

$$d_0(\phi_t(W), \phi_t(Z)) \leqslant \frac{K^2}{2^n} d_0(W, Z) \leqslant \frac{1}{2} d_0(W, Z) \tag{34}$$

for all $t > t_0$.

Therefore $\mathcal{L}^+$ is contracted by the flow on F_ρ.

Let us now consider what happens in the quotient $\mathsf{U}_\rho\Gamma = F_\rho/\Gamma$. For any $Z \in F_\rho$ and $\epsilon > 0$, let

$$\mathcal{L}_\epsilon^\pm(Z) = \mathcal{L}_Z^\pm \cap B(Z, \epsilon).$$

and let

$$K_\epsilon(Z) = \Pi_Z\left(\mathcal{L}_\epsilon^+(Z) \times \mathcal{L}_\epsilon^-(Z) \times (-\epsilon, \epsilon)\right),$$

where Π_Z is the product structure of Proposition 5.4. By Proposition 4.1, there exists $\epsilon_0 > 0$ such that for all $\gamma \in \Gamma \setminus \{1\}$ and $Z \in F_\rho$,

$$\gamma(K_{\epsilon_0}(X)) \cap K_{\epsilon_0} = \emptyset.$$

Let $\epsilon \in (0, \min\{\epsilon_0/2, \alpha\})$ and $\hat{Z} \in \mathsf{U}_\rho\Gamma$. Choose $Z \in F_\rho$ in the pre image of $\hat{Z}$, then inequality (34) holds for the flow on $\mathsf{U}_\rho\Gamma$ for points in the chart which is the projection of $K_\epsilon(Z)$. Therefore, $\mathcal{L}^+$ is contracted by the flow on $\mathsf{U}_\rho\Gamma$.

A symmetric proof holds for the central unstable leaf.

6. Analytic variation of the dynamics

In order to apply the thermodynamic formalism we need to check that if $\{\rho_u\}_{u\in M}$ is an analytic family of projective Anosov representations, then the associated limit maps and reparameterizations of the Gromov geodesic flow may be chosen to vary analytically, at least locally. Our proofs generalize earlier proofs of the stability of Anosov representations, see Labourie [44, Proposition 2.1] and Guichard-Wienhard [29, Theorem 5.13], and that the limit maps vary continuously, see Guichard-Wienhard [29, Theorem 5.13]. In the process, we also see that our limit maps are Hölder.

We will make use of the following concrete description of the analytic structure of $\mathrm{Hom}(\Gamma, \mathsf{G})$. Suppose that Γ is a word hyperbolic group, hence finitely presented, and let $\{g_1, \ldots, g_m\}$ be a generating set for Γ. If G is a real semi-simple Lie group, then $\mathrm{Hom}(\Gamma, \mathsf{G})$ has the structure of a real algebraic variety. An *analytic family* $\beta : M \to \mathrm{Hom}(\Gamma, \mathsf{G})$ of homomorphisms of Γ into G is a map with domain an analytic manifold M so that, for each i, the map $\beta_i : M \to \mathsf{G}$ given by $\beta_i(u) = \beta(u)(g_i)$ is real analytic. If G is a complex Lie group, we may similarly define complex analytic families of homomorphisms of a complex analytic manifold into $\mathrm{Hom}(\Gamma, \mathsf{G})$.

We first show that the limit maps of an analytic family of Anosov homomorphisms vary analytically. We begin by setting our notation. If $\alpha > 0$, X is a compact metric space and D and M are real-analytic manifolds, then we let $C^\alpha(X, M)$ denote the space of α-Hölder maps of X into M and let $C^\omega(D, M)$ denote the space of real analytic maps of D into M. If D and M are complex analytic manifolds, we will abuse notation by letting $C^\omega(D, M)$ denote the space of complex analytic maps.

Theorem 6.1. *Let G be a real (or complex) semi-simple Lie group with finite center and let P be a parabolic subgroup of G. Let $\{\rho_u\}_{u\in D}$ be a real (or complex) analytic family of homomorphisms of Γ into G parameterized by a real (or complex) disk D about 0. If ρ_0 is a (G, P)-Anosov homomorphism with limit map $\xi_0 : \partial_\infty\Gamma \to \mathsf{G}/\mathsf{P}$, then there exists a sub-disk D_0 of D (containing 0), $\alpha > 0$ and a unique continuous map*

$$\xi : D_0 \times \partial_\infty\Gamma \to \mathsf{G}/\mathsf{P}$$

so that $\xi(0, \cdot) = \xi_0(\cdot)$ with the following properties:

1. *If $u \in D_0$, then ρ_u is a (G, P)-Anosov homomorphism with α-Hölder limit map $\xi_u : \partial_\infty\Gamma \to \mathsf{G}/\mathsf{P}$ given by $\xi_u(\cdot) = \xi(u, \cdot)$.*
2. *If $x \in \partial_\infty\Gamma$, then $\xi_x : D_0 \to \mathsf{G}/\mathsf{P}$ given by $\xi_x = \xi(\cdot, x)$ is real (or complex) analytic*
3. *The map from $\partial_\infty\Gamma$ to $C^\omega(D_0, \mathsf{G}/\mathsf{P})$ given by $x \mapsto \xi_x$ is α-Hölder.*
4. *The map from D_0 to $C^\alpha(\partial_\infty\Gamma, \mathsf{G}/\mathsf{P})$ given by $u \to \xi_u$ is real (or complex) analytic.*

Given a projective Anosov representation $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$, we constructed a geodesic flow $U_\rho\Gamma$ which is a reparameterization of the Gromov geodesic flow $\mathsf{U}_0\Gamma$. In Section 6.3, we show that given a real analytic family of projective Anosov representions, one may choose the parameterizing functions to vary analytically.

Proposition 6.2. *Let $\{\rho_u\}_{u\in D}$ be a real analytic family of projective Anosov homomorphisms of Γ into $\mathsf{SL}_m(\mathbb{R})$ parameterized by a disk about 0. Then, there exists a sub-disk D_0 about 0 and a real analytic family $\{f_u : \mathsf{U}_0\Gamma \to \mathbb{R}\}_{u\in D_0}$ of positive Hölder functions such that the reparametrization of $\mathsf{U}_0\Gamma$ by f_u is Hölder conjugate to $\mathsf{U}_{\rho_u}\Gamma$ for all $u \in D_0$.*

We now show that the real analytic case of Theorem 6.1 follows from the complex analytic case, which we will establish in Section 6.2. We first suppose that G is a real semi-simple Lie group with trivial center and let $\mathsf{P}^\pm$ be a pair of opposite parabolic subgroup. (In fact, the argument works, more generally, for any semi-simple subgroup of $\mathsf{SL}_m(\mathbb{R})$.) Let $\mathsf{G}^\mathbb{C}$ be the complexification of G and let $(\mathsf{P}^\pm)^\mathbb{C}$ be the complexifications of $\mathsf{P}^\pm$, i.e. the normalizers of the complexifications $(\mathfrak{p}^\pm)^\mathbb{C} = \mathfrak{p}^\pm \otimes \mathbb{C}$ of $\mathfrak{p}^\pm$. Observe that a $(\mathsf{G},\mathsf{P}^+)$-Anosov representation $\rho : \Gamma \to \mathsf{G}$ with limit maps $\xi^\pm_\rho : \partial\Gamma \to \mathsf{G}/\mathsf{P}^\pm$ is also a $(\mathsf{G}^\mathbb{C}, (\mathsf{P}^+)^\mathbb{C})$-Anosov representation with limit maps $\xi^\pm_\rho$, since the bundles $\mathcal{N}^\pm_{\rho^\mathbb{C}}$, where one regards ρ as a representation into $\mathsf{G}^\mathbb{C}$, are the complexifications of the bundles $\mathcal{N}^\pm_\rho$ obtained by regarding ρ as a representation into G.

The embedding of G into $\mathsf{G}^\mathbb{C}$ gives rise to an anti-holomorphic involution I of $\mathsf{G}_\mathbb{C}$ whose fixed point set is G, which descends to an involution $\bar{I}$ of $\mathsf{G}^\mathbb{C}/(\mathsf{P}^\pm)^\mathbb{C}$ with fixed point set $\mathsf{G}/\mathsf{P}^\pm$. If $\rho : \Gamma \to \mathsf{G}^\mathbb{C}$ has image in G and is $(\mathsf{G}^\mathbb{C}, (\mathsf{P}^+)^\mathbb{C})$-Anosov with limit maps $\xi^\pm_\rho$, then, since $\rho = I \circ \rho \circ I^{-1}$, ρ is $(\mathsf{G}^\mathbb{C}, (\mathsf{P}^+)^\mathbb{C})$-Anosov with limit maps $\bar{I} \circ \xi^\pm_\rho$. Therefore, since limit maps of Anosov representations are unique ([29, Lemma 3.3]), $\bar{I} \circ \xi^\pm_\rho = \xi^\pm_\rho$ which implies that $\xi^\pm_\rho$ has image in $\mathsf{G}/\mathsf{P}^\pm$. It follows that ρ is $(\mathsf{G},\mathsf{P}^+)$-Anosov, again since the bundles $\mathcal{N}^\pm_{\rho^\mathbb{C}}$ are the complexifications of the bundles $\mathcal{N}^\pm_\rho$.

Now suppose that $\{\rho_u\}_{u\in D}$ is an analytic family of representations of Γ into G so that ρ_0 is $(\mathsf{G},\mathsf{P}^+)$-Anosov. On a sub-disk D_1 of D, containing 0, one may extend $\{\rho_u\}_{u\in D_1}$ to a complex analytic family $\{\rho_u\}_{u\in D_1^\mathbb{C}}$ of homomorphisms of Γ into $\mathsf{G}^\mathbb{C}$ defined on the complexification $D_1^\mathbb{C}$ of D_1. Let $\xi^\mathbb{C} : D_0^\mathbb{C} \times \partial_\infty\Gamma \to \mathsf{G}^\mathbb{C}/(\mathsf{P}^+)^\mathbb{C}$ be the map provided by the complex analytic case of Theorem 6.1 and let $\xi = \xi^\mathbb{C}|_{D_0\times\partial_\infty\Gamma}$. If $u \in D_0$, then ρ_u has image in G and is $(\mathsf{G}^\mathbb{C}, (\mathsf{P}^+)^\mathbb{C})$-Anosov with limit map ξ_u, so, by the argument in the above paragraph, ρ_u is $(\mathsf{G},\mathsf{P}^+)$-Anosov with limit map ξ_u which has image in G/P^+, so we may regard ξ as a map into G/P and (1) holds. Notice that the real analyticity in properties (2) and (4) follows from the fact that restrictions of complex analytic functions to real analytic submanifolds are real analytic. The map in (3) is α-Hölder, since it is the restriction of an α-Hölder map. This completes the proof of Theorem 6.1 in the case that G is a real semi-simple Lie group with finite center (or a semi-simple subgroup of $\mathsf{SL}_m(\mathbb{R})$).

Finally, suppose that G is a real semi-simple Lie group with finite center, and that $\{\rho_u\}_{u\in D}$ is an analytic family of representations of Γ into G so that ρ_0 is (G,P)-Anosov. Let $\mathsf{G}_0 = \mathsf{G}/Z(\mathsf{G})$, $\mathsf{P}_0 = \mathsf{P}/Z(\mathsf{G})$, and let $\pi : \mathsf{G} \to \mathsf{G}_0$ be the quotient map. Notice that $\mathsf{G}/\mathsf{P} = \mathsf{G}_0/\mathsf{P}_0$, so $\rho : \Gamma \to \mathsf{G}$ is (G,P)-Anosov with limit maps $\xi^\pm$ if and only if $\pi \circ \rho$ is $(\mathsf{G}_0,\mathsf{P}_0)$-Anosov with limit map $\xi^\pm$, since one may identify $\mathcal{N}^\pm_\rho$ with $\mathcal{N}^\pm_{\pi\circ\rho}$. The above argument implies that there exists a sub-disk D_0 of D

(containing 0) and a map $\xi : D_0 \times \partial_\infty \Gamma \to \mathsf{G}_0/\mathsf{P}_0$ which satisfies properties (1)–(4) for $\{\pi \circ \rho_u\}_{u \in D_0}$. Therefore, again since $\mathsf{G}/\mathsf{P} = \mathsf{G}_0/\mathsf{P}_0$, ξ also satisfies properties (1)–(4) for $\{\rho_u\}_{u\in D}$.

6.1. **Transverse regularity.** In this section, we set up our notation and establish a version of the C^r-section Theorem of Hirsch-Pugh-Shub [32, Theorem 3.8] which keeps track of the transverse regularity of the resulting section. Our version of Hirsch, Pugh and Shub's result will be the main tool in the proof of Theorem 6.1.

Definition 6.3. [TRANSVERSELY REGULAR FUNCTIONS] *Let D be a real (or complex) disk, let X be a compact metric space and let M be a real (or complex) analytic manifold. A continuous function $f : D \times X \to M$ is* transversely real (or complex) analytic *if*

1. *For every $x \in X$, the function $f_x : D \to M$ given by $f_x(\cdot) = f(\cdot, x)$ is real (or complex) analytic, and*
2. *The function from X to $C^\omega(D, M)$ given by $x \to f_x$ is continuous.*

Furthermore, we say that f is α-Hölder (or Lipschitz) transversely real (or complex) analytic *if the map in (2) is α-Hölder (or Lipschitz).*

If we replace M with a C^k manifold, we can similarly define α-Hölder (or Lipschitz) transversely C^k functions by requiring that the maps in (1) are C^k and that the map in (2) from X to $C^p(D, M)$ is α-Hölder (or Lipschitz) for all $p \leq k$,

Similarly, we define transverse regularity of bundles in terms of the transverse regularity of their trivializations.

Definition 6.4. [TRANVERSALLY REGULAR BUNDLES] *Suppose that the fiber of a bundle $\pi : E \to D \times X$ is a real (or complex) analytic manifold M and that D is a real (or complex) disk. We say that E is* transversely real (or complex) analytic *if it admits a family of trivializations of the form $\{D \times U_\alpha \times M\}$ (where $\{U_\alpha\}$ is an open cover of X) so that the the corresponding change of coordinate functions are transversely real (or complex) analytic. We similarly say $\pi : E \to D \times X$ is* α-Hölder (or Lipschitz) transversely real (or complex) analytic *if it admits a family of trivializations which are α-Hölder (or Lipschitz) transversely real (or complex) analytic.*

In this case, a section of E is α-Hölder (or Lipschitz) transversely real (or complex) analytic, if in any of the trivializations the corresponding map to M is α-Hölder (or Lipschitz) transversely real (or complex) analytic.

Clearly, if M is a C^k-manifold, we can similarly define α-Hölder (or Lipschitz) transversely C^k bundles and sections.

We are now ready to state our version of the C^r-Section Theorem.

Theorem 6.5. *Let X be a compact metric space and let M be a complex analytic (or C^k) manifold. Suppose that $\pi : E \to D \times X$ is a Lipschitz transversely complex analytic (or C^k) bundle with fibre M and D is a complex (or real) disk. Let $f : X \to X$ be a Lipschitz homeomorphism and let F be a Lipschitz transversely complex analytic (or C^k) bundle automorphism of E lifting* $\mathrm{id} \times f$.

Suppose that σ_0 is a section of the restriction of E over $\{0\} \times X$ which is fixed by F and that F contracts along σ_0. Then there exists a neighborhood U of 0 in D, a positive number $\alpha > 0$, an α-Hölder transversely complex analytic (or C^k) section η over $D_0 \times X$ and a neighborhood B of $\eta(U \times X)$ in $\pi^{-1}(U \times X)$ such that

(1) *F fixes η,*
(2) *F contracts E along η,*
(3) *$\eta|_{\{0\}\times X} = \sigma_0$, and*
(4) *if $\nu : U \times X \to E$ is a section so that $\nu(U \times X) \subset B$ and ν is fixed by F, then $\nu = \eta$.*

We recall that if U is a subset of D, then a section σ over $U \times X$ is *fixed* by F if $F(\sigma(u,x)) = \sigma(u, f(x))$. In such a case, we further say that F *contracts* along σ if there exists a continuously varying fibrewise Riemannian metric $\|\cdot\|$ on the bundle E such that if

$$\mathrm{D}^f F_{\sigma(u,x)} : T_{\sigma(u,x)}\pi^{-1}(u,x) \to T_{\sigma(u,f(x))}\pi^{-1}(u, f(x))$$

is the fibrewise tangent map, then

$$\|\mathrm{D}^f F_{\sigma(u,x)}\| < 1.$$

We will derive Theorem 6.5 from the following version of the C^r-section theorem which is a natural generalization of the ball bundle version of the C^r-section theorem in Shub [68, Theorem 5.18].

Theorem 6.6. [Fixed sections] *Let X be a compact metric space equipped with a Lipschitz homeomorphism $f : X \to X$. Suppose that $\pi : W \to D \times X$ is a Lipschitz transversely complex analytic (or C^k) Banach space bundle, D is a complex (or real) disk, $B \subset W$ is the closed ball sub-bundle of radius r, and F is a Lipschitz transversely complex analytic (or C^k) bundle morphism of B lifting the homeomorphism* $\mathrm{id} \times f : D \times X \to X$.

If F contracts B, then there exists a unique α-Hölder transversely complex analytic (or C^k) section η of B which is fixed by F (for some $\alpha > 0$).

Notice that we have not assumed that F is either linear or bijective.

Proof. Let σ be the zero section of B. Observe that σ has the same regularity as W and is thus transversally complex analytic (or C^k).

We first assume that $\pi : W \to D \times M$ is a Lipschitz transversely C^k-bundle. The existence of a unique continuous fixed section η is a standard application of the contraction mapping theorem. Explicitly, for all $(u,x) \in D \times X$, we let

$$\eta(u,x) = \lim_{n\to\infty} F^n(\sigma(u, f^{-n}(x))). \tag{35}$$

We must work harder to show that η is α-Hölder transversely complex analytic (or C^k). We first assume that W is transversely C^k –and so is σ– and obtain the C^k-regularity of η. For any $p \in \mathbb{N}$, let Γ^p be the Lipschitz Banach bundle over X whose fibers over a point $x \in X$ is the Banach space Γ^p_x of C^p-sections of the restriction of W to $D \times \{x\}$. Let B^p be the sub-bundle whose fiber B^p_x over x is the set of those sections with values in B.

Notice that each fiber B^p_x can be identified with $C^p(D \times \{x\}, B_0)$ where B_0 is a closed ball of radius r in the fiber Banach space. Let F^p_* be the bundle automorphism of Γ_p given by

$$[F^p_*(\nu)](u,x) = F(\nu(u, f^{-1}(x))).$$

We can renormalize the metric on D, so that all the derivatives of F of order n (with $p \geqslant n \geqslant 1$) along D are arbitrarily small. Thus after this renormalization the metric on D, F^p_* is contracting, since F is contracting. We now apply Theorem 3.8

of Hirsch-Pugh-Shub [32] (see also Shub [68, Theorem 5.18]) to obtain an invariant α-Hölder section ω. By the uniqueness of fixed sections, we see that

$$\eta(u,x) = \omega(x)(u)$$

for all $1 \leq p \leq k$. It follows that η is α-Hölder transversely C^k.

Now suppose that D is a complex disk and $\pi : E \to D \times X$ is Lipschitz transversely complex analytic bundle. We see, from the above paragraph, that there exists a unique α-Hölder transversely C^k section η_k for all k. By the uniqueness η_k is independent of k and we simply denote it by η. Then, by Formula (35), for all $x \in X$, $\eta|_{D\times\{x\}}$ is a C^k-limit of a sequence of complex analytic sections for all k, hence is complex analytic itself. It follows that η is α-Hölder transversely complex analytic. □

We now notice that one may identify a neighborhood of the section σ_0 in the statement of Theorem 6.5 with a ball sub-bundle of a vector bundle.

Lemma 6.7. *Let $\pi : E \to D \times X$ be a transversely complex analytic (or C^k) bundle over $D \times X$, D is a complex (or real) disk about 0, and σ is a section of E defined over $\{0\} \times X$. Then there exists*

(1) *a neighborhood U of zero in D,*
(2) *a transversely complex analytic (or C^k) closed ball bundle B of radius R in a complex (or real) vector bundle F,*
(3) *a transversely complex analytic (or C^k) bijective map from B to a neighborhood of the graph of σ_0 so that*
    - *the graph of $\sigma_0 = \sigma|_{\{0\}\times X}$ is in the image of the graph of the zero section,*
    - *the fibrewise metric on B coincides along σ_0 with the fiberwise metric on E.*

Proof. We first give the proof in the case that σ is defined over $D \times X$. Let Z be the transversely complex analytic (or C^k) vector bundle over $D \times X$ so that the fibre over the point (u,x) is given by $\mathsf{T}_{\sigma(u,x)}(\pi^{-1}(u,x))$. We equip Z with a Riemannian metric coming from E and let $B(r)$ be the closed ball sub-bundle of radius $r > 0$.

Using the trivializations, we can find, after restricting to an open neighborhood U of 0 in D,

- a finite cover $\{O_i\}_{1\leqslant i\leqslant n}$ of X,
- an open neighborhood W of the graph of σ,
- transversely holomorphic (or C^k-diffeomorphic) bundle maps ϕ_i defined on $W|_{U\times O_i}$ with values in $Z|_{U\times O_i}$ so that for all $(u,x) \in U \times O_i$

$$\begin{aligned} \phi_i(\sigma(u,x)) &= 0 \in \mathsf{T}_{\sigma(u,x)}(\pi^{-1}(u,x)) \\ \mathrm{D}^{\mathrm{f}}_{\sigma(\mathrm{u,x})}\phi_{\mathrm{i}} &= \mathrm{Id}. \end{aligned} \tag{36}$$

Let $\{\psi_i\}_{1\leqslant i\leqslant n}$ be a partition of unity on X subordinate to $\{O_i\}_{1\leqslant i\leqslant n}$ and, for each i, let $\hat{\psi}_i : W \to [0,1]$ be obtained by composing the projection of W to X with ψ_i. One may then define $\Phi : W \to Z$ by letting

$$\Phi = \sum_{i=1}^{n} \hat{\psi}_i \phi_i.$$

Since $\hat{\psi}_i$ is constant in the direction of D, Φ is transversely holomorphic (or C^k-diffeomorphic),

$$\Phi(\sigma(u,x)) = 0 \ \text{ and } \ \mathrm{D}^f_{\sigma(y)}\Phi = \mathrm{Id}.$$

It then follows from the implicit function theorem, that one may further restrict U and W so that Φ is a transversely holomorphic (or C^k-diffeomorphic) isomorphism of W with $B(r)$ for some r.

If σ is only defined on $\{0\} \times X$, it now suffices to extend the section σ_0 to a section σ defined over $U \times X$ where U is a neighborhood of 0 in D. Composing π with the projection $\pi_2 : D \times X \to X$, we may consider the bundle $\pi_2 \circ \pi : E \to X$. Then σ_0 is a section of $\pi_2 \circ \pi$. We now apply the result of the previous paragraph, in the case where the disk is 0-dimensional, to identify, in a complex analytic (or C^k) way, a neighborhood of the graph of σ_0 with a ball bundle B in a vector bundle F over X.

Now π restricts to a bundle morphism from $\pi \circ \pi_2 : B \to X$ to $\pi_2 : D \times X \to X$ which is a fiberwise complex analytic (or C^k) submersion and whose fiberwise derivatives vary continuously. Let W be a linear sub-bundle of F, so that if W_x and F_x are the fibers over $x \in X$, then

$$\mathsf{T}(\pi^{-1}(0,x)) \oplus W_x = F_x.$$

Thus, after further restricting B, π becomes a fiberwise complex analytic (or C^k) injective local diffeomorphism from $W \cap B$ to $D \times X$ whose fiberwise derivatives vary continuously.

Applying the Implicit Function Theorem (with parameter), we obtain a neighborhood U of 0 and a map $\sigma : U \times X \to B$ which is fiberwise complex analytic (or C^k) and whose fiberwise derivatives vary continuously, so that $\pi \circ \sigma = \mathrm{Id}$. Thus σ is the desired section of E. □

Theorem 6.5 now follows from Theorem 6.6 and Lemma 6.7.

Proof of Theorem 6.5: Let V be the complex (or real) vector bundle provided by Lemma 6.7. We know that $\|\mathsf{D}^f_{\sigma_0(x)}F\| < 1$ for all x in X. After further restraining U and choosing r small enough, we may assume by continuity that for all y in $B(r)$, $\|\mathsf{D}^f_y F\| < K < 1$.

After further restricting U, we may assume that for all $u \in U$ and $x \in X$, we have

$$\|F(\sigma(u,x)) - \sigma(u, f(x))\| \leqslant (1-K)r,$$

In particular, if $y \in B(r)$ is in the fiber over (u,x),

$$\begin{aligned} \|F(y) - \sigma(u,f(x))\| &\leqslant \|F(y) - F(\sigma(u,x))\| \\ &\quad + \|F(\sigma(u,x)) - \sigma(u,f(x))\| \\ &\leqslant Kr + (1-K)r = r. \end{aligned}$$

Thus F maps $B(r)$ to itself and is contracting. We can therefore apply Theorem 6.6 to complete the proof of Theorem 6.5. □

In the proof of Theorem 6.1, we will also need to use the fact that transverse regularity of a continuous function $f : D \times X \to M$ implies regularity of the associated map of D into $C^\alpha(X, M)$.

Let X be a compact metric space and let M be a complex analytic (or C^k) manifold. If U is an open subset of M and V is a relatively compact open subset of X, then let

$$\mathcal{W}(U,V) = \{g \in C^\alpha(X,M) \mid \overline{g(V)} \subset U\}.$$

We will say that a map f from D to $C^\alpha(X,M)$ is *complex analytic (or C^k)* if for any U and V as above and any complex analytic function $\phi : U \to \mathbb{C}$ (or C^k function $\phi : U \to \mathbb{R}$), the function f^ϕ defined on $f^{-1}(\mathcal{W}(U,V))$, by

$$f^\phi(x) = \phi \circ f(x)|_V,$$

with values in $C^\alpha(V,\mathbb{C})$ (or $C^\alpha(V,\mathbb{R})$) is complex analytic (or C^k). Recall that the function f^ϕ is complex analytic if and only if it has a a $\mathbb{C}$-linear differential at each point, see, for example, Hubbard [34, Thm. A5.3].

The following lemma shows that an α-Hölder transversely complex analytic map from $D \times X$ to M gives rise to a complex analytic map from D to $C^\alpha(X,M)$. The proof is quite standard so we will omit it, see Hubbard [34, Prop. A5.9] for a very similar statement.

Lemma 6.8. *Suppose that D is a complex (or real) disk, M is a complex analytic (or C^k) manifold, X is a compact metric space and $f : D \times X \to M$ is α-Hölder transversely complex analytic (or C^k), then the map $\hat f$ from D to $C^\alpha(X,M)$ given by $u \to f_u$ where $f_u(\cdot) = f(u,\cdot)$ is complex analytic (or C^{k-1}).*

6.2. **Analytic variation of the limit maps.** We are now ready for the proof of Theorem 6.1 in the complex analytic case. Given a complex analytic family of representations which contains an Anosov representation, we construct an associated bundle where we can apply the results of the previous section to produce a family of limit maps.

Let G be a complex Lie group and let P be a parabolic subgroup. Let $\{\rho_u\}_{u\in D}$ be a complex analytic family of homomorphisms of Γ into G parameterized by a complex disk D about 0 so that ρ_0 is (G,P)-Anosov.

We construct a G/P-bundle over $D \times \mathsf{U}_0\Gamma$. Let

$$\tilde A = D \times \widetilde{\mathsf{U}_0\Gamma} \times \mathsf{G}/\mathsf{P}$$

which is a G/P-bundle over $D \times \widetilde{\mathsf{U}_0\Gamma}$. Then $\gamma \in \Gamma$ acts on $\tilde A$, by

$$\gamma(u,x,[g]) = (u,\gamma(x),[\rho_u(\gamma)g])$$

and we let

$$A = \tilde A/\Gamma.$$

The geodesic flows on $\widetilde{\mathsf{U}_0\Gamma}$ and $\mathsf{U}_0\Gamma$ lift to geodesic flows $\{\tilde\Psi_t\}_{t\in\mathbb{R}}$ and $\{\tilde\Psi_t\}_{t\in\mathbb{R}}$ on $\tilde A$ and A. (The flow $\{\tilde\Psi_t\}_{t\in\mathbb{R}}$ acts trivially on the D and G/P factors.)

Since ρ_0 is (G,P)-Anosov there exists a section σ_0 of A over $\{0\}\times\mathsf{U}_0\Gamma$. Concretely, if $\xi_0 : \partial_\infty\Gamma \to \mathsf{G}/\mathsf{P}$ is the limit map, we construct an equivariant section $\tilde\sigma_0$ of $\tilde A$ over $\{0\} \times \widetilde{\mathsf{U}_0\Gamma}$ of the form

$$(0,(x,y,t)) \to (0,(x,y,t),\xi_0(x)).$$

The section $\tilde\sigma_0$ descends to the desired section σ_0 of A over $\{0\} \times \mathsf{U}_0\Gamma$. One may identify the bundle over $\{0\} \times \mathsf{U}_0\Gamma$ with fiber $\mathsf{T}_{\sigma_0(x)}\pi^{-1}(0,x)$ with $\mathcal{N}^-_\rho$. Since the geodesic flow lifts to a flow on $\mathcal{N}^-_\rho$ whose inverse flow is contracting, the inverse flow $\{\Phi_{-t}\}_{t\in\mathbb{R}}$ is contracting along $\sigma_0(\mathsf{U}_0\Gamma)$.

Theorem 6.5 then implies that there exists a sub-disk $D_1 \subset D$ containing 0, $\alpha > 0$, and a unique α-Hölder transversely complex analytic section $\eta : D \times \mathsf{U}_0\Gamma \to$

A that extends σ_0, is fixed by $\{\Phi_t\}_{t\in\mathbb{R}}$ and so that the inverse flow $\{\Phi_{-t}\}_{t\in\mathbb{R}}$ contracts along η. (More concretely, Theorem 6.5 produces, for large enough t_0, a section fixed by Φ_{-t_0} so that Φ_{-t_0} contracts along η. One may then use the uniqueness portion of the statement to show that η is fixed by Φ_t for all t.) We may lift η to a section $\tilde{\eta} : D_1 \times \widetilde{\mathsf{U}_0\Gamma} \to \tilde{A}$ which we may view as a map $\bar{\eta} : D_1 \times \widetilde{\mathsf{U}_0\Gamma} \to \mathsf{G}/\mathsf{P}$.

We next observe that $\bar{\eta}(u,(x,y,t))$ does not depend on either y or t. Since $\bar{\eta}$ is flow-invariant, $\bar{\eta}(u,(x,y,t))$ does not depend on t. Fix $u \in D_1$ and let $\bar{\eta}_u : \widetilde{\mathsf{U}_0\Gamma} \to \mathsf{G}/\mathsf{P}$ be given by $\bar{\eta}_u(\cdot) = \bar{\eta}(u,\cdot)$. Let γ be an infinite order element of Γ whose associated orbit in $\mathsf{U}_0\Gamma$ has period t_γ and let d be an arbitrary metric on G/P. Since $\{\Phi_{-t}\}_{t\in\mathbb{R}}$ is contracting along η, there exists a constant $k_0 > 0$ such that if $\{p_n\}$ is a sequence in G/P with $d(\bar{\eta}_u(\gamma^+,\gamma^-,0),p_n) \le k_0$ for all n, then

$$\lim_{n\to\infty} d(\bar{\eta}_u(\gamma^+,\gamma^-,0),\gamma^n(p_n)) = \lim_{n\to\infty} d(\bar{\eta}_u(\gamma^n(\gamma^+,\gamma^-,-nt_\gamma)),\gamma^n(p_n)) = 0. \quad (37)$$

Given $z \in \partial_\infty\Gamma$, there exists $t_z \in R$, so that, if $\bar{d}$ denotes a Γ-invariant metric on $\widetilde{\mathsf{U}_0\Gamma}$, then

$$\lim_{n\to\infty} \bar{d}(\gamma^n(\gamma^+,\gamma^-,0),(\gamma^+,z,t_z+nt_\gamma)) = 0.$$

Therefore,

$$\lim_{n\to\infty} \bar{d}((\gamma^+,\gamma^-,0),\gamma^{-n}(\gamma^+,z,t_z+nt_\gamma)) = 0.$$

Applying (37) with $p_n = \bar{\eta}(\gamma^{-n}(\gamma^+,z,t_z+nt_\gamma))$, we see that

$$\lim_{n\to\infty} d(\bar{\eta}_u(\gamma^+,\gamma^-,0),\gamma^n\bar{\eta}_u(\gamma^{-n}(\gamma^+,z,t_z+nt_\gamma))) = 0.$$

Since $\bar{\eta}_u$ is Γ-equivariant, this implies that

$$\lim_{n\to\infty} d(\bar{\eta}_u(\gamma^+,\gamma^-,0),\bar{\eta}_u(\gamma^+,z,t_z+nt_\gamma)) = 0.$$

Since $\bar{\eta}_u(\gamma^+,z,t)$ does not depend on t, we finally obtain that

$$\bar{\eta}(u,(\gamma^+,\gamma^-,0)) = \bar{\eta}(u,(\gamma^+z,t))$$

for any $z \in \partial_\infty\Gamma$, $u \in D_1$ and $t \in \mathbb{R}$. Since, fixed points of infinite order elements are dense in $\partial_\infty\Gamma$ and $\bar{\eta}$ is continuous, we see that $\bar{\eta}(u,(x,y,t))$ does not depend on y or t.

Therefore, we obtain an α-Hölder transversely complex analytic map

$$\xi : D_1 \times \partial_\infty\Gamma \to \mathsf{G}/\mathsf{P}$$

which extends ξ_0. The map ξ satisfies properties (2) and (3), since ξ is α-Hölder transversely complex analytic, while property (4) follows from Lemma 6.8.

It remains to prove that we may restrict to a sub disk D_0 of D_1 so that if $u \in D_0$, then ρ_u is (G,P)-Anosov with limit map ξ_u. Let Q be a parabolic subgroup of G which is opposite to P. Then there exists a Lipschitz transversely complex analytic G/Q-bundle A' over $D \times \mathsf{U}_0\Gamma$ and we may lift the geodesic flow to a flow $\{\Phi'_t\}$ on A'. Since ρ_0 is (G,P)-Anosov, there exists a map $\theta_0 : \partial_\infty\Gamma \to \mathsf{G}/\mathsf{Q}$ which gives rise to a section σ'_0 of A' over $\{0\} \times \mathsf{U}_0\Gamma$ such that the flow is contracting on a neighborhood of $\sigma'_0(\{0\} \times \mathsf{U}_0\Gamma)$. We again apply Corollary 6.5 to find an α'-Hölder (for some $\alpha' > 0$) transversely complex analytic flow invariant section $\eta' : D_2 \times \mathsf{U}_0\Gamma \to A'$ that extends σ'_0, for some sub-disk D_2 of D which contains 0, such that the flow $\{\Psi'_t\}_{t\in\mathbb{R}}$ contracts along $\eta'(D_2 \times \mathsf{U}_0\Gamma)$. The section η' lifts to a section of $\tilde{\eta}'$ of $\tilde{A}'$ which we may reinterpret as a map $\bar{\eta}' : D_2 \times \widetilde{\mathsf{U}_0\Gamma} \to \mathsf{G}/\mathsf{Q}$ so that $\bar{\eta}'(u,(x,y,t))$

depends only on u and y. So we obtain an α'-Hölder transversely complex analytic map

$$\theta : D_2 \times \partial_\infty \Gamma \to \mathsf{G}/\mathsf{Q}$$

which restricts to θ_0. Since ξ_0 and θ_0 are transverse, we may find a sub-disk D_0 of $D_1 \cap D_2$ so that ξ_u and θ_u are transverse if $u \in D_0$. It follows that if $u \in D_0$, then ρ_u is (G, P)-Anosov with limit maps ξ_u and θ_u. Notice that ξ is unique, since limit maps of Anosov representations are unique ([29, Lemma 3.3]). This completes the proof of Theorem 6.1 in the complex analytic case.

Remark: Notice that the same proof applies to a C^k-family $\{\rho_u\}_{u\in D}$ of representations of a hyperbolic group Γ into a real semi-simple Lie group G such that ρ_0 is (G, P)-Anosov. It produces a sub-disk D_0 and a α-Hölder transversely C^k map $\xi : D_0 \times \partial_\infty \Gamma \to \mathsf{G}/\mathsf{P}$ so that if $u \in D_0$, then ρ_u is (G, P)-Anosov with limit map ξ_u.

6.3. **Analytic variation of the reparameterization.** We now turn to the proof of Proposition 6.2.

Let $\{\rho_u : \Gamma \to \mathsf{SL}_m(\mathbb{R})\}_{u\in D}$ be a real analytic family of projective Anosov representations and let $D^{\mathbb{C}}$ be the complexification of D. We may extend $\{\rho_u\}_{u\in D}$ to a complex analytic family $\{\rho_u : \Gamma \to \mathsf{SL}_m(\mathbb{C})\}_{u\in D^{\mathbb{C}}}$ of homomorphisms. Theorem 6.1 implies that, after possibly restricting $D^{\mathbb{C}}$, there exists a α-Hölder transversely complex analytic map

$$\xi : D^{\mathbb{C}} \times \partial_\infty \Gamma \to \mathsf{G}^{\mathbb{C}}/\mathsf{P}^{\mathbb{C}} = \mathbb{CP}(m)$$

such that if $u \in D^{\mathbb{C}}$, then ρ_u is Anosov with respect to the parabolic subgroup $\mathsf{P}^{\mathbb{C}}$, which is the stabilizer of a complex line, with limit map ξ_u. (We call such representations *complex projective Anosov*.)

We construct a Lipschitz transversely complex analytic $\mathbb{C}^m$-bundle $W^{\mathbb{C}}$ over $D^{\mathbb{C}} \times \mathsf{U}_0\Gamma$ which is the quotient of $\tilde{W}^{\mathbb{C}} = D^{\mathbb{C}} \times \widetilde{\mathsf{U}_0\Gamma} \times \mathbb{C}^m$ associated to the family $\{\rho_u\}_{u\in D^{\mathbb{C}}}$. We can then lift the Gromov geodesic flow on $\mathsf{U}_0\Gamma$ to a Lipschitz transversely complex analytic flow $\{\Psi_t\}_{t\in\mathbb{R}}$ on $W^{\mathbb{C}}$. Since the functions in the partition of unity for our trivializations of $W^{\mathbb{C}}$ are constant in the the $D^{\mathbb{C}}$ direction, we have:

Proposition 6.9. *After possibly further restricting $D^{\mathbb{C}}$, the bundle $W^{\mathbb{C}}$ is equipped with a Lipschitz transversely complex analytic 2-form ω of type $(1,1)$ such that*

$$\tau(u, v) = \omega(u, v) + \overline{\omega(v, u)},$$

is Hermitian.

Let $L^{\mathbb{C}}$ be the (complex) line sub-bundle of $W^{\mathbb{C}}$ determined by ξ, i.e. $L^{\mathbb{C}}$ is the quotient of the line sub-bundle of $\tilde{W}^{\mathbb{C}}$ whose fiber over $(u, (x, y, t)) \in D^{\mathbb{C}} \times \widetilde{\mathsf{U}_0\Gamma}$ is the complex line $\xi_u(x)$. Then, $L^{\mathbb{C}}$ is a α-Hölder transversely complex analytic line bundle over $D^{\mathbb{C}} \times \mathsf{U}_0\Gamma$. Since each ρ_u is complex projective Anosov with limit map ξ_u, $L^{\mathbb{C}}$ is preserved by the flow $\{\Psi_t\}_{t\in\mathbb{R}}$. We restrict ω and τ to $L^{\mathbb{C}}$ (and still denote them by ω and τ).

Since $L^{\mathbb{C}}$ is a line bundle, we can consider the function

$$a : D^{\mathbb{C}} \times \mathsf{U}_0\Gamma \to \mathbb{C}$$

such that

$$\omega(u, x)(v, v) = a(u, x)\tau(u, x)(v, v).$$

whenever v is in the fiber of $L^{\mathbb{C}}$ over (u,x). Concretely.

$$a(u,x) = \frac{\omega(v,v)}{2\Re(\omega(v,v))}$$

for any non-trivial v in the fiber over (u,x).

We observe that a is α-Holder transversely real analytic. If U is an open subset of $\mathsf{U}_0\Gamma$ in one of our trivializing sets, we can construct a non-zero section

$$V : D^{\mathbb{C}} \times U \to L^{\mathbb{C}}$$

which is α-Hölder transversely complex analytic. Then

$$\omega(V,V) : D^{\mathbb{C}} \times U \to \mathbb{C}$$

is α-Hölder transversely complex analytic. Lemma 6.8 implies that the map from $D^{\mathbb{C}}$ to $C^{\alpha}(U,\mathbb{C})$ given by $u \to \omega(V(u,\cdot),V(u,\cdot))$ is complex analytic. Therefore, the map from $D^{\mathbb{C}}$ to $C^{\alpha}(U,\mathbb{R})$ given by $u \to \Re(\omega(V(u,\cdot),V(u,\cdot)))$ is real analytic. It follows that the map from $D^{\mathbb{C}}$ to $C^{\alpha}(U,\mathbb{C})$ given by $u \to a(u,\cdot)$ is real analytic since

$$a|_{D^{\mathbb{C}}\times U} = \frac{\omega(V,V)}{2\Re(\omega(V,V))}.$$

Since x was arbitrary the map from $D^{\mathbb{C}}$ to $C^{\alpha}(\mathsf{U}_0\Gamma,\mathbb{C})$ given by $u \to a(u,\cdot)$ is real analytic. Similarly, a itself is α-Holder transversely real analytic.

If we define, for all t, the map

$$h_t : D^{\mathbb{C}} \times \mathsf{U}_0\Gamma \to \mathbb{C}$$

so that

$$\Psi_t^*\omega = h_t\omega,$$

then, we may argue, just as above, that h_t is α-Hölder transversely complex analytic. Lemma 6.8 guarantees that the map from $D^{\mathbb{C}}$ to $C^{\beta}(\mathsf{U}_0\Gamma,\mathbb{C})$ given by $u \to h_t(u,\cdot)$ is complex analytic.

If $t \in \mathbb{R}$,

$$\Psi_t^*\tau(\cdot) = 2\Re(\Psi_t^*\omega(\cdot)) = 2\Re(h_t(\cdot)\omega(\cdot)) = 2\Re\left(a(\cdot)h_t(\cdot)\right)G(\cdot).$$

We define $k_t(\cdot) = \Re(ah_t)(\cdot)$ and note that $\Psi_t^*\tau = k_t\tau$. Then, k_t is α-Hölder transversely real analytic and the map from $D^{\mathbb{C}}$ to $C^{\alpha}(\mathsf{U}_0\Gamma,\mathbb{R})$ given by $u \to k_t(u,\cdot)$ is real analytic (since it is the real part of a product of a real analytic and a complex analytic function).

We apply the construction of Lemma 4.3 to produce an α-Hölder transversely real analytic metric τ^0 on $\hat{L}$ such that

$$\Psi_t^*(\tau^0) < e^{-\beta t}\tau^0.$$

for some $\beta > 0$ and all $t > 0$. Concretely,

$$\tau^0 = \int_0^{t_0} e^{\beta s}\Psi_s^*(\tau)\mathrm{d}s = \left(\int_0^{t_0} e^{\beta s}k_s\ \mathrm{d}s\right)\tau$$

for some appropriately chosen $t_0 > 0$.

We define, for all t, $K_t : D^{\mathbb{C}} \times \mathsf{U}_0\Gamma \to \mathbb{R}$ by

$$K_t = e^{-\beta t}\frac{\int_t^{t_0+t} e^{\beta s}k_s\ \mathrm{d}s}{\int_0^{t_0} e^{\beta s}k_s\ \mathrm{d}s}.$$

One then checks that

$$\Psi_t^*(\tau^0) = K_t\tau^0$$

for all t. Then, for each $u \in D^{\mathbb{C}}$ we define $f_u : \mathsf{U}_0\Gamma \to \mathbb{R}$, by setting

$$f_u(\cdot) = \frac{\partial K_t}{\partial t}(u, \cdot, 0) = -\beta + \frac{e^{\beta t_0} k_{t_0}(\cdot) - 1}{\int_0^{t_0} e^{\beta s} k_s(\cdot)\, \mathrm{d}s}.$$

Then, since $u \to k_t(u, \cdot)$ is real analytic for all t, our formula for f_u guarantees that the map from $D^{\mathbb{C}}$ to $C^\beta(\mathsf{U}_0\Gamma, \mathbb{R})$ given by $u \to f_u$ is real analytic. Therefore, the restriction of this map to the real submanifold D is also real analytic.

To complete the proof of Proposition 6.2 we will show that, for each $u \in D$, the periods of the reparameterization of $\mathsf{U}_0\Gamma$ by f_u and the periods of $U_{\rho_u}\Gamma$ agree. Livšic's Theorem 3.3 then implies that the reparameterization of $\mathsf{U}_0\Gamma$ by f_u is Hölder conjugate to $U_{\rho_u}\Gamma$ as desired.

For $u \in D$, let $j_u : \mathsf{U}_0\Gamma \times \mathbb{R}$ be given by $j_u(\cdot, t) = \log K_t(u, \cdot)$. We can differentiate the equality

$$j_u(\cdot, t+s) = j_u(\Psi_s(\cdot), t) + j_u(\cdot, s)$$

with respect to t and evaluate at $t = 0$ to conclude that

$$f_u(\cdot, s) = f_u(\Psi_s(\cdot), 0).$$

In particular, for any t,

$$\int_0^t (f_u(\Psi_s(\cdot), 0)\, \mathrm{d}s = j_u(\cdot, t).$$

Let $\gamma \in \Gamma$ and let $x \in \mathsf{U}_0\Gamma$ be a point on the periodic orbit associated to γ (which is simply the quotient of $(\gamma^+, \gamma^-) \times \mathbb{R} \subset \widetilde{\mathsf{U}_0\Gamma}$). If t_γ is the period of the orbit of $\mathsf{U}_0\Gamma$ containing x, then

$$e^{\int_0^{t_\gamma} f_u(\Psi_s(u,x))\mathrm{d}s}\tau^0(x, u) = \Psi_{t_\gamma}^*\tau^0(u, x) = e^{\Lambda(\rho_u,\gamma)}\tau^0(u, x),$$

so

$$\int_0^{t_\gamma} f_u(\Psi_s(u, x))\mathrm{d}s = \Lambda(\rho_u, \gamma)$$

is the period of the reparameterization of the flow $\mathsf{U}_0\Gamma$ by f_u, which agrees with the period of the orbit in $U_{\rho_u}\Gamma$ associated to γ (see Proposition 4.1). This completes the proof of Proposition 6.2.

Remark: Notice that a simpler version of the above proof establishes that given a C^k family of projective Anosov representations, one may, at least locally, choose the reparameterization functions to vary C^{k-1}.

7. Deformation spaces of projective Anosov representations

In this section, we collect a few facts about the structure of deformation spaces of projective Anosov representations of Γ into $\mathsf{SL}_m(\mathbb{R})$ and their relatives.

7.1. **Irreducible projective Anosov representations.** We first observe that our deformation spaces $\mathcal{C}(\Gamma, m)$ and $\mathcal{C}_g(\Gamma, \mathsf{G})$ are real analytic manifolds. Let $\widetilde{\mathcal{C}}(\Gamma, m) \subset \mathrm{Hom}(\Gamma, \mathsf{SL}_m(\mathbb{R}))$ denote the set of regular, irreducible, projective Anosov representations and let

$$\mathcal{C}(\Gamma, m) = \widetilde{\mathcal{C}}(\Gamma, m)/\mathsf{SL}_m(\mathbb{R}).$$

If G is a reductive subgroup of $\mathsf{SL}_m(\mathbb{R})$, then we similarly let $\widetilde{\mathcal{C}}_g(\Gamma, \mathsf{G}) \subset \mathrm{Hom}(\Gamma, \mathsf{G})$ denote the space of G-generic, regular representations which are irreducible and projective Anosov when viewed as representations into $\mathsf{SL}_m(\mathbb{R})$. Let

$$\mathcal{C}_g(\Gamma, \mathsf{G}) = \widetilde{\mathcal{C}}_g(\Gamma, \mathsf{G})/\mathsf{G}.$$

Proposition 7.1. *Suppose that Γ is a word hyperbolic group. Then*

(1) *The deformation spaces $\mathcal{C}(\Gamma, m)$ and $\mathcal{C}_g(\Gamma, \mathsf{SL}_m(\mathbb{R}))$ have the structure of a real analytic manifold compatible with the algebraic structure on* $\mathrm{Hom}(\Gamma, \mathsf{SL}_m(\mathbb{R}))$
(2) *If G is a reductive subgroup of $\mathsf{SL}_m(\mathbb{R})$, then $\mathcal{C}_g(\Gamma, \mathsf{G})$ has the structure of a real analytic manifold compatible with the algebraic structure on* $\mathrm{Hom}(\Gamma, \mathsf{G})$.

Proof. We may regard $\mathrm{Hom}(\Gamma, \mathsf{SL}_m(\mathbb{R}))$ as a subset of $\mathrm{Hom}(\Gamma, \mathsf{SL}_m(\mathbb{C}))$. We first notice that an irreducible homomorphism in $\mathrm{Hom}(\Gamma, \mathsf{SL}_m(\mathbb{R}))$ is also irreducible when regarded as a homomorphism in $\mathrm{Hom}(\Gamma, \mathsf{SL}_m(\mathbb{C}))$. Lubotzky and Magid ([54, Proposition 1.21 and Theorem 1.28]) proved that the set of irreducible homomorphisms form an open subset of $\mathrm{Hom}(\Gamma, \mathsf{SL}_m(\mathbb{C}))$, so they also form an open subset of $\mathrm{Hom}(\Gamma, \mathsf{SL}_m(\mathbb{R}))$. Results of Labourie [44, Prop. 2.1] and Guichard-Wienhard [29, Theorem 5.13] imply that the set of projective Anosov homomorphisms is an open subset of $\mathrm{Hom}(\Gamma, \mathsf{SL}_m(\mathbb{R}))$ (see also Proposition 6.1). Therefore, $\widetilde{\mathcal{C}}(\Gamma, m)$ is an open subset of $\mathrm{Hom}(\Gamma, \mathsf{SL}_m(\mathbb{R}))$. Since the former consists of regular homomorphisms, it is an analytic manifold.

Lubotzky–Magid ([54, Theorem 1.27]) also proved that $\mathsf{SL}_m(\mathbb{C})$ acts properly (by conjugation) on the set of irreducible representations in $\mathrm{Hom}(\Gamma, \mathsf{SL}_m(\mathbb{C}))$. It follows that $\mathsf{SL}_m(\mathbb{R})$ acts properly on $\widetilde{\mathcal{C}}(\Gamma, m)$. Schur's Lemma guarantees that the centralizer of an irreducible representation is contained in the center of $\mathsf{SL}_m(\mathbb{R})$. Therefore, $\mathsf{PSL}_m(\mathbb{R})$ acts freely, analytically and properly on the analytic manifold $\widetilde{\mathcal{C}}(\Gamma, m)$, so its quotient $\mathcal{C}(\Gamma, m)$ is also an analytic manifold.

Since the set of G-generic elements of G is an open G-invariant subset of G, we may argue exactly as above to show that $\widetilde{\mathcal{C}}_g(\Gamma, \mathsf{G})$ is an open subset of $\mathrm{Hom}(\Gamma, \mathsf{G})$ which is an analytic manifold. The action of $\mathsf{G}/Z(\mathsf{G})$ on $\widetilde{\mathcal{C}}_g(\Gamma, \mathsf{G})$ is again free, analytic and proper, so its quotient $\mathcal{C}_g(\Gamma, \mathsf{G})$ is again an analytic manifold. □

If $\rho \in \widetilde{\mathcal{C}}(\Gamma, m)$, then one may identify $\mathsf{T}_\rho\widetilde{\mathcal{C}}(\Gamma, m)$ with the space $Z^1_\rho(\Gamma, \mathfrak{sl}_m(\mathbb{R}))$ of cocycles and one may then identify $\mathsf{T}_{[\rho]}\mathcal{C}(\Gamma, m)$ with the cohomology group $H^1_\rho(\Gamma, \mathfrak{sl}_m(\mathbb{R}))$ (see [54, 36]). In particular, the space $B^1_\rho(\Gamma, \mathfrak{sl}_m(\mathbb{R}))$ is identified with the tangent space of the $\mathsf{SL}_m(\mathbb{R})$-orbit of ρ. Similarly, if $\rho \in \widetilde{\mathcal{C}}_g(\Gamma, \mathsf{G})$, we identify $\mathsf{T}_\rho\widetilde{\mathcal{C}}_g(\Gamma, \mathsf{G})$ with $Z^1(\Gamma, \mathfrak{g})$ and $\mathsf{T}_{[\rho]}\mathcal{C}_g(\Gamma, \mathsf{G})$ with $H^1_\rho(\Gamma, \mathfrak{g})$. More generally, if ρ is an irreducible representation in $\mathrm{Hom}(\Gamma, \mathsf{G})$, the tangent vector to any analytic path through ρ may be identified with an element of $Z^1_\rho(\Gamma, \mathfrak{g})$ (see [36, Section 2]).

A simple calculation in cohomology gives that irreducible projective Anosov representations of fundamental groups of 3-manifolds with non-empty boundary are regular. These include free groups and fundamental groups of closed surfaces.

Proposition 7.2. *If Γ is isomorphic to the fundamental group of a compact orientable 3-manifold M with non empty boundary, then $\mathcal{C}(\Gamma, m)$ is the set of conjugacy classes of irreducible projective Anosov representations.*

Proof. Let $\Gamma = \pi_1(M)$ where M is a compact orientable 3-manifold with non-empty boundary. It suffices to show that the open subset of $\mathrm{Hom}(\Gamma, \mathsf{SL}_m(\mathbb{R}))$ consisting of irreducible projective Anosov homomorphisms consists entirely of regular points. We recall that $\rho_0 \in \mathrm{Hom}(\Gamma, \mathsf{SL}_m(\mathbb{R}))$ is regular if there exists a neighborhood U of ρ_0 so that $\dim(Z^1_\rho(M, \mathfrak{g}))$ is constant on U and the centralizer of any representation $\rho \in U$ is trivial [54].

If ρ_0 is projective Anosov and irreducible, we can take U to be any open neighborhood of ρ_0 consisting of irreducible projective Anosov representations. Since $\rho \in U$ is irreducible, Schur's Lemma guarantees that the centralizer of $\rho(\Gamma)$ is the center of $\mathsf{SL}_m(\mathbb{R})$. Moreover, if $\rho \in U$, then

$$\dim(H^0_\rho(M, \mathfrak{g})) - \dim(H^1_\rho(M, \mathfrak{g})) + \dim(H^2_\rho(M, \mathfrak{g})) = \chi(M)\dim(\mathsf{G}).$$

Since the centralizer is trivial, $\dim(H^0_\rho(M, \mathfrak{g})) = 0$. By Poincaré duality, $\dim(H^2_\rho(M, \mathfrak{g})) = \dim(H^0_\rho(M, \partial M, \mathfrak{g}))$. Since $\dim(H^0_\rho(M, \mathfrak{g})) = 0$, the long exact sequence for relative homology implies that $\dim(H^0_\rho(M, \partial M, \mathfrak{g})) = 0$. Thus,

$$\dim(H^1_\rho(M, \mathfrak{g})) = -\chi(M)\dim(\mathsf{G}).$$

Therefore, $\dim(Z^1_\rho(M, \mathfrak{g})) = (1 - \chi(M))\dim(\mathsf{G})$ for all $\rho \in U$, so ρ is a regular point. □

7.2. **Virtually Zariski dense representations.** We recall that if Γ is a word hyperbolic group, G is a semi-simple Lie group with finite center and P is a non-degenerate parabolic subgroup, then $\mathcal{Z}(\Gamma; \mathsf{G}, \mathsf{P})$ is the space of (conjugacy classes of) regular virtually Zariski dense (G, P)-Anosov representations of Γ into G. We will prove that $\mathcal{Z}(\Gamma; \mathsf{G}, \mathsf{P})$ is a real analytic orbifold.

Proposition 7.3. *Suppose that Γ is a word hyperbolic group, G is a semi-simple Lie group with finite center and P is a non-degenerate parabolic subgroup of G. Then $\mathcal{Z}(\Gamma; \mathsf{G}, \mathsf{P})$ is a real analytic orbifold.*

Moreover, if G is connected, then $\mathcal{Z}(\Gamma; \mathsf{G}, \mathsf{P})$ is a real analytic manifold.

Proof. Let $\mathrm{Hom}^*(\Gamma, \mathsf{G})$ be the set of regular homomorphisms. By definition, $\mathrm{Hom}^*(\Gamma, \mathsf{G})$ is an open subset of $\mathrm{Hom}(\Gamma, \mathsf{G})$ and hence it is an analytic manifold, since it is the set of smooth points of a real algebraic variety. Results of Labourie [44, Prop. 2.1] and Guichard-Wienhard [29, Theorem 5.13] again imply that the set of (G, P)-Anosov homomorphisms is open in $\mathrm{Hom}^*(\Gamma, \mathsf{G})$. The main difficulty in the proof is to show that the set $\widetilde{\mathcal{Z}}(\Gamma; \mathsf{G}, \mathsf{P})$ of virtually Zariski dense Anosov homomorphisms is open in $\mathrm{Hom}^*(\Gamma, \mathsf{G})$ and hence an analytic manifold.

Once we have shown that $\widetilde{\mathcal{Z}}(\Gamma; \mathsf{G}, \mathsf{P})$ is an analytic manifold, we may complete the proof in the same spirit as the proof of Proposition 7.1. We observe that if $\rho \in \widetilde{\mathcal{Z}}(\Gamma; \mathsf{G}, \mathsf{P})$ then its centralizer is finite, since the Zariski closure of $\rho(\Gamma)$ has finite index in G. Then, $\mathsf{G}/Z(\mathsf{G})$ acts properly and analytically on $\widetilde{\mathcal{Z}}(\Gamma; \mathsf{G}, \mathsf{P})$ with finite point stabilizers, so the quotient $\mathcal{Z}(\Gamma; \mathsf{G}, \mathsf{P})$ is an analytic orbifold. If G^0 is the connected component of G, then the Zariski closure of any representation $\rho \in \widetilde{\mathcal{Z}}(\Gamma; \mathsf{G}, \mathsf{P})$ contains G^0, so the intersection of the centralizer of ρ with G^0 is simply $Z(\mathsf{G}) \cap \mathsf{G}^0$. Therefore, $\widetilde{\mathcal{Z}}(\Gamma; \mathsf{G}, \mathsf{P})/\mathsf{G}^0$ is an analytic manifold. In particular, if G is connected, $\mathcal{Z}(\Gamma; \mathsf{G}, \mathsf{P})$ is an analytic manifold.

We complete the proof by showing that the set of virtually Zariski dense (G, P)-Anosov homomorphisms is open in $\mathrm{Hom}^*(\Gamma, \mathsf{G})$. If not, then there exists a sequence $\{\rho_m\}_{m\in\mathbb{N}}$ of (G, P)-Anosov representations which are not virtually Zariski dense converging to a virtually Zariski dense (G, P)-Anosov representation ρ_0.

Since G has finitely many components, $\rho_n^{-1}(\mathsf{G}^0)$ has bounded finite index for all n. Since Γ is finitely generated, it contains only finitely many subgroups of a given index, so we may pass to a finite index subgroup Γ_0 of Γ so that $\rho_n(\Gamma_0)$ is contained in the identity component G^0 of G for all n. Since each $\rho_n|_{\Gamma_0}$ is (G, P)-Anosov and $\rho_0(\Gamma_0)$ is also virtually Zariski dense, we may assume for the remainder of the proof that G is the Zariski closure of G^0.

Let Z_n be the Zariski closure of $\mathrm{Im}(\rho_n)$ and let $\mathfrak{z}_n$ be the Lie algebra of Z_n. Consider the decomposition of the Lie algebra $\mathfrak{g}$ of G

$$\mathfrak{g} = \bigoplus_{i=1}^{p} \mathfrak{g}_i,$$

where $\mathfrak{g}_i$ are simple Lie algebras. Let $\mathsf{G}_i = \mathrm{Aut}(\mathfrak{g}_i)$. We consider the adjoint representation $\mathrm{Ad} : \mathsf{G} \to \mathrm{Aut}(\mathfrak{g})$. Let H be the subgroup of G consisting of all $g \in \mathsf{G}$ so that $\mathrm{Ad}(g)$ preserves the factors of $\mathfrak{g}$. Then H is a finite index, Zariski closed subgroup of G. Hence, with our assumptions, $\mathsf{H} = \mathsf{G}$. Therefore, we get a well-defined projection map $\pi_i : \mathsf{G} \to \mathsf{G}_i$. If $\mathfrak{p}$ is the Lie algebra of P, then $\mathfrak{p} = \bigoplus_{i=1}^{p} \mathfrak{p}_i$, where $\mathfrak{p}_i$ is a Lie subalgebra of $\mathfrak{g}_i$. Let P_i be the stabilizer of $\mathfrak{p}_i$ in G_i. Then we also obtain a G-equivariant projection, also denoted π_i,

$$\pi_i : \mathsf{G}/\mathsf{P} \to \mathsf{G}_i/\mathsf{P}_i = \mathsf{G}\mathfrak{p}_i \subset \mathrm{Gr}_{\dim(\mathfrak{p}_i)}(\mathfrak{g}_i)$$

where $\mathrm{Gr}_{\dim(\mathfrak{p}_i)}(\mathfrak{g}_i)$ is the Grassmanian space of $\dim(\mathfrak{p}_i)$-dimensional vector spaces in $\mathfrak{g}_i$.

If $\xi_n : \partial_\infty\Gamma \to \mathsf{G}/\mathsf{P}$ is the limit map of ρ_n, $\pi_i \circ \xi_n$ is a ρ_n-equivariant map from $\partial_\infty\Gamma$ to $\mathsf{G}_i/\mathsf{P}_i$. If $\pi_i \circ \xi_n$ is constant, then $\rho_n(\Gamma)$ would normalize a conjugate of $\mathfrak{p}_i$. So, if $\pi_i \circ \xi_n$ is constant for infinitely many n, then $\rho_0(\Gamma)$ would normalize a conjugate of $\mathfrak{p}_i$,which is impossible since $\rho_0(\Gamma)$ is Zariski dense and P_i is a proper parabolic subgroup of G_i. Therefore, we may assume that $\pi_i \circ \xi_n$ is non-constant for all i and all n. Since Γ acts topologically transitively on $\partial_\infty\Gamma$, we then know that the image must then be infinite. Therefore, for all i and n,

$$\dim(\pi_i(\mathfrak{z}_n)) > 0. \tag{38}$$

We may thus assume that $\{\mathfrak{z}_n\}$ converges to a proper Lie subalgebra $\mathfrak{z}_0$ which is normalized by $\rho_0(\Gamma)$ with

$$\dim(\mathfrak{z}_0) > 0. \tag{39}$$

Since ρ_0 is virtually Zariski dense, $\mathfrak{z}_0$ must be a strict factor in the Lie algebra $\mathfrak{g}$ of G. Thus, after reordering, we may assume that

$$\mathfrak{z}_0 = \bigoplus_{i=1}^{q} \mathfrak{g}_i. \tag{40}$$

For n large enough, $\mathfrak{z}_n$ is thus a graph of an homomorphim

$$f_n : \mathfrak{z}_0 \to \mathfrak{h} = \bigoplus_{i=q+1}^{p} \mathfrak{g}_i.$$

Since there are only finitely many conjugacy classes (under the adjoint representation) of homomorphisms of $\mathfrak{z}_0$ into $\mathfrak{h}$, we may pass to a subsequence such that

$$f_n = \mathrm{Ad}(g_n) \circ f_0 \circ \pi_{\mathfrak{h}_1},$$

where f_0 is a fixed isomorphism from an ideal $\mathfrak{h}_1$ in $\mathfrak{z}_0$ to an ideal $\mathfrak{h}_2$ in $\mathfrak{h}$, $\pi_{\mathfrak{h}_1}$ is the projection from $\mathfrak{z}_0$ to $\mathfrak{h}_1$ and $g_n \in \mathsf{H}_2$ where H_i is the subgroup of G whose Lie algebra is $\mathfrak{h}_i$.

Let Z_0 be the subgroup of G whose Lie algebra is $\mathfrak{z}_0$ and consider $\mathsf{A}_1 = \exp \mathfrak{a}_{\mathsf{Z}_0}$, where $\mathfrak{a}_{\mathsf{Z}_0}$ is a Cartan subspace of $\mathfrak{z}_0$, and let $\mathsf{A}_2 = \exp \mathfrak{a}_{\mathsf{H}_2}$, where the Cartan subspace $\mathfrak{a}_{\mathsf{H}_2}$ is chosen so that $f_0(\pi_{\mathfrak{h}_1}(\mathsf{A}_1)) = \mathsf{A}_2$. Considering the Cartan decomposition $\mathsf{H}_2 = \mathsf{K}\mathsf{A}_2\mathsf{K}$ of H_2 where K is a maximal compact subgroup, we may write $g_n = k_n a_n c_n$ with $a_n \in \mathsf{A}_2$ and $k_n, c_n \in \mathsf{K}$. Moreover we may write $\mathrm{Ad}(c_n) = f_0(\mathrm{Ad}(d_n))$, where d_n lies in a fixed compact subgroup of H_1. Thus, if $u \in \mathfrak{a}_{\mathsf{Z}_0}$, since A_2 is commutative, we have

$$f_n(\mathrm{Ad}(d_n^{-1})u) = \mathrm{Ad}(g_n) f_0(\mathrm{Ad}(d_n^{-1})u) = \mathrm{Ad}(k_n) f_0(u).$$

We may extract a subsequence so that that $\{k_n\}_{n\in\mathbb{N}}$ and $\{d_n\}_{n\in\mathbb{N}}$ converge respectively to k_0 and d_0. Therefore,

$$\{(\mathrm{Ad}(d_0^{-1})u, \mathrm{Ad}(k_0) f_0(u)) \mid u \in \mathfrak{a}_{\mathsf{Z}_0}\} \subset \mathfrak{z}_0,$$

which contradicts the fact that $\mathfrak{z}_0 = \bigoplus_{i=1}^q \mathfrak{g}_i$. This contradiction establishes the fact that the set of Anosov, virtually Zariski dense regular homomorphisms is open, which completes the proof. □

We record the following observation, established in the proof of Proposition 7.3 which will be useful in the proof of Corollary 1.9.

Proposition 7.4. *Suppose that Γ is a word hyperbolic group, G is a semi-simple Lie group with finite center and P is a non-degenerate parabolic subgroup of G. Then $\widetilde{\mathcal{Z}}(\Gamma; \mathsf{G}, \mathsf{P})/\mathsf{G}^0$ is an analytic manifold.*

7.3. **Kleinian groups.** Let $\mathcal{C}_c(\Gamma, \mathsf{PSL}_2(\mathbb{C}))$ be the set of (conjugacy classes of) convex cocompact representations of Γ into $\mathsf{PSL}_2(\mathbb{C})$). We say that a convex cocompact representation ρ in $\mathsf{PSL}_2(\mathbb{C})$ is *Fuchsian* if its image is conjugate into $\mathsf{PSL}_2(\mathbb{R})$. Since every non-elementary Zariski closed, connected subgroup of $\mathsf{PSL}_2(\mathbb{C})$ is conjugate to $\mathsf{PSL}_2(\mathbb{R})$, we note that $\rho \in \mathcal{C}_c(\Gamma, \mathsf{PSL}_2(\mathbb{C}))$ is Zariski dense unless ρ is *virtually Fuchsian*, i.e. there exists a finite index subgroup of $\rho(\Gamma)$ which is conjugate into $\mathsf{PSL}_2(\mathbb{R})$ (see also Johnson-Millson [36, Lemma 3.2]). Notice that if ρ is virtually Fuchsian, then $\rho(\Gamma)$ contains a finite index subgroup which is isomorphic to a free group or a closed surface group.

Bers [8] proved that $\mathcal{C}_c(\Gamma, \mathsf{PSL}_2(\mathbb{C}))$ is a complex analytic manifold. which has real dimension $-6\chi(\Gamma)$ if Γ is torsion-free. (See also Kapovich [37, Section 8.8] where a proof of this is given in the spirit of Proposition 7.1.) We summarize these results in the following proposition.

Proposition 7.5. *Let Γ be a word hyperbolic group. Then*

(1) *$\mathcal{C}_c(\Gamma, \mathsf{PSL}_2(\mathbb{C}))$ is a smooth analytic manifold.*
(2) *$\rho \in \mathcal{C}_c(\Gamma, \mathsf{PSL}_2(\mathbb{C}))$ is Zariski dense if and only if ρ is not virtuallyFuchsian.*
(3) *If Γ is torsion-free, then $\mathcal{C}_c(\Gamma, \mathsf{PSL}_2(\mathbb{C}))$ has dimension $-6\chi(\Gamma)$.*

7.4. **Hitchin components.** Let S be a closed orientable surface of genus at least 2 and let $\tau_m : \mathsf{PSL}_2(\mathbb{R}) \to \mathsf{PSL}_m(\mathbb{R})$ be an irreducible homomorphism. If $\rho : \pi_1(S) \to \mathsf{PSL}_2(\mathbb{R})$ is discrete and faithful, hence uniformizes S, then $\tau_m \circ \rho$ is called a *Fuchsian representation*. A representation $\rho : \pi_1(S) \to \mathsf{PSL}_m(\mathbb{R})$ that can be deformed into a Fuchsian representation is called a *Hitchin representation*. Lemma 10.1 of [44] implies that all Hitchin representations are irreducible.

Let $H_m(S)$ be the space of Hitchin representations into $\mathsf{PSL}_m(\mathbb{R})$ and let

$$\mathcal{H}_m(S) = H_m(S)/\mathsf{PGL}_m(\mathbb{R}).$$

Each $\mathcal{H}_m(S)$ is called a *Hitchin component* and Hitchin [33] proved that $H_m(S)$ is an analytic manifold diffeomorphic to $\mathbb{R}^{(m^2-1)|\chi(S)|}$.

One may identify the Teichmüller space $\mathcal{T}(S)$ with $\mathcal{H}_2(S)$. The irreducible representation gives rise to an analytic embedding that we also denote τ_m, of $\mathcal{T}(S)$ into the Hitchin component $\mathcal{H}_m(S)$ and we call its image the *Fuchsian locus* of the Hitchin component.

Each Hitchin representation lifts to a representation into $\mathsf{SL}_m(\mathbb{R})$. Labourie [44] showed that all lifts of Hitchin representations are irreducible and $(\mathsf{SL}_m(\mathbb{R}), \mathsf{B})$-Anosov where B is a minimal parabolic subgroup of $\mathsf{SL}_m(\mathbb{R})$. In particular, lifts of Hitchin representations are projective Anosov. Moreover, Labourie [44] showed that the image of every non-trivial element of $\pi_1(S)$ under the lift of a Hitchin representation is diagonalizable with distinct eigenvalues. In particular, every lift of a Hitchin representation is $\mathsf{SL}_m(\mathbb{R})$-generic, so is contained in $\mathcal{C}_g(\pi_1(S), \mathsf{SL}_m(\mathbb{R}))$. Moreover, notice that distinct lifts of a given Hitchin representation must be contained in distinct components of $\mathcal{C}_g(\pi_1(S), \mathsf{SL}_m(\mathbb{R}))$.

We summarize what we need from Hitchin and Labourie's work in the following result.

Theorem 7.6. *Every Hitchin component lifts to a component of the analytic manifold* $\mathcal{C}_g(\pi_1(S), \mathsf{SL}_m(\mathbb{R}))$.

8. Thermodynamic formalism on the deformation space of projective Anosov representations

In Section 8.1, we show that entropy, intersection and renormalized intersection vary analytically over $\mathcal{C}(\Gamma, m)$, then in section 8.2 we construct the thermodynamic mapping of $\mathcal{C}(\Gamma, m)$ into the space of Livšic cohomology classes of pressure zero functions on $\mathsf{U}_0\Gamma$ and use it to define the pressure form on $\mathcal{C}(\Gamma, m)$ and $\mathcal{C}_g(\Gamma, \mathsf{G})$.

8.1. **Analyticity of entropy and intersection.** Let Γ be a word hyperbolic group admitting a projective Anosov representation. By Proposition 5.7, the Gromov geodesic flow on $\mathsf{U}_0\Gamma$ admits a Hölder reparametrization which turns it into a topologically transitive metric Anosov flow. Since the Gromov geodesic flow is only well defined up to reparametrization, we choose a fixed Hölder reparametrization which gives rise to a topologically transitive metric Anosov flow, and use the corresponding flow, denoted by $\psi = \{\psi_t\}_{t\in\mathbb{R}}$, as a background flow on $\mathsf{U}_0\Gamma$.

Let $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ be a projective Anosov representation. By Proposition 4.1, the geodesic flow $(\mathsf{U}_\rho\Gamma, \{\phi_t\}_{t\in\mathbb{R}})$ of ρ is Hölder conjugate to a Hölder reparametrization of the flow $\{\psi_t\}_{t\in\mathbb{R}}$. Periodic orbits of $\{\phi_t\}_{t\in\mathbb{R}}$ are in one-to-one correspondence with conjugacy classes of infinite order elements of Γ. The periodic orbit associated to the conjugacy class $[\gamma]$ has period $\Lambda(\rho)(\gamma)$.

If $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ is projective Anosov, let $f_\rho : \mathsf{U}_0\Gamma \to \mathbb{R}$ be a Hölder function such that the reparameterization of $\mathsf{U}_0\Gamma$ by f_ρ is Hölder conjugate to $\mathsf{U}_\rho\Gamma$. Livšic's theorem 3.3 implies that the correspondence $\rho \mapsto f_\rho$ is well defined modulo Livšic cohomology and invariant under conjugation of the homomorphism ρ. Therefore, we may define

$$h(\rho_1) = h(f_{\rho_1}), \tag{41}$$

$$\mathbf{I}(\rho_1, \rho_2) = \mathbf{I}(f_{\rho_1}, f_{\rho_2}), \text{ and} \tag{42}$$

$$\mathbf{J}(\rho_1, \rho_2) = \mathbf{J}(f_{\rho_1}, f_{\rho_2}) = \frac{h(\rho_2)}{h(\rho_1)}\mathbf{I}(\rho_1, \rho_2), \tag{43}$$

for projective Anosov representations $\rho_1 : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ and $\rho_2 : \Gamma \to \mathsf{SL}_m(\mathbb{R})$. These quantities are well defined and agree with the definition given in the Introduction. Proposition 7.3.1 implies that

$$h(f_{\rho_1}) = \lim_{T\to\infty} \frac{1}{T} \log \sharp(R_T(\rho_1))$$

while equation (12) implies that

$$\mathbf{I}(f_{\rho_1}, f_{\rho_2}) = \lim_{T\to\infty} \left(\frac{1}{\sharp(R_T(\rho_1))} \sum_{[\gamma]\in R_T(\rho_1)} \frac{\log(\Lambda(\gamma)(\rho_2))}{\log(\Lambda(\gamma)(\rho_1))} \right).$$

Proposition 6.2 implies that if $\{\rho_u\}_{u\in D}$ is an analytic family of of projective Anosov homomorphisms defined on a disc D, then we can choose, at least locally, the map $u \mapsto f_{\rho_u}$ to be analytic. Proposition 3.12 then implies that entropy, intersection and renormalized intersection all vary analytically.

Proposition 8.1. *Given two analytic families $\{\rho_u\}_{u\in D}$ and $\{\eta_v\}_{v\in D'}$ of projective Anosov homomorphisms, the functions $u \mapsto h(\rho_u)$, $(u, v) \mapsto \mathbf{I}(\rho_u, \eta_v)$ and $(u, v) \mapsto \mathbf{J}(\rho_u, \eta_v)$ are analytic on their domains of definition.*

Combining Propositions 3.8, 3.9 and 3.11 one obtains the following.

Corollary 8.2. *For every pair $\rho_1 : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ and $\rho_2 : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ of projective Anosov representations, one has*

$$\mathbf{J}(\rho_1, \rho_2) \geqslant 1.$$

If $\mathbf{J}(\rho_1, \rho_2) = 1$, then there exists a constant $c \geqslant 1$ such that

$$\Lambda_{\rho_1}(\gamma)^c = \Lambda_{\rho_2}(\gamma)$$

for every $\gamma \in \Gamma$.

Moreover, if $\{\rho_t\}$ is a smooth one parameter family of projective Anosov representations and $\{f_t\}$ is an associated smooth family of reparametrizations, then

$$\left.\frac{\partial^2}{\partial t^2}\right|_{t=0} \mathbf{J}(\rho_0, \rho_t) = 0$$

if and only if

$$\left.\frac{\partial}{\partial t}\right|_{t=0} (h_{\rho_t} f_t)$$

is Livšic cohomologous to 0.

8.2. **The thermodynamic mapping and the pressure form.** If $\rho \in \mathcal{C}(\Gamma, m)$ and f_ρ is a reparametrization of the Gromov geodesic flow giving rise to the geodesic flow of ρ, we define $\Phi_\rho : \mathsf{U}_0\Gamma \to \mathbb{R}$ by

$$\Phi_\rho(x) = -h(\rho) f_\rho(x).$$

Lemma 3.1 implies that $\Phi_\rho \in \mathcal{P}(\mathsf{U}_0\Gamma)$. Let $\mathcal{H}(\mathsf{U}_0\Gamma)$ be the set of Livšic cohomology classes of pressure zero function, we saw that the class of Φ_ρ in $\mathcal{H}(\mathsf{U}_0\Gamma)$ only depends on ρ. We define the *thermodynamic mapping* to be

$$\mathfrak{T} : \left\{ \begin{array}{ccc} \mathcal{C}(\Gamma, m) & \to & \mathcal{H}(\mathsf{U}_0\Gamma) \\ \rho & \mapsto & [\Phi_\rho] \end{array} \right.$$

By Proposition 6.2, the thermodynamic mapping is "analytic" in the following sense: for every representation ρ in the analytic manifold $\mathcal{C}(\Gamma, m)$, there exists a neighborhood U of ρ in $\mathcal{C}(\Gamma, m)$ and an analytic mapping from U to $\mathcal{P}(\mathsf{U}_0\Gamma)$ which lifts the thermodynamic mapping.

We use the thermodynamic mapping to define a 2-tensor on our deformation spaces.

Definition 8.3. [Pressure Form] *Let $\{\rho_u\}_{u\in M}$ be an analytic family of projective Anosov homomorphims parametrized by an analytic manifold M. If $z \in M$, we define $\mathbf{J}_z : M \to \mathbb{R}$ by letting*

$$\mathbf{J}_z(u) = \mathbf{J}(\rho_z, \rho_u).$$

The associated pressure form $\mathbf{p}$ *on M is the 2-tensor such that if $v, w \in \mathsf{T}_z M$, then*

$$\mathbf{p}(v, w) = \mathrm{D}^2_z \mathbf{J}_z(v, w).$$

Notice that, by Corollary 8.2, the pressure form is non-negative.

In particular, we get pressure forms on $\widetilde{\mathcal{C}}(\Gamma, m)$ and on $\widetilde{\mathcal{C}}(\Gamma, \mathsf{G})$ when G is a reductive subgroup of $\mathsf{SL}_m(\mathbb{R})$. Since $\mathbf{J}$ is invariant under the action of conjugation on each variable, these pressure forms descend to 2-tensors, again called pressure forms, on the analytic manifolds $\mathcal{C}(\Gamma, m)$ and $\mathcal{C}_g(\Gamma, \mathsf{G})$.

9. Degenerate vectors for the pressure metric

In this section, we analyze the norm zero vectors for the pressure metric. If Γ is a word hyperbolic group, α is an infinite order element of Γ and $\{\rho_u\}_{u\in M}$ is an analytic family of projective Anosov homomorphisms parameterized by an analytic manifold M, one may view $\mathsf{L}(\alpha)$ as an analytic function on M where we abuse notation by letting $\mathsf{L}(\alpha)(u) = \mathsf{L}(\alpha)(\rho_u)$ denote the eigenvalue of $\rho_u(\alpha)$ of maximal modulus. The following is the main result of the section.

Proposition 9.1. *Let Γ be a word hyperbolic group and let G be a reductive subgroup of $\mathsf{SL}_m(\mathbb{R})$. Suppose that $\{\rho_u : \Gamma \to \mathsf{G}\}_{u\in D}$ is an analytic family of projective Anosov G-generic homomorphisms defined on a disc D with associated pressure form $\mathbf{p}$. Suppose that $z \in D$, $v \in \mathsf{T}_z D$ and*

$$\mathbf{p}(v, v) = 0.$$

Then, for every element α of infinite order in Γ,

$$\mathrm{D}_z \mathsf{L}(\alpha)(v) = 0.$$

9.1. **Log-type functions.** We begin by showing that if v is a norm zero vector, then each $\mathsf{L}(\alpha)$ is of log-type Kat v for some fixed K.

Definition 9.2. *We say that an analytic function f has* log-type K *at $v \in \mathsf{T}_u M$, if $f(u) \neq 0$ and*

$$\mathrm{D}_u \log(|f|)(v) = K \log(|f(u)|),$$

and is of log-type *if it is of log-type K for some K.*

Lemma 9.3. *Let $\{\rho_u\}_{u \in M}$ be an analytic family of projective Anosov homomorphims parametrized by an analytic manifold M and let $\mathbf{p}$ be the associated pressure form. If $v \in \mathsf{T}_z M$ and*

$$\mathbf{p}(v, v) = 0,$$

then there exists $K \in \mathbb{R}$ such that if α is any element of infinite order in Γ, then $\mathsf{L}(\alpha)$ is of log-type K at v.

Proof. Consider a smooth one parameter family $\{u_s\}_{s \in (-1,1)}$ in M such that $u_0 = z$ and $\dot{u}_0 = v$. Let $\rho_s = \rho_{u_s}$ and let $f_s = f_{u_s}$ where $\{f_{u_s}\}$ is a smooth family of reparametrizations obtained from Proposition 6.2. We define, for all $s \in (-1, 1)$,

$$\Phi_s = \Phi_{\rho_s} = -h(\rho_s) f_s,$$

By Corollary 8.2, $\dot{\Phi}_0$ is Livšic cohomologous to zero. In particular, the integral of $\dot{\Phi}_0$ is zero on any ϕ_s-invariant measure. Thus for any infinite order element $\alpha \in \Gamma$ one has

$$\langle \delta_\alpha | \dot{\Phi}_0 \rangle = 0.$$

By definition, $\Phi_s = -h(\rho_s) f_{\rho_s}$ and thus

$$\langle \delta_\alpha | \Phi_s \rangle = -h(\rho_s) \log \Lambda(\alpha)(u_s).$$

It then follows that

$$0 = \langle \delta_\alpha | \left. \frac{\mathrm{d}\Phi_s(x)}{\mathrm{d}s} \right|_{s=0} \rangle = \left. \frac{\mathrm{d}(\langle \delta_\alpha | \Phi_s \rangle)(x)}{\mathrm{d}s} \right|_{s=0} = \left. \frac{\mathrm{d}\left(h(\rho_s) \log(\Lambda(\alpha)(u_s))\right.}{\mathrm{d}s} \right|_{s=0}.$$

Applying the chain rule we get

$$0 = \left(\left. \frac{\mathrm{d}h(\rho_s)}{\mathrm{d}s} \right|_{s=0} \right) \log(\Lambda(\alpha)(u_s)) + h(\rho_s) \left(\left. \frac{\mathrm{d}\log(\Lambda(\alpha)(u_s)}{\mathrm{d}s} \right|_{s=0} \right).$$

It follows that setting

$$K = -\frac{1}{h(\rho_0)} \left. \frac{\mathrm{d}\left(h(\rho_s)\right)}{\mathrm{d}s} \right|_{s=0},$$

we get that for all $\alpha \in \Gamma$,

$$\mathrm{D}_z \log(\Lambda(\alpha))(v) = \left. \frac{\mathrm{d}}{\mathrm{d}s} \right|_{s=0} (\log(\Lambda(\alpha)(\rho_s)) = K \log\left(\Lambda(\alpha)(z)\right).$$

Since $\Lambda(\alpha) = |\mathsf{L}(\alpha)|$, $\mathsf{L}(\alpha)$ has log-type K at v. $\square$

9.2. **Trace functions.** Recall, from Proposition 2.6, that if α is an infinite order element of Γ and ρ is a projective Anosov representation in $\mathcal{C}(\Gamma, m)$, then we may write

$$\rho(\alpha) = \mathsf{L}(\alpha)(\rho)\mathbf{p}(\rho(\alpha)) + \mathbf{m}(\rho(\alpha)) + \frac{1}{\mathsf{L}(\alpha^{-1})(\rho)}\mathbf{q}(\rho(\alpha)),$$

where

1. $\mathsf{L}(\alpha)(\rho)$ is the eigenvalue of $\rho(\alpha)$ of maximum modulus and $\mathbf{p}(\rho(\alpha))$ is the projection on $\xi(\alpha^+)$ parallel to $\theta(\alpha^-)$
2. $\mathsf{L}(\alpha^{-1})(\rho)$ is the eigenvalue of $\rho(\alpha^{-1})$ of maximal modulus and $\mathbf{q}(\rho(\alpha))$ is the projection onto the line $\xi(\alpha^-)$ parallel to $\theta(\alpha^+)$, and
3. the spectral radius of $\mathbf{m}(\rho(\alpha))$ is less than $\delta^{l(\alpha)}\Lambda(\alpha)(\rho)$ for some $\delta = \delta(\rho) \in (0,1)$ which depends only on ρ.

It will be useful to define

$$\mathbf{r}(\rho(\alpha)) = \mathbf{m}(\rho(\alpha)) + \frac{1}{\mathsf{L}(\alpha^{-1})(\rho)}\mathbf{q}(\rho(\alpha))$$

which also has spectral radius less than $\delta^{l(\alpha)}\Lambda(\alpha)(\rho)$.

If $\{\rho_u\}_{u\in D}$ is an analytic family of projective Anosov G-generic homomorphisms defined on a disc D and α and β are infinite order elements of Γ, we consider the following analytic functions on D:

$$\begin{aligned}
\mathsf{T}(\alpha,\beta): &\quad u \mapsto \quad \mathrm{Tr}(\rho_u(\alpha)\rho_u(\beta))\\
\mathsf{T}(\mathbf{p}(\alpha),\beta): &\quad u \mapsto \quad \mathrm{Tr}(\mathbf{p}(\rho_u(\alpha))\rho_u(\beta)),\\
\mathsf{T}(\mathbf{p}(\alpha),\mathbf{p}(\beta)): &\quad u \mapsto \quad \mathrm{Tr}(\mathbf{p}(\rho_u(\alpha))\mathbf{p}(\rho_u(\beta))),\\
\mathsf{T}(\mathbf{p}(\alpha),\mathbf{r}(\beta)): &\quad u \mapsto \quad \mathrm{Tr}(\mathbf{p}(\rho_u(\alpha))\mathbf{r}(\rho_u(\beta))),\\
\mathsf{T}(\mathbf{r}(\alpha),\mathbf{p}(\beta)): &\quad u \mapsto \quad \mathrm{Tr}(\mathbf{r}(\rho_u(\alpha))\mathbf{p}(\rho_u(\beta))),\\
\mathsf{T}(\mathbf{r}(\alpha),\mathbf{r}(\beta)): &\quad u \mapsto \quad \mathrm{Tr}(\mathbf{r}(\rho_u(\alpha))\mathbf{r}(\rho_u(\beta))).
\end{aligned}$$

We say that two infinite order elements of Γ are *coprime* if they have distinct fixed points in $\partial_\infty\Gamma$ (i.e. they do not share a common power).

We then have

Proposition 9.4. *Let $\{\rho_u\}_{u\in D}$ be an analytic family of projective Anosov homomorphisms defined on a disc D. If α and β are infinite order, coprime elements of Γ, then*

$$\mathsf{T}(\mathbf{p}(\alpha),\mathbf{p}(\beta)) = \lim_{n\to\infty}\frac{\mathsf{L}(\alpha^n\beta^n)}{\mathsf{L}(\alpha)^n\mathsf{L}(\beta)^n}$$

and

$$\mathsf{T}(\mathbf{p}(\alpha),\beta) = \lim_{n\to\infty}\frac{\mathsf{L}(\alpha^n\beta)}{\mathsf{L}(\alpha)^n}.$$

Moreover, if $\mathsf{L}(\gamma)$ has log-type K at $v \in \mathsf{T}_uD$ for all infinite order $\gamma \in \Gamma$, then both $\mathsf{T}(\mathbf{p}(\alpha),\mathbf{p}(\beta))$ and $\mathsf{T}(\mathbf{p}(\alpha),\beta)$ have log-type K at v.

We say that a family $\{f_n\}_{n\in\mathbb{N}}$ of analytic functions defined on a disk D *decays* at $v \in \mathsf{T}_zD$ if

$$\lim_{n\to\infty} f_n(z) = 0 \quad\text{and}\quad \lim_{n\to\infty} \mathrm{D}_z f_n(v) = 0.$$

The following observation will be useful in the proof of Proposition 9.4.

Lemma 9.5. *Let G be an analytic function that may be written, for all positive integers n, as*

$$G = G_n(1 + h_n),$$

where G_n has log-type K and $\{h_n\}_{n\in\mathbb{N}}$ decays at $v \in \mathsf{T}_u M$, then G has log-type K.

Proof. Notice that

$$\begin{aligned} D_u \log(G)(v) &= D_u \log(G_n)(v) + D_u \log(1 + h_n)(v) \\ &= K \log G_n(u) + \frac{D_u h_n(v)}{1 + h_n(u)}. \end{aligned}$$

We now simply notice that the right hand side of the equation converges to $K \log G(u)$ □

Proof of Proposition 9.4: First notice that

$$\mathsf{T}(\alpha^n, \beta^n) = \mathsf{L}(\alpha^n\beta^n)(1 + g_n)$$

where

$$g_n = \frac{\mathrm{Tr}(\mathbf{r}(\alpha^n\beta^n))}{\mathsf{L}(\alpha^n\beta^n)}.$$

Since $\mathbf{r}(\alpha^n\beta^n)(\rho_u)$ has spectral radius at most $\delta(\rho_u)^{\ell(\alpha^n\beta^n)}|\mathsf{L}(\alpha^n\beta^n)|$, $\delta(\rho_u) \in (0,1)$, and $\lim_{n\to\infty} \ell(\alpha^n\beta^n) = +\infty$, we see that $\lim_{n\to\infty} g_n(\rho_u) = 0$ for all $\rho_u \in \mathcal{C}(\Gamma, m)$. Since $\{g_n\}$ is a sequence of analytic functions, g_n decays at v.

On the other hand,

$$\rho_u(\alpha^n\beta^n) = \mathsf{L}(\alpha)^n\mathsf{L}(\beta)^n\mathbf{p}(\alpha)\mathbf{p}(\beta) + \mathsf{L}(\alpha)^n\mathbf{p}(\alpha)\mathbf{r}(\beta^n) + \mathsf{L}(\beta)^n\mathbf{r}(\alpha^n)\mathbf{p}(\beta) + \mathbf{r}(\alpha^n)\mathbf{r}(\beta^n),$$

so

$$\mathsf{T}(\alpha^n, \beta^n) = \mathsf{L}(\alpha)^n\mathsf{L}(\beta)^n\mathsf{T}(\mathbf{p}(\alpha), \mathbf{p}(\beta))(1 + \hat{g}_n)$$

where

$$\hat{g}_n = \frac{\mathsf{L}(\alpha)^n\mathsf{T}(\mathbf{p}(\alpha), \mathbf{r}(\beta^n)) + \mathsf{L}(\beta)^n\mathsf{T}(\mathbf{r}(\alpha^n), \mathbf{p}(\beta)) + \mathsf{T}(\mathbf{r}(\alpha^n), \mathbf{r}(\beta^n))}{\mathsf{T}(\mathbf{p}(\alpha), \mathbf{p}(\beta))\mathsf{L}(\alpha)^n\mathsf{L}(\beta)^n}.$$

and again $\hat{g}_n$ decays at v. (Notice that, since α and β are co-prime, $\xi_{\rho_u}(\beta^+)$ is not contained in $\theta_{\rho_u}(\alpha^-)$ for any $u \in D$, so $\mathsf{T}(\mathbf{p}(\alpha), \mathbf{p}(\beta))$ is non-zero on D.)

Combining, we see that

$$\mathsf{T}(\mathbf{p}(\alpha), \mathbf{p}(\beta)) = \frac{\mathsf{L}(\alpha^n\beta^n)(1 + g_n)}{\mathsf{L}(\alpha)^n\mathsf{L}(\beta)^n(1 + \hat{g}_n)},$$

which implies, since $\lim g_n = 0$ and $\lim \hat{g}_n = 0$, that

$$\mathsf{T}(\mathbf{p}(\alpha), \mathbf{p}(\beta)) = \lim_{n\to\infty} \frac{\mathsf{L}(\alpha^n\beta^n)}{\mathsf{L}(\alpha)^n\mathsf{L}(\beta)^n}.$$

Moreover, if $\mathsf{L}(\gamma)$ has log-type K at v for all infinite order $\gamma \in \Gamma$, then $G_n = \frac{\mathsf{L}(\alpha^n\beta^n)}{\mathsf{L}(\alpha)^n\mathsf{L}(\beta)^n}$ has log-type K, being the ratio of log-type K functions and we may apply Lemma 9.5 to see that $\mathsf{T}(\mathbf{p}(\alpha), \mathbf{p}(\beta))$ has log-type K.

We similarly derive the claimed facts about $\mathsf{T}(\mathbf{p}(\alpha), \beta)$ by noting that

$$\mathsf{T}(\alpha^n, \beta) = \mathsf{L}(\alpha^n\beta)(1 + h_n)$$

where

$$h_n = \frac{\mathrm{Tr}(\mathbf{r}(\alpha^n\beta))}{\mathsf{L}(\alpha^n\beta)},$$

and that

$$\mathsf{T}(\alpha^n,\beta) = \mathsf{L}(\alpha)^n\mathsf{T}(\mathbf{p}(\alpha),\beta)(1+\hat{h}_n)$$

where

$$\hat{h}_n = \frac{\mathsf{T}(\mathbf{r}(\alpha^n),\beta)}{\mathsf{L}(\alpha^n)\mathsf{T}(\mathbf{p}(\alpha),\beta)}$$

and applying an argument similar to the one above. ☐

Remark: Dreyer [25] previously established that

$$\left\{\frac{\Lambda(\alpha^n\beta)(\rho)}{\Lambda(\alpha)(\rho)^n}\right\}$$

has a finite limit when ρ is a Hitchin representation.

9.3. **Technical lemmas.** We will need a rather technical lemma, Lemma 9.7, in the proof of Lemma 9.8, which is itself the main ingredient in the proof of Proposition 9.1.

We first prove a preliminary lemma, which may be viewed as a complicated version of the fact that exponential functions grow faster than polynomials. If a_s is a polynomial in q variables and their conjugates, we will use the notation

$$\|a_s\| = \sup\{|a_s(z_1,\ldots,z_q)| \mid |z_i| = 1\}.$$

Lemma 9.6. *Let $(f_1,\ldots,f_q)$ and $(\theta_1,\ldots,\theta_q)$ be two q-tuples of real numbers and let $(g_1,\ldots,g_q)$ be a q-tuple of complex numbers, such that*

$$1 > f_1 > \cdots > f_q > 0.$$

Suppose that there exists a strictly decreasing sequence $\{\mu_s\}_{s\in\mathbb{N}}$ of positive real numbers so that $\mu_1 < 1$ and a sequence of complex-valued polynomials $\{a_s\}_{s\in\mathbb{N}}$ in q variables and their conjugates, such that, for all $n \in \mathbb{N}$,

$$\sum_{p=1}^{q} n f_p^n\, \Re(e^{in\theta_p}g_p) = \sum_{s=1}^{\infty} \mu_s^n \Re(a_s(e^{in\theta_1},\ldots,e^{in\theta_q})), \tag{44}$$

and there exists N such that

$$\sum_{s=1}^{\infty} |\mu_s|^n \|a_s\|$$

is convergent for all $n \geq N$. Then, for all $p = 1,\ldots,q$,

$$\begin{aligned} \Re(g_p) = 0 \quad &\text{if} \quad \theta_p \in 2\pi\mathbb{Q},\\ g_p = 0 \quad &\text{if} \quad \theta_p \notin 2\pi\mathbb{Q}.\end{aligned}$$

Proof. There exists $r \in \mathbb{N}$, so that, for all i, either $r\theta_i \in 2\pi\mathbb{Z}$ or $r\theta_i \notin 2\pi\mathbb{Q}$. Equation (44) remains true if we replace $(\theta_1,\ldots,\theta_q)$ with $(r\theta_1,\ldots,r\theta_q)$, so we may assume that either $\theta_i \notin 2\pi\mathbb{Q}$ or $\theta_i \in \mathbb{Z}$.

Let V be the set of accumulation points of $\{(e^{in\theta_1},\ldots,e^{in\theta_q}) \mid n \in \mathbb{N}\}$. We first show that if $(z_1,\ldots,z_q) \in V$, then $\Re(g_1z_1) = 0$. This will suffice to prove our claim if $p = 1$, since if $\theta_i \in 2\pi\mathbb{Z}$, then $z_1 = 1$ and $\Re(g_1) = 0$. If not, any $z_1 \in S^1$ can arise in such a limit, so $\Re(z_1g_1) = 0$ for all $z_1 \in S^1$, which implies that $g_1 = 0$.

So, suppose that $\{n_m\}$ is an increasing sequence in $\mathbb{N}$ and $\{(e^{in_m\theta_1},\ldots,e^{in_m\theta_q})\}$ converges to $(z_1,\ldots,z_q)$. Then either

(1)
$$\Re(a_s(z_1,z_2,\ldots,z_q)) = 0$$
for all s, or

(2) there exists $s_0 \in \mathbb{N}$ so that

$$\mathfrak{A}_0 = \Re(a_{s_0}(z_1, z_2, \dots, z_q)) \neq 0,$$

and for all $s < s_0$

$$\Re(a_s(z_1, z_2, \dots, z_q)) = 0.$$

If (1) holds, then Equation (44) implies

$$\lim_{m\to\infty} n_m \Re(e^{in_m\theta_1} g_1) + \epsilon_0(n_m) = 0. \tag{45}$$

where

$$\epsilon_0(n_m) = \sum_{p=2}^{q} n_m \left(\frac{f_p}{f_1}\right)^{n_m} \Re(e^{in_m\theta_p} g_p).$$

Since, $\lim_{m\to\infty} \Re(e^{in_m\theta_1} g_1) = \Re(z_1 g_1)$ and $\lim_{m\to\infty} \epsilon_0(n_m) = 0$, we conclude that $\Re(z_1 g_1) = 0$.

If (2) holds, then Equation (44) implies that

$$\lim_{m\to\infty} n_m \Re(z_1 g_1) + \epsilon_0(n_m) - \left(\frac{\mu_{s_0}}{f_1}\right)^{n_m} \mathfrak{A}_m (1 + \epsilon_1(n_m)) = 0$$

where

$$\begin{aligned} \mathfrak{A}_m &= \Re(a_{s_0}(e^{in_m\theta_1}, e^{in_m\theta_2}, \dots, e^{in_m\theta_q})), \\ A_{m,s} &= \frac{1}{\mathfrak{A}_m}\left(\frac{\mu_s}{\mu_{s_0}}\right)^{n_m} \Re\left(a_s(e^{in_m\theta_1}, \dots, e^{in_m\theta_q})\right), \text{ and} \\ \epsilon_1(n_m) &= \sum_{s=s_0+1}^{\infty} A_{m,s}. \end{aligned} \tag{46}$$

Observe that

$$\lim_{m\to\infty} \mathfrak{A}_m = \mathfrak{A}_0 \neq 0$$

If m is large enough that $|\mathfrak{A}_m| \geqslant \frac{1}{2}|\mathfrak{A}_0|$ and $n_m > N$, then

$$|A_{m,s}| \leqslant \frac{\mu_{s_0+1}^{n_m - N}}{\mu_{s_0}^{n_m}} B_s \quad \text{where} \quad B_s = \frac{2}{\mathfrak{A}_0} |\mu_s|^N \|a_s\|.$$

Since $\lim_{m\to\infty} \frac{\mu_{s_0+1}^{n_m-N}}{\mu_{s_0}^{n_m}} = 0$ and $\sum_{s=1}^{\infty} B_s$ is convergent, $\lim_{n\to\infty} \epsilon_1(n_m) = 0$. It then follows that the sequence

$$\left\{ \frac{1}{n_m} \left(\frac{\mu_{s_0}}{f_1}\right)^{n_m} \right\}_{m\in\mathbb{N}}$$

is bounded. Thus $\mu_{s_0} \leqslant f_1$ and it follows that $\Re(z_1 g_1) = 0$.

Once we have proved that $\Re(z_1 g_1) = 0$ for all $(z_1, \dots, z_q) \in V$, we may use the same argument to prove that $\Re(z_2 g_2) = 0$ for all $(z_1, z_2, \dots, z_q)$ and proceed iteratively to complete the proof for all p. ☐

We are now read to prove the technical lemma used in the proof of Lemma 9.8

Lemma 9.7. *Let* $\{f_p\}_{p=1}^q$ *and* $\{\theta_p\}_{p=1}^q$ *be 2 families of real analytic functions defined on* $(-1, 1)$ *such that, for all* $t \in (-1, 1)$,

$$1 > |f_1(t)| > \cdots > |f_q(t)| > 0 \quad \textit{and} \quad \dot{\theta}_q(0) = 0$$

Let $\{g_p\}_{p=1}^q$ *be a family of complex valued analytic functions defined on* $(-1,1)$ *so that* $g_q(0) \in \mathbb{R} \setminus \{0\}$*. For all* $n \in \mathbb{N}$*, let*

$$F_n = 1 + \sum_{p=1}^{q} f_p^n \Re(e^{in\theta_p} g_p).$$

If there exists a constant K *such that for all large enough* n*,*

$$\dot{F_n}(0) = KF_n(0)\log(F_n(0)).$$

Then, $\dot{f}_q(0) = 0$*.*

Proof. We first notice that it suffices to prove the lemma in the restricted setting where $f_p(t) > 0$ for all p and all t. In general, we can then replace each f_p with f_p^2 and each θ_p with $2\theta_p$ and apply the restricted form of the lemma to conclude that $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f_q^2 = 0$, which implies that $\dot{f}_q(0) = 0$. For the remainder of the proof, we will assume that $f_p(t) > 0$ for all p and all t.

Let $g(x) = K(1+x)log(1+x)$. Then g is analytic at 0. Consider the expansion

$$g(x) = \sum_{n>0} a_m x^m$$

with radius of convergence $\delta > 0$. Notice that there exists N such that if $n \geq N$, then

$$\sum_{p=1}^{q} f_p(0)^n |g_p(0)| < \frac{\delta}{2}.$$

If $n \geq N$, then

$$\begin{aligned} KF_n(0)\log(F_n(0)) &= g\left(\sum_{p=1}^{q} f_p(0)^n \Re(e^{in\theta_p(0)} g_p(0))\right) \\ &= \sum_{m>0} a_m \left(\sum_{p=1}^{q} f_p(0)^n \Re(e^{in\theta_p(0)} g_p(0))\right)^m. \end{aligned}$$

If we expand this out, for each q-tuple of non-negative integers $\vec{m} = (m_1, \ldots, m_q)$, we get a term of the form

$$a_{m_1+\cdots+m_q} \left(\Pi_{p=1}^q f_p(0)^{m_p}\right)^n \binom{m_1 + \cdots + m_q}{m_1\ m_2\ \cdots\ m_q} \left(\Pi_{p=1}^q (\Re(g_p(0)e^{in\theta_p(0)})^{m_p}\right). \quad (47)$$

Let

$$h_{\vec{m}} = \Pi_{p=1}^q f_p(0)^{m_p} < 1.$$

Using the equality $\Re(z(w+\bar{w})) = 2\Re(z)\Re(w)$ repeatedly, we may rewrite the term in (47) in the form

$$h_{\vec{m}}^n \Re(H_{\vec{m}}(e^{in\theta_1(0)}, \ldots, e^{in\theta_q(0)})$$

where $H_{\vec{m}}$ is a complex polynomial in q variables and their conjugates. Since the series $\sum_{\vec{m}} h_{\vec{m}}^n \|H_{\vec{m}}\|$ is convergent for all $n \geq N$, we are free to re-arrange the terms. We group all terms where the coefficient $h_{\vec{m}}$ agrees (of which there are only finitely many for each value of $h_{\vec{m}}$) and order the resulting terms in decreasing order of co-efficient to express

$$KF_n(0)\log(F_n(0)) = \sum_{s=0}^{\infty} h_s^n \Re(H_s(e^{in\theta_1}, \ldots, e^{in\theta_q})),$$

where each H_s is a complex polynomial in q variables and their conjugates and $\{h_s\}_{s\in\mathbb{N}}$ is a strictly decreasing sequence of positive numbers less than 1. Moreover, for all $n \geq N$ the series

$$\sum_{s=0}^{\infty} h_s^n \|H_s\|$$

is convergent.

On the other hand,

$$\dot{F}_n(0) = \sum_{p=1}^{q} n f_p^n \Re\left(e^{in\theta_p} g_p \left(\frac{\dot{f}_p}{f_p} + i\dot{\theta}_p\right)\right) + \sum_{p=1}^{q} f_p^n \Re(e^{in\theta_p}\dot{g}_p)$$

where all functions on the right hand side are evaluated at 0. Since $\dot{F}_n(0) = KF_n(0)\log(F_n(0))$ we see that

$$\sum_{p=1}^{q} n f_p^n \Re\left(e^{in\theta_p} g_p \left(\frac{\dot{f}_p}{f_p} + i\dot{\theta}_p\right)\right) = \sum_{s=1}^{\infty} h_s^n \Re(H_s(e^{in\theta_1}, \ldots, e^{in\theta_q})).$$

The previous lemma then implies that for all p

$$\Re\left(g_p\left(\frac{\dot{f}_p}{f_p} + i\dot{\theta}_p\right)\right) = 0$$

Since $g_q(0)$ is a non zero real number, $f_q(0) \neq 0$ and $\dot{\theta}_q(0) = 0$, we get that $\dot{f}_q(0) = 0$. □

9.4. **Degenerate vectors have log-type zero.** Proposition 9.1 then follows from the following lemma and Lemma 9.3.

Lemma 9.8. *Let Γ be a word hyperbolic group and let G be a reductive subgroup of $\mathsf{SL}_m(\mathbb{R})$. If $\{\rho_u\}_{u\in D}$ is an analytic family of projective Anosov G-generic homomorphisms defined on a disc D and $\mathsf{L}(\alpha)$ has log-type K at $v \in \mathsf{T}_z D$ for all infinite order $\alpha \in \Gamma$, then $\mathsf{D}_z\mathsf{L}(\alpha)(v) = 0$ for all infinite order $\alpha \in \Gamma$.*

Proof. Notice that if we replace the family $\{\rho_u\}_{u\in D}$ by a conjugate family $\{\rho'_u = g_u\rho_u g_u^{-1}\}_{u\in D}$ where $\{g_u\}_{u\in D}$ is an analytic family of elements of $\mathsf{SL}_m(\mathbb{R})$, then $\mathsf{L}(\alpha)(\rho_u) = \mathsf{L}(\alpha)(\rho'_u)$ for all $u \in D$. Therefore, we are free to conjugate our original family when proving the result.

By Proposition 2.21, we may choose $\beta \in \Gamma$, so that $\rho_u(\beta)$ is generic. After possibly restricting to a smaller disk about z, we may assume that $\rho_u(\beta)$ is generic for all $u \in D$. We may then conjugate the family so that $\rho_u(\beta)$ lies in the same maximal torus for all u, we can write

$$\rho_u(\beta^n) = \mathsf{L}(\beta)^n \mathsf{p} + \sum_{p=1}^{q-1} \lambda_p^n \left(\cos(n\theta_p)\mathsf{p}_p + \sin(n\theta_p)\widehat{\mathsf{p}_p}\right) + \frac{1}{\mathsf{L}(\beta^{-1})^n}\mathsf{q},$$

where $\mathsf{L}(\beta)$, $\mathsf{L}(\beta^{-1})$, λ_p, and θ_p are analytic functions of u and

$$|\mathsf{L}(\beta)(u)| > |\lambda_1(u)| > |\lambda_2(u)| > \cdots > |\lambda_{q-1}(u)| > \frac{1}{|\mathsf{L}(\beta^{-1}(u)|} > 0$$

for all $u \in D$.

Choose an infinite order element $\alpha \in \Gamma$ which is coprime to β. Proposition 9.4, implies that, for all n,

$$\frac{\mathsf{T}(\mathbf{p}(\alpha),\beta^n)}{\mathsf{L}(\beta^n)\mathsf{T}(\mathbf{p}(\alpha),\mathbf{p}(\beta))} = 1 + \left(\frac{1}{\mathsf{L}(\beta)\mathsf{L}(\beta^{-1})}\right)^n \left(\frac{\mathrm{Tr}(\mathbf{p}(\rho(\alpha))\mathsf{q}))}{\mathsf{T}(\mathbf{p}(\alpha),\mathbf{p}(\beta))}\right)$$
$$+\sum_{p=1}^{q-1}\left(\frac{\lambda_p}{\mathsf{L}(\beta)}\right)^n \Re\left(e^{in\theta_p}\left(\frac{\mathrm{Tr}(\mathbf{p}(\rho(\alpha))\mathsf{p}_p)}{\mathsf{T}(\mathbf{p}(\alpha),\mathbf{p}(\beta))} + i\frac{\mathrm{Tr}(\mathbf{p}(\rho(\alpha))\widehat{\mathsf{p}_p})}{\mathsf{T}(\mathbf{p}(\alpha),\mathbf{p}(\beta))}\right)\right).$$

has log-type K at v, since the numerator has log-type K at v and the denominator is a product of two functions which have log-type K at v.

Since α and β are coprime and ρ is projective Anosov, $\xi(\beta^-)\oplus\theta(\alpha^-) = \mathbb{R}^m$, so $\mathrm{Tr}(\mathbf{p}(\rho(\alpha)),\mathsf{q}) \neq 0$ (since $\mathbf{p}(\rho(\alpha))$ is a projection onto the line $\xi(\alpha^+)$ parallel to $\theta(\alpha^-)$ and $\mathsf{q} = \mathbf{q}(\rho(\beta))$ is a projection onto the line $\xi(\beta^-)$). Similarly, $\mathsf{T}(\mathbf{p}(\alpha),\mathbf{p}(\beta)) \neq 0$, since $\xi(\beta^+)\oplus\theta(\alpha^-) = \mathbb{R}^m$.

Let $\{u_s\}_{s\in(-1,1)}$ be a smooth family in D so that $u_0 = z$ and $\dot{u}_0 = v$. We now apply Lemma 9.7, taking

$$\begin{aligned} f_p(s) &= \frac{\lambda_p(u_s)}{\mathsf{L}(\beta)(u_s)}, \\ g_p(s) &= \left(\frac{\mathrm{Tr}(\mathbf{p}(\rho_{u_s}(\alpha))\mathsf{p}_p)}{\mathsf{T}(\mathbf{p}(\alpha),\mathbf{p}(\beta))(u_s)} + i\frac{\mathrm{Tr}(\mathbf{p}(\rho_{u_s}(\alpha))\widehat{\mathsf{p}_p})}{\mathsf{T}(\mathbf{p}(\alpha),\mathbf{p}(\beta))(u_s)}\right), \end{aligned}$$

if $p = 1,\dots,q-1$, and taking

$$\begin{aligned} f_q(s) &= \frac{1}{\mathsf{L}(\beta)(u_s)\mathsf{L}(\beta^{-1})(u_s)}, \\ g_q(s) &= \frac{\mathrm{Tr}(\mathbf{p}(\rho_{u_s}(\alpha))\mathsf{q})}{\mathsf{T}(\mathbf{p}(\alpha),\mathbf{p}(\beta))(u_s)}, \text{ and} \\ \theta_q(s) &= 0. \end{aligned}$$

We conclude from Lemma 9.7 that $\dot{f}_q = 0$. Thus

$$\mathrm{D}_z\mathsf{L}(\beta)(v)\cdot\mathsf{L}(\beta^{-1})(z) = -\mathsf{L}(\beta)(z)\cdot\mathrm{D}_z\mathsf{L}(\beta^{-1})(v). \tag{48}$$

Since $\mathsf{L}(\beta)$ and $\mathsf{L}(\beta^{-1})$ both have log-type K at v, we get that

$$\frac{\mathrm{D}_z\mathsf{L}(\beta)(v)}{\mathsf{L}(\beta)(z)} = K\log(|\mathsf{L}(\beta)(z)|) \quad\text{and}\quad \frac{\mathrm{D}_z\mathsf{L}(\beta^{-1})(v)}{\mathsf{L}(\beta^{-1})(z)} = K\log(|\mathsf{L}(\beta^{-1})(z)|). \tag{49}$$

Combining (48) and (49) we see that

$$K\log(|\mathsf{L}(\beta)(z)|) = \frac{\mathrm{D}_z\mathsf{L}(\beta)(v)}{\mathsf{L}(\beta)(z)} = -\frac{\mathrm{D}_z\mathsf{L}(\beta^{-1})(v)}{\mathsf{L}(\beta^{-1})(z)} = -K\log(|\mathsf{L}(\beta^{-1})(z)|).$$

Since $\log|\mathsf{L}(\beta)(z)| > 0$ and $\log|\mathsf{L}(\beta^{-1})(z)| > 0$, this implies that $K = 0$. Therefore, $\mathsf{L}(\alpha)$ has log-type 0 at v for all infinite order $\alpha\in\Gamma$, so $\mathrm{D}_z\mathsf{L}(\alpha)(v) = 0$ for all infinite order $\alpha\in\Gamma$. □

10. Variation of length and cohomology classes

The aim of this section is to prove the following proposition.

Proposition 10.1. *Let Γ be a word hyperbolic group and let G be a reductive subgroup of $\mathsf{SL}_m(\mathbb{R})$. Suppose that $\eta : D \to \mathrm{Hom}(\Gamma,\mathsf{G})$ is an analytic map such that for each $u\in D$, $\eta(u) = \rho_u$ is irreducible, projective Anosov, and G-generic. If $v \in \mathsf{T}_zD$ and*

$$\mathrm{D}_z\mathsf{L}(\alpha)(v) = 0$$

for all infinite order elements $\alpha \in \Gamma$*, then* $\mathrm{D}_z\eta(v)$ *defines a zero cohomology class in* $H^1_{\eta(z)}(\Gamma, \mathfrak{g})$.

We recall that $\mathrm{D}_z\eta(v)$ defines a zero cohomology class in $H^1_{\eta(z)}(\Gamma, \mathfrak{g})$ if and only if it is tangent to the orbit $\mathsf{G}\eta(z)$ in $\mathrm{Hom}(\Gamma, \mathsf{G}) \subset \mathsf{G}^r$.

Propositions 9.1 and 10.1 together imply that the pressure form is non-degenerate on $\mathcal{C}_g(\Gamma, \mathsf{G})$. More generally, we obtain the following corollary.

Corollary 10.2. *Let* Γ *be a word hyperbolic group and let* G *be a reductive subgroup of* $\mathsf{SL}_m(\mathbb{R})$*. Suppose that* $\eta : D \to \widetilde{\mathcal{C}}_g(\Gamma, \mathsf{G})$ *is an analytic map and* $\mathbf{p}$ *is the associated pressure form on* D*. If* $v \in \mathsf{T}_z D$ *and*

$$\mathbf{p}(v, v) = 0,$$

then $\mathrm{D}_z\eta(v)$ *defines a zero cohomology class in* $H^1_{\eta(z)}(\Gamma, \mathfrak{g})$.

In the course of the proof of Proposition 10.1 we also obtain the following fact which is of independent interest.

Proposition 10.3. *Suppose that* G *is a reductive subgroup of* $\mathsf{SL}_m(\mathbb{R})$ *and* $\rho \in \mathcal{C}_g(\Gamma, \mathsf{G})$*. Then the set*

$$\{D_\rho \mathsf{L}(\alpha) \mid \alpha \text{ infinite order in } \Gamma\},$$

generates the cotangent space $\mathsf{T}^*_\rho \mathcal{C}_g(\Gamma, \mathsf{G})$.

Both propositions will be established in section 10.3.

10.1. **Invariance of the cross-ratio.** We recall the definition of the *cross ratio* of a pair of hyperplanes and a pair of lines. First define

$$\mathbb{RP}(m)^{(4)} = \{(\varphi, \psi, u, v) \in \mathbb{RP}(m)^{*2} \times \mathbb{RP}(m)^2 : (\varphi, v) \text{ and } (\psi, u) \text{ span } \mathbb{R}^m\}.$$

We then define $\mathbf{b} : \mathbb{RP}(m)^{(4)} \to \mathbb{R}$ by

$$\mathbf{b}(\varphi, \psi, u, v) = \frac{\langle \varphi | u \rangle \, \langle \psi | v \rangle}{\langle \varphi | v \rangle \, \langle \psi | u \rangle}.$$

Notice that for this formula to make sense we must make choices of elements in φ, ψ, u, and v, but that the result is independent of our choices.

If ρ is a projective Anosov representation with limit curves $\xi : \partial_\infty \Gamma \to \mathbb{RP}(m)$ and $\theta : \partial_\infty \Gamma \to \mathbb{RP}(m)^*$, we define the associated *cross ratio* on $\partial_\infty \Gamma^{(4)}$, as in [45], to be

$$\mathrm{b}_\rho(x, y, z, w) = \mathbf{b}(\theta(x), \theta(y), \xi(z), \xi(w)). \tag{50}$$

We first derive a formula for the cross-ratio at points associated to co-prime elements. This formula generalizes the formula in Corollary 1.6 from Benoist [4].

Proposition 10.4. *If* $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ *is a projective Anosov representation and* α *and* β *are infinite order co-prime elements of* Γ*, then*

$$\mathbf{b}_\rho(\alpha^-, \beta^-, \beta^+, \alpha^+) = \mathsf{T}(\mathbf{p}(\alpha), \mathbf{p}(\beta)) = \lim_{n \to \infty} \frac{\mathsf{L}(\alpha^n \beta)}{\mathsf{L}(\alpha)^n}.$$

Proof. Choose $a^+ \in \xi(\alpha^+)$, $a^- \in \theta(\alpha^-)$, $b^+ \in \xi(\beta^+)$ and $b^- \in \theta(\beta^-)$. Observe that

$$\mathbf{p}(\alpha)(u) = \frac{\langle a^- | u \rangle}{\langle a^- | a^+ \rangle} \, a^+.$$

for all $u \in \mathbb{R}^m$. In particular,

$$\mathbf{p}(\beta)\mathbf{p}(\alpha)(u) = \frac{\langle b^-|a^+\rangle}{\langle a^-|a^+\rangle \langle b^-|b^+\rangle} \langle a^-|u\rangle\, b^+.$$

Therefore,

$$\mathsf{T}(\mathbf{p}(\alpha)\mathbf{p}(\beta)) = \frac{\langle a^-|b^+\rangle \langle b^-|a^+\rangle}{\langle a^-|a^+\rangle \langle b^-|b^+\rangle} = \mathbf{b}_\rho(\alpha^-, \beta^-, \beta^+, \alpha^+).$$

The last equality in the formula follows immediately from Proposition 9.4. ☐

As a corollary, we see that if $\mathsf{L}(\alpha)$ has log-type zero for all infinite order $\alpha \in \Gamma$, then the cross-ratio also has log-type zero.

Corollary 10.5. *Let Γ be a word hyperbolic group and let G be a reductive subgroup of $\mathsf{SL}_m(\mathbb{R})$. Suppose that $\{\rho_u : \Gamma \to \mathsf{G}\}_{u\in D}$ is an analytic family of projective Anosov G-generic homomorphisms parametrized by a disc D. If $\mathsf{L}(\alpha)$ has log-type 0 at $v \in \mathsf{T}_z D$ for all infinite order $\alpha \in \Gamma$, then for all distinct collections of points $x, y, z, w \in \partial_\infty\Gamma$, the function*

$$u \mapsto \mathrm{b}_{\rho_u}(x, y, z, w),$$

is of log-type 0 at v.

Proof. Suppose that $\alpha, \beta \in \Gamma$ have infinite order. Propositions 9.4 and 10.4 imply that $\mathbf{b}_\rho(\alpha^-, \beta^-, \beta^+, \alpha^+)$ has log-type 0.

Since pairs of fixed points of infinite order elements are dense in $\partial_\infty\Gamma^{(2)}$ and ξ_u and θ_u vary analytically by Proposition 6.1, we see that

$$\rho \mapsto \mathrm{b}_\rho(x, y, z, w),$$

has log-type 0 for all pairwise distinct $x, y, z, w \in \partial_\infty\Gamma$. ☐

10.2. **An useful immersion.** We define a mapping from $\mathsf{PSL}_m(\mathbb{R})$ into a quotient $\mathsf{W}(m)$ of the vector space M^{m+1} of all $(m+1)\times(m+1)$-matrices and use it to encode a collection of cross ratios.

Consider the action of the multiplicative group $(\mathbb{R}\setminus\{0\})^{2(m+1)}$ on M^{m+1} given by

$$(a_0, \ldots, a_m, b_0, \ldots, b_m)(M_{i,j}) = (a_i b_j M_{i,j}).$$

We denote the quotient by

$$\mathsf{W}(m) = \mathsf{M}^{m+1}/(\mathbb{R}\setminus\{0\})^{2(m+1)}.$$

Given a projective frame $F = (x_0, \ldots, x_m)$ for $\mathbb{RP}(m)$ and a projective frame $F^* = (X_0, \ldots, X_m)$ for the dual $\mathbb{RP}(m)^*$, let

- $\hat{x}_i$ be non zero vectors in x_i, such that
$$0 = \sum_{i=0}^m \hat{x}_i, \tag{51}$$
- $\hat{X}_i$ be non zero covectors in X_i such that
$$0 = \sum_{i=0}^m \hat{X}_i. \tag{52}$$

Observe that $\hat{x}_i$, respectively $\hat{X}_i$, are uniquely defined up to a common multiple. Then, the mapping

$$\mu_{F,F^*} : \mathsf{PSL}_m(\mathbb{R}) \to \mathsf{W}(m)$$

given by

$$\mu_{F,F^*} : A \mapsto \hat{X}_i(A(\hat{x}_j))$$

is well defined, independent of the choice of $\hat{x}_i$ and $\hat{X}_i$.

Lemma 10.6. *The mapping μ_{F,F^*} is a smooth injective immersion.*

Proof. Since $\mu_{F,F^*}(A)$ determines the projective coordinates of the image of the projective frame $(x_0, \ldots, x_n)$ by A, μ_{F,F^*} is injective.

Let $\mu = \mu_{F,F^*}$. Let $\{A_t\}_{t\in(-1,1)}$ be a smooth one-parameter family in $\mathsf{PSL}_m(\mathbb{R})$ such that

$$\dot{A} \in \mathsf{T}_{A_0}(\mathsf{PSL}_m(\mathbb{R})) \text{ and } \mathsf{D}\mu(\dot{A}) = 0.$$

Let $\{\hat{X}_i^t\}_{t\in(-1,1)}$ and $\{\hat{x}_j^t\}_{t\in(-1,1)}$ be time dependent families of covectors in X_i and vectors x_j respectively, and let

$$a_{i,j}^t = \hat{X}_i^t(A_t(\hat{x}_j^t)).$$

If $\mathsf{D}\mu(\dot{A}) = 0$, then there exists λ_i and μ_j such that

$$\dot{a}_{i,j} = \lambda_i a_{i,j} + \mu_j a_{i,j}.$$

Multiplying each $\hat{X}_i^t$ by $e^{-\lambda_0 t}$ and each $\hat{x}_i^t$ by $e^{-\mu_0 t}$ has the effect of replacing λ_i and μ_j by $\lambda_i - \lambda_0$ and $\mu_j - \mu_0$ respectively. Thus, we may assume that $\lambda_0 = \mu_0 = 0$.

We now use the normalization (51) and (52), to see that

$$\sum_{i=1}^m \lambda_i a_{i,j} = 0 = \sum_{j=1}^m \mu_j a_{i,j}.$$

On the other hand, since the collections of vectors $\{v_i = (a_{i,j})_{1\leqslant j\leqslant m}\}$ and $\{w_j = (a_{i,j})_{1\leqslant i\leqslant m}\}$ are linearly independent, this implies that $\lambda_i = \mu_j = 0$ for all i and j. □

The following lemma relates the immersion μ and the cross ratio.

Lemma 10.7. *Let $\{x_0, \ldots, x_m\}$ and $\{y_0, \ldots, y_m\}$ be collections of $m+1$ pairwise distinct points in $\partial_\infty\Gamma$. Suppose that $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ is projective Anosov with limit maps ξ and θ and that*

$$\begin{aligned} F &= (\xi(x_0), \ldots, \xi(x_m)), \\ F^* &= (\theta(y_0), \ldots, \theta(y_m)). \end{aligned}$$

are projective frames for $\mathbb{RP}(m)$ and $\mathbb{RP}(m)^*$. If $\alpha \in \Gamma$, then

$$\mu_{F,F^*}(\pi_m(\rho(\alpha))) = [\mathrm{b}_\rho(y_i, z, \alpha(x_j), w)]$$

where z and w are arbitrary points in $\partial_\infty\Gamma$.

Proof. Choose, for each $i = 0, \ldots, m$, $\phi_i \in \theta(y_i)$ and $v_i \in \xi(x_i)$, and choose $\phi \in \theta(z)$ and $v \in \xi(w)$. Then

$$\mu_{F,F^*}(\pi_m(\rho(\alpha))) = [\langle \phi_i | \alpha(v_j)\rangle]$$

while

$$[\mathrm{b}_\rho(y_i, z, \alpha(x_j), w)] = \left[\frac{\langle \phi_i | \alpha(v_j)\rangle \, \langle \phi | v\rangle}{\langle \phi_i | v\rangle \, \langle \phi | \alpha(v_j)\rangle}\right].$$

The equivalence is given by taking $a_i = \frac{\langle \phi | v\rangle}{\langle \phi_i | v\rangle}$ and $b_j = \frac{1}{\langle \phi | \alpha(v_j)\rangle}$. □

10.3. **Vectors with log type zero.** Propositions 10.1 and 10.3 follow from Proposition 9.1 and the following lemma.

Lemma 10.8. *Let Γ be a word hyperbolic group and let G be a reductive subgroup of $\mathsf{SL}_m(\mathbb{R})$. Suppose that $\eta : D \to \mathrm{Hom}(\Gamma, \mathsf{G})$ is an analytic map such that for each $u \in D$, $\eta(u) = \rho_u$ is irreducible, projective Anosov and G-generic. Suppose that $v \in \mathsf{T}_z D$ and that $\mathrm{D}_z\mathsf{L}(\alpha)(v) = 0$ for all infinite order $\alpha \in \Gamma$. Then the cohomology class of $\mathrm{D}\eta(v)$ vanishes in $H^1_{\eta(z)}(\Gamma, \mathfrak{g})$.*

Proof. Let $\{u_t\}_{t\in(-1,1)}$ be a path in D so that $u_0 = z$ and $\dot u_0 = v$. Let $\rho_t = \rho_{u_t}$. By Corollary 10.5,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathrm{b}_{\rho_t}(x, y, z, w)) = 0$$

for any pairwise distinct (x, y, z, w) in $\partial_\infty\Gamma$.

Lemma 2.18 allows us to choose collections $\{x_0, \dots, x_m\}$ and $\{y_0, \dots, y_m\}$ of pairwise distinct points in $\partial_\infty\Gamma$ such that if

$$\begin{aligned} F_t &= (\xi_t(x_0), \dots \xi_t(x_m)),\\ F_t^* &= (\theta_t(y_0), \dots \theta_t(y_m)). \end{aligned}$$

then F_0 and F_0^* are both projective frames. For some $\epsilon > 0$, F_t and F_t^* are projective frames for all $t \in (-\epsilon, \epsilon)$. (We will restrict to this domain for the remainder of the argument.) We may then normalize, by conjugating ρ_t by an appropriate element of $\mathsf{SL}_m(\mathbb{R})$, so that $F_t = F_0$ for all $t \in (-\epsilon, \epsilon)$.

Let

$$\mu_t = \mu_{F_t, F_t^*} \circ \pi_m.$$

Then, by Lemma 10.7,

$$\mu_t(\rho_t(\alpha)) = [\mathrm{b}_{\rho_t}(x_i, z, \alpha(y_j), w)]$$

for all $\alpha \in \Gamma$. Therefore,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mu_t(\rho_t(\alpha)) = 0.$$

for all $\alpha \in \Gamma$. Notice that if χ and χ^* are projective frames, then

$$\mu_{\chi, B^*\chi^*}(A) = \mu_{\chi,\chi^*}(B^{-1} \circ A),$$

for all $A, B \in \mathsf{SL}_m(\mathbb{R})$. If we choose $C_t \in \mathsf{SL}_m(\mathbb{R})$ so that $(C_t^{-1})^*(F_t^*) = F_0^*$, then

$$\begin{aligned} 0 &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mu_t(\rho_t(\alpha))) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mu_0(C_t\rho_t(\alpha)))\\ &= \mathrm{D}\mu_0\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (C_t \circ \rho_t(\alpha))\right). \end{aligned}$$

Lemma 10.6 implies that μ_0 is an immersion, so

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (C_t \circ \rho_t(\alpha)) = 0$$

Thus,

$$C_0 \circ \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \rho_t(\alpha) + \dot C_0 \circ \rho(\alpha) = 0. \tag{53}$$

Taking $\alpha = \mathrm{id}$ in Equation (53), we see that $\dot{C}_0 = 0$. Since $C_0 = I$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \rho_t(\alpha) = 0$$

for all $\alpha \in \Gamma$. Therefore the cohomology class of $\mathrm{D}\eta(v)$ vanishes in $H^1_{\eta(z)}(\Gamma, \mathfrak{sl}_m(\mathbb{R}))$. Since G is a reductive subgroup of $\mathsf{SL}_m(\mathbb{R})$, $\mathfrak{sl}_m\mathbb{R} = \mathfrak{g} \oplus \mathfrak{g}^\perp$, so $H^1_{\eta(z)}(\Gamma, \mathfrak{g})$ injects into $H^1_{\eta(z)}(\Gamma, \mathfrak{sl}_n(\mathbb{R}))$. Therefore, $\mathrm{D}\eta(v)$ vanishes in $H^1_{\eta(z)}(\Gamma, \mathfrak{g})$ as claimed. □

11. Rigidity results

In this section, we establish two rigidity results for projective Anosov representations. We first establish Theorem 1.2 which states that the signed spectral radii determine the limit map of a projective Anosov representation, up to the action of $\mathsf{SL}_m(\mathbb{R})$, and that they determine the conjugacy class, in $\mathsf{GL}_m(\mathbb{R})$, of an irreducible projective Anosov representation.

Theorem 11.1. [Spectral rigidity] *Let Γ be a word hyperbolic group and let $\rho_1 : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ and $\rho_2 : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ be projective Anosov representations such that*

$$\mathsf{L}(\gamma)(\rho_1) = \mathsf{L}(\gamma)(\rho_2)$$

for all infinite order $\gamma \in \Gamma$. Then there exists $g \in \mathsf{GL}_m(\mathbb{R})$ such that $g \circ \xi_1 = \xi_2$.

Moreover, if ρ_1 is irreducible, then $\rho_2 = g\rho_1 g^{-1}$.

We next establish our rigidity result for renormalized intersection. If H is a Lie group, denote by $Z(\mathsf{H})$ its center and by H^0 the connected component of the identity. We denote by π_m the projection from $\mathsf{SL}_m(\mathbb{R})$ to $\mathsf{PSL}_m(\mathbb{R})$. If $\mathsf{H} \subset \mathsf{SL}_m(\mathbb{R})$ denote by $\mathsf{PH} = \pi_m(\mathsf{H})$ the projectivised group. Finally, if $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ is a representation, denote by G_ρ the Zariski closure of $\rho(\Gamma)$.

Theorem 11.2. [Intersection rigidity] *Let Γ be a word hyperbolic group and let $\rho_1 : \Gamma \to \mathsf{SL}_{m_1}(\mathbb{R})$ and $\rho_2 : \Gamma \to \mathsf{SL}_{m_2}(\mathbb{R})$ be projective Anosov representations such that*

$$\mathbf{J}(\rho_1, \rho_2) = 1.$$

If G_{ρ_1} and G_{ρ_2} are connected, then there exists an isomorphism $\sigma : \mathsf{G}_{\rho_1}/Z(\mathsf{G}_{\rho_1}) \to \mathsf{G}_{\rho_2}/Z(\mathsf{G}_{\rho_2})$ such that

$$\sigma\bar{\rho}_1 = \bar{\rho}_2,$$

where $\bar{\rho}_i : \Gamma \to \mathsf{G}_{\rho_i}/Z(\mathsf{G}_{\rho_i})$ is the composition of ρ_i and the projection of G_{ρ_i} onto $\mathsf{G}_{\rho_i}/Z(\mathsf{G}_{\rho_i})$.

Remarks:

(1) If either G_{ρ_1} or G_{ρ_2} is not connected, then Theorem 11.2 holds for the finite index subgroup

$$\Gamma_0 = \Gamma \cap \rho_1^{-1}(\mathsf{G}^0_{\rho_1}) \cap \rho_2^{-1}(\mathsf{G}^0_{\rho_2}).$$

Indeed, each $\rho_i|\Gamma_0$ is again projective Anosov (see [29, Cor. 3.4]), and Corollary 8.2 implies that $\mathbf{J}(\rho_1|_{\Gamma_0}, \rho_2|_{\Gamma_0}) = 1$.

(2) Consequently, if $\mathsf{G}^0_{\rho_1}$ and $\mathsf{G}^0_{\rho_2}$ are not isomorphic, then Theorem 11.2 implies that $\mathbf{J}(\rho_1, \rho_2) > 1$.

(3) The representations need not actually be conjugate if $\mathbf{J}(\rho_1,\rho_2)=1$. Let $\rho : \pi_1(S)\to \mathsf{PSL}_2(\mathbb{R})$ be a Fuchsian representation and let $\tau_k : \mathsf{PSL}_2(\mathbb{R})\to \mathsf{PSL}_k(\mathbb{R})$ be the irreducible representation, then

$$\mathbf{J}(\tau_n\circ\rho,\tau_m\circ\rho)=1$$

but $\tau_n\circ\rho$ and $\tau_m\circ\rho$ are not conjugate if $n\neq m$.

11.1. **Spectral rigidity.** Our spectral rigidity theorem will follow from Proposition 10.4 and work of Labourie [45].

Recall, from Section 10.1, that we defined the cross ratio $\mathbf{b}$ of a pair of hyperplanes and a pair of lines. Then, given a projective Anosov representation ρ with limit maps ξ and θ, we defined a cross ratio b_ρ on $\partial_\infty\Gamma^{(4)}$ by letting

$$\mathrm{b}_\rho(x,y,z,w)=\mathbf{b}(\theta(x),\theta(y),\xi(z),\xi(w)). \tag{54}$$

Labourie [45, Theorem 5.1] showed that if ρ is a projective Anosov representation with limit map ξ, then the dimension $\dim\langle\xi(\partial_\infty\Gamma)\rangle$ can be read directly from the cross ratio b_ρ. (In [45], Labourie explicitly handles the case where $\Gamma=\pi_1(S)$, but his proof generalizes immediately.) Consider S^p_* the set of pairs $(e,u)=(e_0,\ldots,e_p,u_0,\ldots,u_p)$ of $(p+1)$-tuples in $\partial_\infty\Gamma$ such that $e_j\neq e_i\neq u_0$ and $u_j\neq u_i\neq e_0$ when $j>i>0$. If $(e,u)\in S^p_*$, he defines

$$\chi^p_{\mathrm{b}_\rho}(e,u)=\det_{i,j>0}(\mathrm{b}_\rho(e_i,e_0,u_j,u_0)).$$

Lemma 11.3. *If $\rho:\Gamma\to\mathsf{SL}_m(\mathbb{R})$ is projective Anosov, then*

$$\dim\langle\xi(\partial_\infty\Gamma)\rangle=\inf\{p\in\mathbb{N}:\chi^p_{\mathrm{b}_\rho}\equiv 0\}-1.$$

Lemma 4.3 of Labourie [45] extends in our setting to give:

Lemma 11.4. *If $\rho_1:\Gamma\to\mathsf{SL}_m(\mathbb{R})$ and $\rho_2:\Gamma\to\mathsf{SL}_m(\mathbb{R})$ are projective Anosov and $\mathrm{b}_{\rho_1}=\mathrm{b}_{\rho_2}$, then there exists $g\in\mathsf{GL}_m(\mathbb{R})$ such that $g\circ\xi_1=\xi_2$.*

Moreover, if ρ_1 is irreducible, then $g\,(\pi_m\circ\rho_1)\,g^{-1}=\pi_m\circ\rho_2$.

Proof. Lemma 11.3 implies that

$$\dim\langle\xi_1(\partial_\infty\Gamma)\rangle=\dim\langle\xi_2(\partial_\infty\Gamma)\rangle=p.$$

Choose $\{x_0,\ldots,x_p\}\subset\partial_\infty\Gamma$ so that

$$\{\xi_1(x_0),\ldots,\xi_1(x_p)\}\text{ and }\{\xi_2(x_0),\ldots,\xi_2(x_p)\}$$

are projective frames for $\langle\xi_1(\partial_\infty\Gamma)\rangle$ and $\langle\xi_2(\partial_\infty\Gamma)\rangle$ (see Lemma 2.17).

Choose $u_0\in\xi_1(x_0)$ and $\{\varphi_1,\ldots,\varphi_p\}\subset(\mathbb{R}^m)^*$ such that $\varphi_i\in\theta_1(x_i)$ and $\varphi_i(u_0)=1$. One may check that $\{\varphi_1,\ldots,\varphi_p\}$ is a basis for $\langle\theta_1(\partial_\infty\Gamma)\rangle$. Complete $\{\varphi_1,\ldots,\varphi_p\}$ to a basis

$$\mathcal{B}_1=\{\varphi_1,\ldots,\varphi_p,\varphi_{p+1},\ldots,\varphi_m\}$$

for $(\mathbb{R}^m)^*$ such that $\varphi_i(\langle\xi_1(\partial_\infty\Gamma)\rangle)=0$ for all $i>p$. For $y\in\partial_\infty\Gamma$, the projective coordinates of $\xi_1(y)$ with respect to the dual basis of $\mathcal{B}_1$ are given by

$$[\ldots:\langle\varphi_i|\xi_1(y)\rangle:\ldots]=[\ldots:\frac{\langle\varphi_i|\xi_1(y)\rangle}{\langle\varphi_1|\xi_1(y)\rangle}\frac{\langle\varphi_1|u_0\rangle}{\langle\varphi_i|u_0\rangle}:\ldots]$$

which reduces to

$$[\mathrm{b}_{\rho_1}(x_1,x_1,y,x_0),\ldots,\mathrm{b}_{\rho_1}(x_p,x_1,y,x_0),0,\ldots,0].$$

Now choose $v_0 \in \xi_2(x_0)$ and $\{\psi_1, \ldots, \psi_p\}$ such that $\psi_i \in \theta_2(x_i)$ and $\psi_i(v_0) = 1$. One sees that $\{\psi_1, \ldots, \psi_p\}$ is a basis of $\langle \theta_2(\partial_\infty \Gamma) \rangle$. One can then complete $\{\psi_1, \ldots, \psi_p\}$ to a basis

$$\mathcal{B}_2 = \{\psi_1, \ldots, \psi_p, \psi_{p+1}, \ldots, \psi_m\}$$

for $(\mathbb{R}^m)^*$ such that $\psi_i(\langle \xi_2(\partial_\infty \Gamma) \rangle) = 0$ for all $i > p$. One checks, as above, that if $y \in \partial_\infty \Gamma$, then the projective coordinates $\xi_2(y)$ with respect to the dual basis of $\mathcal{B}_2$ are given by

$$[\mathrm{b}_{\rho_2}(x_1, x_1, y, x_0), \ldots, \mathrm{b}_{\rho_2}(x_p, x_1, y, x_0), 0, \ldots, 0].$$

We now choose $g \in \mathsf{GL}_m(\mathbb{R})$ so that $g\varphi_i = \psi_i$ for all i. It follows from the fact that $\mathrm{b}_{\rho_1}(x_i, x_1, y, x_0) = \mathrm{b}_{\rho_2}(x_i, x_1, y, x_0)$ for all $i \leqslant p$, that $g \circ \xi_1 = \xi_2$.

Assume now that ρ_1 is irreducible, so that $p = m$. Lemma 2.17 implies that there exists a $(m+1)$-tuple $(x_0, \ldots, x_m)$ of points in $\partial_\infty \Gamma$ such that $F = (\xi_1(x_0), \ldots, \xi_1(x_m))$ is a projective frame for $\mathbb{RP}(m)$ and $F^* = (\theta_1(x_0), \ldots, \theta_1(x_m))$ is a projective frame for $\mathbb{RP}(m)^*$. Thus, using the notation of Lemma 10.7, we have that, given arbitrary distinct points $z, w \in \partial_\infty \Gamma$,

$$\mu_{F,F^*}(\pi_m(\rho_1(\gamma))) = [\mathrm{b}_{\rho_1}(x_i, z, \gamma(x_j), w)]$$

Similarly

$$\mu_{F,F^*}(g^{-1}\pi_m(\rho_2(\gamma))g) = \mu_{gF,gF^*}(\pi_m(\rho_2(\gamma))) = [\mathrm{b}_{\rho_2}(x_i, z, \gamma(x_j), w)]$$

Thus, since $\mathrm{b}_{\rho_1} = \mathrm{b}_{\rho_2}$,

$$\mu_{F,F^*}(\rho_1(\gamma)) = \mu_{F,F^*}(g^{-1}\rho_2(\gamma)g).$$

Since $\mu_{F,F*}$ is injective, see Lemma 10.6, it follows that

$$g\,(\pi_m \circ \rho_1)\,g^{-1} = \pi_m \circ \rho_2.$$

□

We can now prove our spectral rigidity theorem:

Proof of Theorem 11.1: Consider two projective Anosov representations $\rho_1 : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ and $\rho_2 : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ such that $\mathsf{L}(\gamma)(\rho_1) = \mathsf{L}(\gamma)(\rho_2)$ for all $\gamma \in \Gamma$. Suppose that α and β are infinite order, co-prime elements of Γ. Proposition 10.4 implies that

$$\begin{aligned}
\mathrm{b}_{\rho_1}(\beta^-, \alpha^-, \alpha^+, \beta^+) &= \lim_{n\to\infty} \frac{\mathsf{L}(\alpha^n\beta^n)(\rho_1)}{\mathsf{L}(\alpha)(\rho_1)^n \mathsf{L}(\beta)(\rho_1)^n} \\
&= \lim_{n\to\infty} \frac{\mathsf{L}(\alpha^n\beta^n)(\rho_2)}{\mathsf{L}(\alpha)(\rho_2)^n \mathsf{L}(\beta)(\rho_2)^n} \\
&= \mathrm{b}_{\rho_2}(\beta^-, \alpha^-, \alpha^+, \beta^+).
\end{aligned}$$

Since pairs of fixed points of infinite order elements of Γ are dense in $\partial_\infty \Gamma^{(2)}$ [27] and b_{ρ_1} and b_{ρ_2} are continuous, we see that $\mathrm{b}_{\rho_1} = \mathrm{b}_{\rho_2}$.

Lemma 11.4 implies that there exists $g \in \mathsf{GL}_m(\mathbb{R})$ such that $g \circ \xi_1 = \xi_2$. If ρ_1 is irreducible, then Lemma 11.4 guarantees that $g\,(\pi_m \circ \rho_1)\,g^{-1} = \pi_m \circ \rho_2$, so

$$\pi_m \circ (g\rho_1 g^{-1}) = \pi_m \circ \rho_2.$$

Notice that if A and B are proximal matrices such that $\pi(A) = \pi(B)$ and that the eigenvalues of A and B of maximal absolute value have the same sign, then $A = B$. Therefore, if α is any infinite order element of Γ, $g\rho_2(\alpha)g^{-1} = \rho_1(\alpha)$. It follows that $g\rho_2 g^{-1} = \rho_1$ as claimed. □

11.2. **Renormalized intersection rigidity.** Theorem 11.2 follows from Corollary 2.20, Corollary 8.2 and Corollary 11.6 below, which is a consequence of a deep result of Benoist [3].

If G is a real-algebraic semi-simple Lie group, let $\mathfrak{a}_{\mathsf{G}}$ be a Cartan subspace of the Lie algebra $\mathfrak{g}$ of G and let $\mathfrak{a}_{\mathsf{G}}^{+}$ be a Weyl Chamber. Let $\mu_{\mathsf{G}} : \mathsf{G} \to \mathfrak{a}_{\mathsf{G}}^{+}$ be the Jordan projection.

Let

$$(\mathfrak{a}_{\mathsf{G}}^{+})^{*} = \{\varphi \in \mathfrak{a}_{\mathsf{G}}^{*} : \varphi|\mathfrak{a}_{\mathsf{G}}^{+} \geqslant 0\}.$$

If φ lies in the interior $\mathrm{int}(\mathfrak{a}_{\mathsf{G}}^{+})^{*}$ of $(\mathfrak{a}_{\mathsf{G}}^{+})^{*}$, then if $v \in \mathfrak{a}_{\mathsf{G}}^{+}$ and $\varphi(v) = 0$, then $v = 0$.

For a subgroup Δ of G the *limit cone* $\mathscr{L}_{\Delta}$ of Δ is the smallest closed cone in $\mathfrak{a}_{\mathsf{G}}^{+}$ that contains

$$\{\mu(g) : g \in \Delta\}.$$

Benoist [3] proved that Zariski dense subgroups have limit cones with non-empty interior.

Theorem 11.5. [Benoist] *If Δ is a Zariski dense subgroup of a connected real-algebraic semi-simple Lie group G, then $\mathscr{L}_{\Delta}$ has non empty interior.*

Benoist's theorem implies the following corollary, which was explained to us by J.-F. Quint. This corollary is a stronger version of a result of Dal'Bo-Kim [23] (see also Labourie [50, Prop. 5.3.6]).

Corollary 11.6. [Quint] *Suppose that Δ is a group, G_{ρ} and G_{η} are center-free connected real-algebraic semi-simple Lie groups without compact factors, and $\rho : \Delta \to \mathsf{G}_{\rho}$ and $\eta : \Delta \to \mathsf{G}_{\eta}$ are Zariski dense representations. If there exist $\varphi_1 \in \mathrm{int}(\mathfrak{a}_{G_{\rho}}^{+})^{*}$ and $\varphi_2 \in \mathrm{int}(\mathfrak{a}_{\mathsf{G}_{\eta}}^{+})^{*}$ such that for all $g \in \Delta$ one has*

$$\varphi_1(\mu_{\mathsf{G}_{\rho}}(\rho(g))) = \varphi_2(\mu_{\mathsf{G}_2}(\eta(g))),$$

then $\eta \circ \rho^{-1} : \Delta \to \Delta$ extends to an isomorphism $\mathsf{G}_{\rho} \to \mathsf{G}_{\eta}$.

Proof. Let H be the Zariski closure of the image of the product representation $\rho \times \eta : \Delta \to \mathsf{G}_{\rho} \times \mathsf{G}_{\eta}$, defined by $g \mapsto (\rho g, \eta g)$. Since the equation

$$\varphi_1(\mu_{\mathsf{G}_{\rho}}(g_1)) = \varphi_2(\mu_{\mathsf{G}_{\eta}}(g_2)) \tag{55}$$

holds for every pair $(g_1, g_2) \in \rho \times \eta\,(\Delta)$, Benoist's [3] Theorem 11.5 implies that the same relation holds for every pair $(g_1, g_2) \in \mathsf{H}$.

The group $\mathsf{H} \cap (\mathsf{G}_{\rho} \times \{e\})$ is a normal subgroup of G_{ρ}, it is hence (up to finite index) a product of simple factors. Equation (55) implies that for all $(g, e) \in \mathsf{H} \cap (\mathsf{G}_{\rho} \times \{e\})$ necessarily one has $\varphi_1(\mu_{\mathsf{G}_{\rho}} g) = 0$. Since $\varphi_1(v) > 0$ for all $v \in \mathfrak{a}_{\mathsf{G}_{\rho}}^{+} - \{0\}$, one has $\mu_{\mathsf{G}_{\rho}}(g) = 0$. This implies that $\mathsf{H} \cap (\mathsf{G}_{\rho} \times \{e\})$ is a normal compact subgroup of G_{ρ}. Since G_{ρ} does not have compact factors and is center free one concludes that $\mathsf{H} \cap (\mathsf{G}_{\rho} \times e) = \{e\}$.

The same argument implies that $\mathsf{H} \cap (\{e\} \times \mathsf{G}_{\eta}) = \{e\}$ and hence H is the graph of an isomorphism extending $\eta \circ \rho^{-1}$. $\square$

11.3. **Rigidity for Hitchin representations.** O. Guichard [30] has announced a classification of the Zariski closures of lifts of Hitchin representations.

Theorem 11.7. [Guichard] *If $\rho : \pi_1(S) \to \mathsf{SL}_m(\mathbb{R})$ is the lift of a Hitchin representation and H is the Zariski closure of $\rho(\pi_1(S))$, then*

- *If $m = 2n$ is even,* H *is conjugate to either* $\tau_m(\mathsf{SL}_2(\mathbb{R}))$, $\mathsf{Sp}(2n,\mathbb{R})$ *or* $\mathsf{SL}_{2n}(\mathbb{R})$.
- *If $m = 2n+1$ is odd and $m \neq 7$, then* H *is conjugate to either* $\tau_m(\mathsf{SL}_2(\mathbb{R}))$, $\mathsf{SO}(n, n+1)$ *or* $\mathsf{SL}_{2n+1}(\mathbb{R})$.
- *If $m = 7$, then* H *is conjugate to either* $\tau_7(\mathsf{SL}_2(\mathbb{R}))$, G_2, $\mathsf{SO}(3,4)$ *or* $\mathsf{SL}_7(\mathbb{R})$.

where $\tau_m : \mathsf{SL}_2(\mathbb{R}) \to \mathsf{SL}_m(\mathbb{R})$ *is the irreducible representation.*

Notice in particular, that the Zariski closure of the lift of a Hitchin representation is always simple and connected. We can then apply our rigidity theorem for renormalized intersection to get a rigidity statement which is independent of dimension in the Hitchin setting.

Corollary 11.8. [Hitchin rigidity] *Let S be a closed, orientable surface and let $\rho_1 \in \mathcal{H}_{m_1}(S)$ and $\rho_2 \in \mathcal{H}_{m_2}(S)$ be two Hitchin representations such that*

$$\mathbf{J}(\rho_1,\rho_2) = 1.$$

Then,

- *either $m_1 = m_2$ and $\rho_1 = \rho_2$ in $\mathcal{H}_{m_1}(S)$,*
- *or there exists an element ρ of the Teichmüller space $\mathcal{T}(S)$ so that $\rho_1 = \tau_{m_1}(\rho)$ and $\rho_2 = \tau_{m_2}(\rho)$.*

Observe that the second case in the corollary only happens if both ρ_1 and ρ_2 are Fuchsian.

Proof. In order to apply our renormalized intersection rigidity theorem, we will need the following analysis of the outer automorphism groups of the Lie algebras of Lie groups which arise as Zariski closures of lifts of Hitchin representations. This analysis was carried about by Gündoğan [31] (see Corollary 2.15 and its proof).

Theorem 11.9. [Gündoğan [31]] *Let $\mathrm{Out}(\mathfrak{g})$ be the group of exterior automorphism of the Lie algebra $\mathfrak{g}$. Then, if $n > 0$,*

1. *If $\mathfrak{g} = \mathfrak{sl}_{2n+2}(\mathbb{R})$, then $\mathrm{Out}(\mathfrak{g})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and is generated by $X \mapsto -X^t$, and conjugation by an element of* $\mathsf{GL}_{2n+2}(\mathbb{R})$.
2. *If $\mathfrak{g} = \mathfrak{sl}_{2n+1}(\mathbb{R})$, then $\mathrm{Out}(\mathfrak{g})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and is generated by $X \mapsto -X^t$.*
3. *If $\mathfrak{g} = \mathfrak{so}(n, n+1, \mathbb{R})$, then $\mathrm{Out}(\mathfrak{g})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and is generated by conjugation by an element of* $\mathsf{SL}_{2n+1}(\mathbb{R})$.
4. *If $\mathfrak{g} = \mathfrak{sp}(2n+2, \mathbb{R})$, then $\mathrm{Out}(\mathfrak{g})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and is generated by conjugation by an element of* $\mathsf{GL}_{2n+2}(\mathbb{R})$.
5. *If $\mathfrak{g} = \mathfrak{g}_2$ then $\mathrm{Out}(\mathfrak{g})$ is trivial.*
6. *If $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$, then $\mathrm{Out}(\mathfrak{g})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and is generated by conjugation by an element of* $\mathsf{GL}_2(\mathbb{R})$.
7. *If $\mathfrak{g} = \mathfrak{so}(n, 1, \mathbb{R})$, then $\mathrm{Out}(\mathfrak{g})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and is generated by conjugation by an element of* $\mathsf{GL}_{n+1}(\mathbb{R})$.

Let $\rho_1 : \pi_1(S) \to \mathsf{PSL}_{m_1}(\mathbb{R})$ and $\rho_2 : \pi_1(S) \to \mathsf{PSL}_{m_2}(\mathbb{R})$ be two Hitchin representations such that

$$\mathbf{J}(\rho_1,\rho_2) = 1.$$

Theorem 11.7 implies that G_{ρ_1} and G_{ρ_2} are simple and connected and have center contained in $\{\pm I\}$.

Theorem 11.2 implies that there exists an isomorphism $\sigma : \mathsf{G}_{\rho_1} \to \mathsf{G}_{\rho_2}$ such that $\rho_2 = \sigma \circ \rho_1$. If G_1 is not conjugate to $\tau_{m_1}(\mathsf{SL}_2(\mathbb{R}))$, then it follows from Theorem 11.7, that $m_1 = m_2 = m$, and that, after conjugation of ρ_1, $\mathsf{G}_{\rho_1} = \mathsf{G}_{\rho_2} = \mathsf{H}$ so that σ is an automorphism of H.

We first observe that, since H is connected, there is an injective map from $\mathrm{Out}(\mathsf{H})$ to $\mathrm{Out}(\mathfrak{h})$. We now analyze the situation in a case-by-case manner using Gündoğan's Theorem 11.9.

(1) If $\mathsf{H} = \mathsf{PG}_2$, then σ is an inner automorphism, so $\rho_1 = \rho_2$ in $\mathcal{H}_7(S)$.

(2) If $\mathsf{H} = \mathsf{PSO}(n, n+1)$ or $\mathsf{H} = \mathsf{PSp}(2n, \mathbb{R})$, σ is either the identity or the conjugation by an element of $\mathsf{PGL}_{2n+1}(\mathbb{R})$ or $\mathsf{PGL}_{2n}(\mathbb{R})$, so $\rho_1 = \rho_2$ in $\mathcal{H}_{2n+1}(S)$ or $\mathcal{H}_{2n}(S)$.

(3) If $\mathsf{H} = \mathsf{SL}_m(\mathbb{R})$, then, after conjugation of ρ_1 by an element of $\mathsf{PGL}_m(\mathbb{R})$, σ is either trivial or $\rho_2 = \eta \circ \rho_1$ where $\eta(g) = \mathrm{transpose}(g^{-1})$. If σ is non-trivial, then since $\mathbf{J}(\rho_1, \rho_2) = 1$ Corollary 8.2 implies that there exists $c > 0$ so that

$$c\mu_1(\rho_1(\gamma)) = \mu_1((\rho_2(\gamma)) = -\mu_m(\rho_1(\gamma))$$

for all $\gamma \in \Gamma$, where

$$(\mu_1, \dots, \mu_m) : \mathsf{SL}_m(\mathbb{R}) \to \{(a_1, \dots, a_m) \in \mathbb{R}^m : \sum a_i = 0 \text{ and } a_1 \geqslant \cdots \geqslant a_m\}$$

is the Jordan projection of $\mathsf{SL}_m(\mathbb{R})$. Thus, the limit cone of $\rho_1(\Gamma)$ has empty interior. Since $\rho_1(\Gamma)$ is Zariski dense, this contradicts Benoist's Theorem 11.5. Therefore, $\rho_1 = \rho_2$ in $\mathcal{H}_m(S)$ in this case as well.

(4) If G_{ρ_1} is conjugate to $\tau_{m_1}(\mathsf{SL}_2(\mathbb{R}))$, then G_{ρ_2} is conjugate to $\tau_{m_2}(\mathsf{SL}_2(\mathbb{R}))$. So, after conjugation, there exist Fuchsian representations, $\eta_1 : \pi_1(S) \to \mathsf{SL}_2(\mathbb{R})$ and $\eta_2 : \pi_1(S) \to \mathsf{SL}_2(\mathbb{R})$, such that $\rho_1 = \tau_{m_1} \circ \eta_1$, $\rho_2 = \tau_{m_1} \circ \eta_1$ and there exists an automorphism of σ of $\mathsf{SL}_2(\mathbb{R})$ such that $\sigma \circ \eta_1 = \eta_2$. Case (6) of Gündoğan's Theorem then implies that η_1 is conjugate to η_2 by an element of $\mathsf{GL}_2(\mathbb{R})$. Therefore, we are in the second case of Theorem 11.8. This completes the proof.

□

11.4. **Benoist representations.** We say that an open subset Ω of $\mathbb{RP}(m)$ is *properly convex* if its intersection with any projective line is connected and its closure $\bar{\Omega}$ is contained in the complement of a projective hyperplane. Moreover, a properly convex open set Ω is said to be *strictly convex* if its boundary $\partial\Omega$ does not contain a projective line segment. A subgroup $\Delta \subset \mathrm{Aut}(\Omega) = \{g \in \mathsf{PGL}_m(\mathbb{R}) : g\Omega = \Omega\}$ is said to *divide* the open properly convex set Ω if the quotient $\Delta\backslash\Omega$ is compact. Benoist [6, Thm. 1.1] proved that if Δ divides the properly convex open set Ω, then Ω is strictly convex if and only if Δ is hyperbolic.

Definition 11.10. *If Γ is a torsion-free hyperbolic group, a faithful representation $\rho : \Gamma \to \mathsf{PGL}_m(\mathbb{R})$ is a* Benoist representation *if $\rho(\Gamma)$ divides an open strictly convex set $\Omega \subset \mathbb{RP}(m)$.*

It is a consequence of Benoist's work [6] that a Benoist representation is irreducible and projective Anosov (see Guichard-Wienhard [29, Proposition 6.1] for a detailed explanation).

Benoist [7, Corollary 1.2] (see also Koszul [43]) proved that the space $B_m(\Gamma)$ of Benoist representations of Γ into $\mathsf{PSL}_m(\mathbb{R})$ is a collection of components of $\mathrm{Hom}(\Gamma, \mathsf{PSL}_m(\mathbb{R}))$. Let

$$\mathcal{B}_m(\Gamma) = B_m(\Gamma)/\mathsf{PGL}_m(\mathbb{R}).$$

We call the components of $\mathcal{B}_m(\Gamma)$ *Benoist components.*

Benoist [5, Theorem 1.3] proved that the Zariski closure of any Benoist representation is either $\mathsf{PSL}_m(\mathbb{R})$ or is conjugate to $\mathsf{PSO}(m-1,1)$. We may thus apply the technique of proof of Theorem 11.8 to prove:

Corollary 11.11. [Benoist rigidity] *Let $\rho_1, \rho_2 \in \mathcal{B}_m(\Gamma)$. If $\mathbf{J}(\rho_1,\rho_2) = 1$, then $\rho_1 = \rho_2$ in $\mathcal{B}_m(\Gamma)$.*

The same techniques also provide the following related rigidity result for Benoist representations. Observe that if ρ is a projective Anosov representation, then so is $\operatorname{Ad}\rho : \Gamma \to \mathsf{PGL}(\mathfrak{sl}(m,\mathbb{R}))$ (see the discussion in Guichard-Wienhard [29, Section 10.2]) If $\eta(g) = (g^{-1})^t$ for all $g \in \mathsf{PGL}_m(\mathbb{R})$, and $\rho \in \mathcal{B}_m(\Gamma)$, then $\eta \circ \rho$ is the dual (or contragredient) representation of ρ.

Corollary 11.12. *If $\rho_1, \rho_2 \in \mathcal{B}_m(\Gamma)$, then $\mathbf{J}(\operatorname{Ad}\rho_1, \operatorname{Ad}\rho_2) = 1$ if and only if either $\rho_1 = \rho_2$ or $\rho_2 = \eta \circ \rho_1$.*

As a consequence, we recover a result of Cooper-Delp [20] and Kim [41] which asserts that if $\rho_1, \rho_2 \in \mathcal{B}_m(\Gamma)$ are the holonomies of strictly convex projective structures with the same Hilbert marked length spectrum, then ρ_1 and ρ_2 either agree or are dual. Recall that if $\rho \in \mathcal{B}_m(\Gamma)$ and $\gamma \in \Gamma$, then the length, in the Hilbert metric, of the closed geodesic on $\rho(\Gamma)\backslash\Omega_\rho$ associated to $[\gamma]$ is

$$\frac{\mu_1(\rho(\gamma)) - \mu_m(\rho(\gamma))}{2}$$

(see, for example, Benoist [6, Proposition 5.1]). Furthermore, if $g \in \mathsf{PGL}_m(\mathbb{R})$ then

$$\log(\Lambda(\operatorname{Ad} g)) = \mu_1(g) - \mu_m(g).$$

Hence if ρ_1 and ρ_2 are the holonomies of strictly convex projective structures with the same Hilbert marked length spectrum, then $\Lambda(\operatorname{Ad}\rho_1(\gamma)) = \Lambda(\operatorname{Ad}\rho_2(\gamma))$ for all $\gamma \in \Gamma$. Hence, $\mathbf{J}(\operatorname{Ad}\rho, \operatorname{Ad}\rho_2) = 1$, so the result follows from Corollary 11.12.

12. Proofs of main results

In this section, we assemble the proofs of the results claimed in the introduction. Several of the results have already been established.

The inequality in Theorem 1.1 follows from Corollary 8.2 and rigidity follows from Theorem 11.2. Theorem 1.2 is proven in Section 11 as Theorem 11.1, while Corollary 1.5 is proven as Corollary 11.8.

Theorem 1.3 follows from Proposition 8.1 and Corollary 8.2. Theorem 1.10 combines the results of Propositions 4.1 and 5.7.

The proof of Theorem 1.4 is easily assembled.

Proof of Theorem 1.4: Consider the pressure form defined on $\mathcal{C}_g(\Gamma, \mathsf{G})$ as in Definition 8.3. Recall that by Corollary 8.2 the pressure form is non-negative. Moreover, by Corollary 10.2 the pressure form is positive definite, so gives a Riemannian metric. The invariance with respect to $\operatorname{Out}(\Gamma)$ follows directly from the definition.

Proof of Corollary 1.6: Corollary 7.6 implies that every Hitchin component lifts to a component of $\mathcal{C}_g(\pi_1(S), \mathsf{SL}_m(\mathbb{R}))$ which is an analytic manifold. Theorem 1.4 then assures that the pressure form is an analytic Riemannian metric which is invariant under the action of the mapping class group. Entropy is constant on the Fuchsian locus, so if $\rho_1, \rho_2 \in \mathcal{T}(S)$, the renormalized intersection has the form

$$\begin{aligned}\mathbf{J}(\tau_m \circ \rho_1, \tau_m \circ \rho_2) &= \lim_{T\to\infty} \frac{1}{\#(R_{\tau_m\circ\rho_1}(T))} \sum_{[\gamma]\in R_{\tau_m\circ\rho_1}} \frac{\log \Lambda(\tau_m \circ \rho_2)(\gamma)}{\log \Lambda(\tau_m \circ \rho_1)(\gamma)} \\ &= \lim_{T\to\infty} \frac{1}{\#(R_{\rho_1}(T))} \sum_{[\gamma]\in R_{\rho_1}} \frac{\log \Lambda(\rho_2)(\gamma)}{\log \Lambda(\rho_1)(\gamma)}\end{aligned}$$

Wolpert [71] showed that the Hessian of the final expression, regarded as a function on $\mathcal{T}(S)$, is a multiple of the Weil-Petersson metric (see also Bonahon [11] and McMullen [56, Theorem 1.12]).

Proof of Corollary 1.7: We may assume that Γ is the fundamental group of a compact 3-manifold with non-empty boundary, since otherwise $\mathcal{C}_c(\Gamma, \mathsf{PSL}_2(\mathbb{C}))$ consists of 0 or 2 points.

We recall, from Theorem 7.5, that the deformation space $\mathcal{C}_c(\Gamma, \mathsf{PSL}_2(\mathbb{C}))$ is an analytic manifold. Let $\alpha : \mathsf{PSL}_2(\mathbb{C}) \to \mathsf{SL}_m(\mathbb{R})$ be the Plücker representation given by Proposition 2.13.

If we choose co-prime infinite order elements α and β of Γ, we may define a global analytic lift

$$\omega : \mathcal{C}_c(\Gamma, \mathsf{PSL}_2(\mathbb{C})) \to \mathrm{Hom}(\Gamma, \mathsf{PSL}_2(\mathbb{C}))$$

by choosing $\omega([\rho])$ to be a representative $\rho \in [\rho]$ so that $\rho(\alpha)$ has attracting fixed point 0 and repelling fixed point ∞ and $\rho(\beta)$ has attracting fixed point 1. Then

$$A = \alpha \circ \omega : \mathcal{C}_c(\Gamma, \mathsf{PSL}_2(\mathbb{C})) \to \mathrm{Hom}(\Gamma, \mathsf{SL}_m(\mathbb{R}))$$

is an analytic family of projective Anosov homomorphisms.

We define the associated entropy $\bar{h}$ and renormalized intersection $\bar{\mathbf{J}}$ functions on $\mathcal{C}_c(\Gamma, \mathsf{PSL}_2(\mathbb{C}))$ by setting

$$\bar{h}([\rho]) = h(A([\rho])) \quad \text{and} \quad \bar{\mathbf{J}}([\rho_1], [\rho_2]) = \mathbf{J}(A([\rho_1]), A([\rho_2])).$$

Since ω is analytic, both $\bar{h}$ and $\bar{\mathbf{J}}$ vary analytically over $\mathcal{C}_c(\Gamma, \mathsf{PSL}_2(\mathbb{C}))$ and we may again define a non-negative 2-tensor on the tangent space $\mathsf{T}\mathcal{C}_c(\Gamma, \mathsf{PSL}_2(\mathbb{C}))$ which we again call the pressure form, by considering the Hessian of $\bar{\mathbf{J}}$.

Let $\mathsf{G} = \alpha(\mathsf{PSL}_2(\mathbb{C}))$. Then G is a reductive subgroup of $\mathsf{SL}_m(\mathbb{R})$. If $\rho(\Gamma)$ is Zariski dense, then $A(\rho)(\Gamma)$ is Zariski dense in G, so Lemma 2.21 implies that $\rho(\Gamma)$ contains a G-generic element. Since α is an immersion,

$$\alpha_* : H^1_\rho(\Gamma, \mathfrak{sl}_2(\mathbb{C})) \to H^1_{\alpha([\rho])}(\Gamma, \mathfrak{g})$$

is injective where $\mathfrak{g}$ is the Lie algebra of G. Corollary 10.2 then implies that the pressure form on $\mathsf{T}_\rho\mathcal{C}_c(\Gamma, \mathsf{PSL}_2(\mathbb{C}))$ is Riemannian if ρ is Zariski dense.

If $\rho = \omega([\rho])$ is not Zariski dense, then its limit set is a subset of $\hat{\mathbb{R}} \subset \hat{\mathbb{C}}$, and the Zariski closure of $\rho(\Gamma)$ is either $\mathsf{H}_1 = \mathsf{PSL}(2, \mathbb{R})$ or $\mathsf{H}_2 = \mathsf{PSL}(2, \mathbb{R}) \cup (z \to -z)\mathsf{PSL}(2, \mathbb{R})$. Since each H_i is a real semi-simple Lie group, Proposition 7.2 then implies that the subset of non-Zariski dense representations in $\mathcal{C}_c(\Gamma, \mathsf{PSL}_2(\mathbb{C}))$ is an analytic submanifold. We then again apply Corollary 10.2 to see that the restriction of the pressure form to the submanifold of non-Zariski dense representations is Riemannian.

The pressure form determines a path pseudo-metric on the deformation space $\mathcal{C}_c(\Gamma, \mathsf{PSL}_2(\mathbb{C}))$, which is a Riemannian metric off the analytic submanifold of non-Zariski dense representations and restricts to a Riemannian metric on the submanifold. Lemma 13.1 then implies that the path metric is actually a metric. This establishes the main claim.

Theorem 7.5 implies that if Γ is not either virtually free or virtually a surface group, then every $\rho \in \mathcal{C}_c(\Gamma, \mathsf{PSL}_2(\mathbb{C}))$ is Zariski dense. Auxiliary claim (1) then follows from our main claim.

In the case that Γ is the fundamental group of a closed orientable surface, then the restriction of the pressure metric to the Fuchsian locus is given by the Hessian of the intersection form **I**. It again follows from work of Wolpert [71] that the restriction to the Fuchsian locus is a multiple of the Weil–Petersson metric. This establishes auxiliary claim (2).

Proof of Corollary 1.8: Let $\alpha : \mathsf{G} \to \mathsf{SL}_m(\mathbb{R})$ be the Plücker representation given by Proposition 2.13. An analytic family $\{\rho_u : \Gamma \to \mathsf{G}\}_{u \in M}$ of convex cocompact homomorphisms parameterized by an analytic manifold M, gives rise to an analytic family $\{\alpha \circ \rho_u\}_{u \in M}$ of projective Anosov homomorphisms of Γ into $\mathsf{SL}_m(\mathbb{R})$. Theorem 1.3, and Corollary 2.14 then imply that topological entropy varies analytically for this family. Results of Patterson [59], Sullivan [69], Yue [72] and Corlette-Iozzi [22] imply that the topological entropy agrees with the Hausdorff dimension of the limit set, so Corollary 1.8 follows.

Proof of Corollary 1.9: Given a semi-simple real Lie group G with finite center and a non-degenerate parabolic subgroup P, let $\alpha : \mathsf{G} \to \mathsf{SL}_m(\mathbb{R})$ be the Plücker representation given by Proposition 2.13. Then $\mathsf{H} = \alpha(\mathsf{G})$ is a reductive subgroup of $\mathsf{SL}_m(\mathbb{R})$.

We will adapt the notation of Proposition 7.3. Let

$$\widehat{\mathcal{Z}}(\Gamma; \mathsf{G}, \mathsf{P}) = \tilde{\mathcal{Z}}(\Gamma; \mathsf{G}, \mathsf{P})/\mathsf{G}_0$$

where G_0 is the connected component of G. Then, $\widehat{\mathcal{Z}}(\Gamma; \mathsf{G}, \mathsf{P})$ is a finite analytic manifold cover of the analytic orbifold $\mathcal{Z}(\Gamma; \mathsf{G}, \mathsf{P})$ with covering transformations given by G/G_0, see Proposition 7.4. Since G^0 acts freely on $\tilde{\mathcal{Z}}(\Gamma; \mathsf{G}, \mathsf{P})$, the slice theorem implies that if $[\rho] \in \widehat{\mathcal{Z}}(\Gamma; \mathsf{G}, \mathsf{P})$, then there exists a neighborhood U of $[\rho]$ and a lift

$$\beta : U \to \tilde{\mathcal{Z}}(\Gamma; G, \mathsf{P}) \subset \mathrm{Hom}(\Gamma, \mathsf{G}).$$

Then $\omega = \alpha \circ \beta$ is an analytic family of H-generic projective Anosov homomorphisms parameterized by U. The Hessian of the pull-back of the renormalized intersection gives rise to an analytic 2-tensor, again called the pressure form, on $\mathsf{T}U$. Suppose that $v \in \mathsf{T}_z\tilde{U}$ has pressure norm zero. Then Corollary 10.2 implies that $\mathrm{D}\omega(v)$ is trivial in $H^1_{\omega(z)}(\Gamma, \mathfrak{h})$ where $\mathfrak{h}$ is the Lie algebra of H. Since α is an immersion,

$$\alpha_* : H^1_{\beta(z)}(\Gamma, \mathfrak{g}) \to H^1_{\omega(z)}(\Gamma, \mathfrak{h})$$

is an isomorphism. Since β_* identifies T_zU with $H^1_{\beta(z)}(\Gamma, \mathfrak{g})$ this implies that $v = 0$, so the pressure form on $\mathsf{T}U$ is non-degenerate. Therefore, the pressure form is an analytic Riemannian metric on $\widehat{\mathcal{Z}}(\Gamma; \mathsf{G}, \mathsf{P})$. Since the pressure form is invariant under the action of G/G_0 it descends to a Riemannian metric on $\widehat{\mathcal{Z}}(\Gamma; \mathsf{G}, \mathsf{P})$ This completes the proof.

13. Appendix

We used the following lemma in the proof of Corollary 1.7.

Lemma 13.1. *Let M be a smooth manifold and let W be a submanifold of M. Suppose that g is a smooth non negative symmetric 2-tensor g such that*

- *g is positive definite on T_xM if $x \in M \setminus W$,*
- *the restriction of g to T_xW is positive definite if $x \in W$.*

Then the path pseudo metric defined by g is a metric.

Proof. It clearly suffices to show that if $x \in M$, then there exists an open neighborhood U of M such that the restriction of g to U gives a path metric on U. If $x \in M \setminus W$, then we simply choose a neighborhood U of x contained in $M \setminus W$ and the restriction of g to U is Riemannian, so determines a path metric.

If $x \in W$ we can find a neighborhood U which is identified with a ball B in $\mathbb{R}^n$ so that $W \cap U$ is identified with $B \cap (\mathbb{R}^k \times \{0^{n-k}\})$. Possibly after restricting to a smaller neighborhood, we can assume that there exists $r > 0$ so that if $v \in \mathsf{T}_zB$ and v is tangent to $\mathbb{R}^k \times \{(z_{k+1}, \ldots, z_n)\}$, then $g(v,v) \geq r^2||v||^2$, where $||v||$ is the Euclidean norm of v. If $z, w \in B$, $z \neq w$ and one of them, say z, is contained in $M \setminus W$, then g is Riemannian in a neighborhood of z, so $d_{U,g}(z,w) > 0$ where $d_{U,g}$ is the path pseudo-metric on U induced by g. If $z, w \in W$, then the estimate above implies that $d_{U,g}(z,w) \geq r d_B(z,w)$ where d_B is the Euclidean metric on B. Therefore, $d_{U,g}$ is a metric on U and we have completed the proof. □

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