PRESSURE METRICS FOR DEFORMATION SPACES OF QUASIFUCHSIAN GROUPS WITH PARABOLICS

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Abstract. In this paper, we produce a mapping class group invariant pressure metric on the space $QF(S)$ of quasiconformal deformations of a co-finite area Fuchsian group uniformizing S. Our pressure metric arises from an analytic pressure form on $QF(S)$ which is degenerate only on pure bending vectors on the Fuchsian locus. Our techniques also show that the Hausdorff dimension of the limit set varies analytically.

Contents

1. Introduction

We construct a pressure metric on the quasifuchsian space $QF(S)$ of quasiconformal deformations, within $\mathsf{PSL}(2,\mathbb{C})$, of a Fuchsian group Γ in $\mathsf{PSL}(2,\mathbb{R})$ whose quotient $\mathbb{H}^2/\Gamma$ has finite area and is homeomorphic to the interior of a compact surface S. Our pressure metric is a mapping class group invariant path metric, which is a Riemannian metric on the complement of the submanifold of Fuchsian representations. Our metric and its construction generalize work of Bridgeman [9] in the case that $\mathbb{H}^2/\Gamma$ is a closed surface.

McMullen [31] initiated the study of pressure metrics, by constructing a pressure metric on the Teichmüller space of a closed surface. His pressure metric is one way of formalizing Thurston's notion of constructing a metric on Teichmüller space as the "Hessian of the length of a random geodesic" (see also Wolpert [49], Bonahon [4] and Fathi-Flaminio [18]) and like Thurston's metric it agrees with the classical Weil-Petersson metric. Subsequently, Bridgeman

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[9] constructed a pressure metric on quasifuchsian space, Bridgeman, Canary, Labourie and Sambarino [10] constructed pressure metrics on deformation spaces of Anosov representations, and Pollicott and Sharp [34] constructed pressure metrics on spaces of metric graphs (see also Kao [21]). The main tool in the construction of these pressure metrics is the Thermodynamic Formalism for topologically transitive, Anosov flows with compact support and their associated well-behaved finite Markov codings.

The major obstruction to extending the constructions of pressure metrics to deformation spaces of geometrically finite (rather than convex compact) Kleinian groups and related settings is that the support of the non-wandering portion of the geodesic flow is not compact and hence there is not a well-behaved finite Markov coding. Mauldin-Urbanski [30] and Sarig [40] extended the Thermodynamical Formalism to the setting of topologically mixing Markov shifts with countable alphabet and the (BIP) property. In the case of finite area hyperbolic surfaces, Stadlbauer [44] and Ledrappier and Sarig [27] construct and study a topologically mixing countable Markov coding with the (BIP) property for the non-wandering portion of the geodesic flow of the surface. In previous work, Kao [23] showed how to adapt the Thermodynamic Formalism in the setting of the Stadlbauer-Ledrappier-Sarig coding to construct pressure metrics on Teichmüller spaces of punctured surfaces.

We adapt the techniques developed by Bridgeman [9] and Kao [23] into our setting to construct a pressure metric which can again be naturally interpreted as the Hessian of the (renormalized) length of a random geodesic.

Theorem (Theorem 9.1)**.** *If S is a compact surface with non-empty boundary, the pressure form $\mathbb{P}$ on $QF(S)$ induces a $\mathrm{Mod}(S)$-invariant path metric, which is an analytic Riemannian metric on the complement of the Fuchsian locus.*

Moreover, if $v \in T_\rho(QF(S))$, then $\mathbb{P}(v,v) = 0$ if and only if ρ is Fuchsian and v is a pure bending vector.

The control obtained from the Thermodynamic Formalism allows us to see that the topological entropy of the geodesic flow of the quasifuchsian hyperbolic 3-manifold varies analytically over $QF(S)$. We recall that the topological entropy $h(\rho)$ of ρ is the exponential growth rate of the number of closed orbits of the geodesic flow of $N_\rho = \mathbb{H}^3/\rho(\Gamma)$ of length at most T. More precisely, if

$$R_T(\rho) = \{[\gamma] \in [\Gamma] \mid 0 < \ell_\rho(\gamma) \leq T\},$$

where $[\Gamma]$ is the collection of conjugacy classes in Γ and $\ell_\rho(\gamma)$ is the translation length of the action of $\rho(\gamma)$ on $\mathbb{H}^3$, then the topological entropy is given by

$$h(\rho) = \lim_{T\to\infty} \frac{\#R_T(\rho)}{T}.$$

Sullivan [47] showed that the topological entropy and the Hausdorff dimension of the limit set agree for quasifuchsian groups. So we see that the Hausdorff dimension of the limit set varies analytically over $QF(S)$, generalizing a result of Ruelle [37] for quasifuchsian deformation spaces of closed surfaces. Schapira and Tapie [41, Thm. 6.2] previously established that the entropy is C^1 on $QF(S)$ and computed its derivative (as a special case of a much more general result).

Corollary (Corollary 5.3)**.** *If S is a compact surface with non-empty boundary, then the Hausdorff dimension of the limit set varies analytically over $QF(S)$.*

Concretely, the pressure form $\mathbb{P}$ at a representation ρ_0 is the Hessian of the renormalized pressure intersection $J(\rho_0,\cdot)$ at ρ_0. The pressure intersection of $\rho,\eta \in QF(S)$ is given by

$$I(\rho,\eta) = \lim_{T\to\infty} \frac{1}{|R_T(\rho)|} \sum_{[\gamma]\in R_T(\rho)} \frac{\ell_\eta(\gamma)}{\ell_\rho(\gamma)}$$

and the renormalized pressure intersection is given by

$$J(\rho,\eta) = \frac{h(\eta)}{h(\rho)} \lim_{T\to\infty} \frac{1}{|R_T(\rho)|} \sum_{[\gamma]\in R_T(\rho)} \frac{\ell_\eta(\gamma)}{\ell_\rho(\gamma)}.$$

The pressure intersection was first defined by Burger [12] for pairs of convex cocompact Fuchsian representations. Schapira and Tapie [41] defined an intersection function for negatively curved manifolds with an entropy gap at infinity, by generalizing the geodesic stretch considered by Knieper [26] in the compact setting. Their definition applies in a much more general framework, but agrees with our notion in this setting, see [41, Prop. 2.17].

Let (Σ^+,σ) be the Stadlbauer-Ledrappier-Sarig coding of a Fuchsian group Γ giving a finite area uniformization of S. If $\rho \in QF(S)$ we construct a roof function $\tau_\rho : \Sigma^+ \to \mathbb{R}$ whose periods are translation lengths of elements of $\rho(\Gamma)$. The key technical work in the paper is a careful analysis of these roof functions. In particular, we show that they vary analytically over $QF(S)$, see Proposition 3.1. If P is the Gurevich pressure function (on the space of all well-behaved roof functions), then the topological entropy $h(\rho)$ of ρ is the unique solution of $P(-t\tau_\rho) = 0$. Our actual working definition of the intersection function will be expressed in terms of equilibrium states on Σ^+ for the functions $-h(\rho)\tau_\rho$, but we will show in Theorem 10.3 that this thermodynamical definition agrees with the more geometric definition given above.

Following Burger [12], if $\rho,\eta \in QF(S)$, we define, the *Manhattan curve*

$$\mathcal{C}(\rho,\eta) = \{(a,b)\ |a,b \geq 0,\ a+b>0, \text{ and } P(-a\tau_\rho - b\tau_\eta) = 0\}.$$

The following result generalizes work of Burger [12] and Kao [22].

Theorem (Theorems 6.1 and 10.3)**.** *If S is a compact surface with non-empty boundary, and $\rho,\eta \in QF(S)$, then $\mathcal{C}(\rho,\eta)$*

(1) *is a closed subsegment of an analytic curve,*
(2) *has endpoints $(h(\rho),0)$ and $(0,h(\eta))$,*
(3) *and is strictly convex, unless ρ and η are conjugate in* $\mathrm{Isom}(\mathbb{H}^3)$.

Moreover, the tangent line to $\mathcal{C}(\rho,\eta)$ at $(h(\rho),0)$ has slope $-I(\rho,\eta)$.

We use Theorem 6.1 in our proof of a rigidity result for the renormalized pressure intersection, see Corollary 7.2 , and in our proof that pressure intersection is analytic on $QF(S)\times QF(S)$, see Proposition 7.1. We also use it to obtain a rigidity theorem for weighted entropy in the spirit of the Bishop-Steger rigidity theorem for Fuchsian groups, see [3]. If $a,b>0$ and $\rho,\eta \in QF(S)$, we define the weighted entropy

$$h^{a,b}(\rho,\eta) = \lim \frac{1}{T} \#\{[\gamma] \in [\Gamma]\ |\ a\ell(\rho(\gamma)) + b\ell(\eta(\gamma)) \leq T\}.$$

Corollary (Corollary 6.3)**.** *If S is a compact surface with non-empty boundary, $\rho,\eta \in QF(S)$ and $a,b>0$, then*

$$h^{a,b}(\rho,\eta) \leq \frac{h(\rho)h(\eta)}{bh(\rho)+ah(\eta)}$$

with equality if and only if $\rho = \eta$.

Other viewpoints: If $\rho \in QF(S)$, then $N_\rho = \mathbb{H}^3/\rho(\Gamma)$ is a geometrically finite hyperbolic 3-manifold. As such its dynamics may be analyzed using techniques from dynamics which do not rely on symbolic dynamics. For example, it naturally fits into the frameworks for geometrically finite negatively curved manifolds developed by Dal'bo-Otal-Peigné [14], negatively curved Riemannian manifolds with bounded geometry as studied by Paulin-Pollicott-Schapira [33] and negatively curved manifolds with an entropy gap at infinity as studied by Schapira-Tapie [41]. In particular, the existence of equilibrium states and their continuous variation in our setting also follows from the work of Schapira and Tapie [41].

Since all the geodesic flows of manifolds in $QF(S)$ are Hölder orbit equivalent, one should be able to think of them all as arising from an analytically varying family of Hölder potential functions on the geodesic flow of a fixed hyperbolic 3-manifold. However, for the construction of the pressure metric it will be necessary to know that the pressure function is at least twice differentiable. Results of this form do not yet seem to be available without symbolic dynamics. We have therefore chosen to develop the theory entirely from the viewpoint of the coding throughout the paper.

Iommi, Riquelme and Velozo [20] have previously used the Dal'bo-Peigné coding [16] to study negatively curved manifolds of extended Schottky type. These manifolds include the hyperbolic 3-manifolds associated to all quasiconformal deformations of finitely generated Fuchsian groups whose quotients have infinite area. In particular, they perform a phase transition analysis and show the existence and uniqueness of equilibrium states in their setting. The symbolic approach to phase transition analysis can be traced back to Iommi-Jordan [19]. Riquelme and Velozo [35] work in a more general setting which includes quasifuchsian groups with parabolics, but without a coding, and obtain a phase transition analysis for the pressure function as well as the existence of equilibrium measures.

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2. Background

2.1. **Quasifuchsian space.** Let S be a compact orientable surface with non-empty boundary and suppose that $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is a discrete torsion-free group so that $\mathbb{H}^2/\Gamma$ is a finite area hyperbolic surface homeomorphic to the interior of S. We say that $\rho : \Gamma \to \mathsf{PSL}(2,\mathbb{C})$ is *quasifuchsian* if there exists a quasiconformal homeomorphism $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\rho(\gamma) = \phi\gamma\phi^{-1}$ for all $\gamma \in \Gamma$. Equivalently, ρ is quasifuchsian if and only if there is an orientation-preserving bilipschitz homeomorphism from $N_\rho = \mathbb{H}^3/\rho(\Gamma)$ to $N = \mathbb{H}^3/\Gamma$ in the homotopy class determined by ρ (see Douady-Earle [17]). Let $QC(\Gamma) \subset \mathrm{Hom}(\Gamma, \mathsf{PSL}(2,\mathbb{C}))$ denote the space of all quasifuchsian representations. We recall, see Maskit [28, Thm. 2], that $\rho : \Gamma \to \mathsf{PSL}(2,\mathbb{C})$ is quasifuchsian if and only if ρ is discrete and faithful, $\rho(\partial S)$ is parabolic and $\rho(\Gamma)$ preserves a Jordan curve in $\widehat{\mathbb{C}}$.

The *quasifuchsian space* is given by

$$QF(S) = QC(\Gamma)/\mathsf{PSL}(2,\mathbb{C}) \subset X(S) = \mathrm{Hom}_{tp}(\Gamma, \mathsf{PSL}(2,\mathbb{C}))//\mathsf{PSL}(2,\mathbb{C})$$

where $\mathrm{Hom}_{tp}(\Gamma, \mathsf{PSL}(2,\mathbb{C}))$ is the space of type-preserving representations of Γ into $\mathsf{PSL}(2,\mathbb{C})$ (i.e. representations taking parabolic elements of Γ to parabolic elements of $\mathsf{PSL}(2,\mathbb{C})$). We call $X(S)$ the relative character variety and it has the structure of a projective variety. The space $QF(S)$ is a smooth open subset of $X(S)$, so is naturally a complex analytic manifold.

(See Kapovich [24, Section 4.3] for details.) Bers [2] showed that $QF(S)$ admits a natural identification with $\mathcal{T}(S) \times \mathcal{T}(S)$.

If $\rho \in QC(\Gamma)$ and ϕ is a quasiconformal map such that $\rho(\gamma) = \phi\gamma\phi^{-1}$ for all $\gamma \in \Gamma$, then ϕ restricts to a ρ-equivariant map $\xi_\rho : \Lambda(\Gamma) \to \Lambda(\rho(\Gamma))$ where $\Lambda(\rho(\Gamma))$ is the limit set of $\rho(\Gamma)$, i.e. the smallest closed $\rho(\Gamma)$-invariant subset of $\widehat{\mathbb{C}}$. Notice that since ξ_ρ is ρ-equivariant, it must take the attracting fixed point γ^+ of a hyperbolic element $\gamma \in \Gamma$ to the attracting fixed point $\rho(\gamma)^+$ of $\rho(\gamma)$. Since attracting fixed points of hyperbolic elements are dense in $\Lambda(\Gamma)$, ξ_ρ depends only on ρ (and not on the choice of quasiconformally conjugating map ϕ). We now record well-known fundamental properties of this limit map.

Lemma 2.1. *If $\rho \in QC(\Gamma)$, then there exists a ρ-equivariant bi-Hölder continuous map*

$$\xi_\rho : \Lambda(\Gamma) \to \Lambda(\rho(\Gamma)).$$

Moreover, if $x \in \Lambda(\Gamma)$, then $\xi_\rho(x)$ varies complex analytically over $QC(\Gamma)$.

Proof. Since each ξ_ρ is the restriction of a quasiconformal map $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and quasiconformal maps are bi-Hölder (see [1, Thm. 10.3.2]), ξ_ρ is also bi-Hölder.

Suppose that $\{\rho_z\}_{z\in\Delta}$ is a complex analytic family of representations parameterized by the unit disk Δ. If γ is a hyperbolic element of Γ, then $\{\xi_{\rho_z}(\gamma^+) = \rho_z(\gamma)^+\}$ varies complex analytically over Δ, since $\{\rho_z(\gamma)\}$ varies complex analytically. Since attracting fixed points of hyperbolic elements are dense in $\Lambda(\Gamma)$, Slodkowski's λ-lemma [43] implies that $\{\xi_{\rho_z}(x)\}$ varies complex analytically over Δ for all $x \in \Lambda(\Gamma)$. Hartogs' Theorem then implies that $\xi_\rho(x)$ varies complex analytically over all of $QC(\Gamma)$. □

2.2. **Countable Markov Shifts.** A two-sided *countable Markov shift* with alphabet $\mathcal{A}$ and transition matrix $\mathbb{T} \in \{0,1\}^{\mathcal{A}\times\mathcal{A}}$ is the set

$$\Sigma = \{x = (x_i) \in \mathcal{A}^{\mathbb{Z}} \mid t_{x_i x_{i+1}} = 1 \text{ for all } i \in \mathbb{Z}\}$$

equipped with a shift map $\sigma : \Sigma \to \Sigma$ which takes $(x_i)_{i\in\mathbb{Z}}$ to $(x_{i+1})_{i\in\mathbb{Z}}$. Notice that the shift simply moves the letter in place i into place $i-1$, i.e. it shifts every letter one place to the left.

Associated to any two-sided countable Markov shift Σ is the one-sided countable Markov shift

$$\Sigma^+ = \{x = (x_i) \in \mathcal{A}^{\mathbb{N}} \mid t_{x_i x_{i+1}} = 1 \text{ for all } i \in \mathbb{N}\}$$

equipped with a shift map $\sigma : \Sigma^+ \to \Sigma^+$ which takes $(x_i)_{i\in\mathbb{N}}$ to $(x_{i+1})_{i\in\mathbb{N}}$. In this case, the shift deletes the letter x_1 and moves every other letter one place to the left. There is a natural projection map $p^+ : \Sigma \to \Sigma^+$ given by $p^+(x) = x^+ = (x_i)_{i\in\mathbb{N}}$ which simply forgets all the terms to the left of x_1. Notice that $p^+ \circ \sigma = \sigma \circ p^+$. We will work entirely with one-sided shifts, except in the final section.

One says that (Σ^+, σ) is *topologically mixing* if for all $a, b \in \mathcal{A}$, there exists $N = N(a,b)$ so that if $n \geq N$, then there exists $x \in \Sigma$ so that $x_1 = a$ and $x_n = b$. The shift (Σ^+, σ) has the big images and pre-images property (BIP) if there exists a finite subset $\mathcal{B} \subset \mathcal{A}$ so that if $a \in \mathcal{A}$, then there exists $b_0, b_1 \in \mathcal{B}$ so that $t_{b_0,a} = 1 = t_{a,b_1}$.

Given a one-sided countable Markov shift (Σ^+, σ) and a function $g : \Sigma^+ \to \mathbb{R}$, let

$$V_n(g)(x) = \sup\{|g(x) - g(y)| \mid x, y \in \Sigma^+,\ x_i = y_i \text{ for all } 1 \leq i \leq n\}$$

be the n^{th} *variation* of g. We say that g is *locally Hölder continuous* if there exists $C > 0$ and $\theta \in (0,1)$ so that

$$V_n(g) \leq C\theta^n$$

for all $n \in \mathbb{N}$. We say that two locally Hölder continuous functions $f : \Sigma^+ \to \mathbb{R}$ and $g : \Sigma^+ \to \mathbb{R}$ are *cohomologous* if there exists a locally Hölder continuous function $h : \Sigma^+ \to \mathbb{R}$ so that

$$f - g = h - h \circ \sigma.$$

Sarig [38] considers the associated *Gurevich pressure* of a locally Hölder continuous function $g : \Sigma^+ \to \mathbb{R}$, given by

$$P(g) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in \mathrm{Fix}^n \,\mid\, x_1 = a} e^{S_n g(x)}$$

for some (any) $a \in \mathcal{A}$ where

$$S_n(g)(x) = \sum_{i=1}^{n} g(\sigma^i(x))$$

is the *ergodic sum* and $\mathrm{Fix}^n = \{x \in \Sigma^+ \mid \sigma^n(x) = x\}$.

A Borel probability measure m on Σ^+ is said to be a *Gibbs state* for a locally Hölder continuous function $g : \Sigma^+ \to \mathbb{R}$ if there exists a constant $B > 1$ and $C \in \mathbb{R}$ so that

$$\frac{1}{B} \le \frac{m([a_1, \ldots, a_n])}{e^{S_n g(x) - nC}} \le B$$

for all $x \in [a_1, \ldots, a_n]\}$, where $[a_1, \ldots, a_n]$ is the *cylinder* consisting of all $x \in \Sigma^+$ so that $x_i = a_i$ for all $1 \le 1 \le n$.

The *transfer operator* is a central tool in the Thermodynamic Formalism. Recall that the *transfer operator* $\mathcal{L}_f \colon C^b(\Sigma^+) \to C^b(\Sigma^+)$ of a locally Hölder continuous function f over Σ^+ is defined by

$$\mathcal{L}_f(g)(x) = \sum_{y \in \sigma^{-1}(x)} e^{f(y)} g(y) \qquad \text{for all } x \in \Sigma^+.$$

If (Σ^+, σ) is topologically mixing and has the BIP property, ν is a Borel probability measure for Σ^+ and $(\mathcal{L}_f)^*(\nu) = e^{P(f)}\nu$ (where $(\mathcal{L}_f)^*$ is the dual of transfer operator), then ν is a Gibbs state for f, see Mauldin-Urbanski [30, Theorem 2.3.3].

A σ-invariant Borel probability measure m on Σ^+ is said to be an *equilibrium measure* for a locally Hölder continuous function $g : \Sigma^+ \to \mathbb{R}$ if

$$P(g) = h_\sigma(m) + \int_{\Sigma^+} g \, dm$$

where $h_\sigma(m)$ is the measure-theoretic entropy of σ with respect to the measure m. Mauldin and Urbanski [30, Thm. 2.2.9] and Sarig [40, Thm 4.9] show that if Σ^+ is topologically mixing and has BIP, f is locally Hölder continuous, f admits a shift invariant Gibbs state ν_f and $-\int f \nu_f < +\infty$, then ν_f is the unique equilibrium measure for f.

We say that $\{g_u : \Sigma^+ \to \mathbb{R}\}_{u \in M}$ is a *real analytic family* if M is a real analytic manifold and for all $x \in \Sigma^+$, $u \to g_u(x)$ is a real analytic function on M. Mauldin and Urbanski [30, Thm. 2.6.12, Prop. 2.6.13 and 2.6.14], see also Sarig ([39, Cor. 4],[40, Thm 5.10 and 5.13]), prove real analyticity properties of the pressure function and evaluate its derivatives. Here the *variance* of a locally Hölder continuous function $f : \Sigma^+ \to \mathbb{R}$ with respect to a probability measure m on Σ^+ is given by

$$\mathrm{Var}(f, m) = \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma^+} S_n\Big(\big(f - \int_{\Sigma^+} f \, dm\big)^2\Big) \, dm.$$

Theorem 2.2. (Mauldin-Urbanski, Sarig) *Suppose that (Σ^+, σ) is a one-sided countable Markov shift which has BIP and is topologically mixing. If $\{g_u : \Sigma^+ \to \mathbb{R}\}_{u \in M}$ is a real analytic family* *of locally Hölder continuous functions such that $P(g_u) < \infty$ for all u, then $u \to P(g_u)$ is real analytic.*

Moreover, if $v \in T_{u_0}M$ and there exists a neighborhood U of u_0 in M so that if $u \in U$, then $-\int_{\Sigma^+} g_u dm_{g_{u_0}} < \infty$, then

$$D_v P(g_u) = \int_{\Sigma^+} D_v(g_u(x)) \; dm_{g_{u_0}}$$

and

$$D_v^2 P(g_u) = \mathrm{Var}(D_v g_u, m_{g_{u0}}) + \int_{\Sigma^+} D_v^2 g_u dm_{g_{u_0}}$$

where $m_{g_{u_0}}$ is the unique equilibrium state for g_{u_0}.

2.3. **The Stadlbauer-Ledrappier-Sarig coding.** Stadlbauer [44] and Ledrappier-Sarig [27] describe a one-sided countable Markov shift (Σ^+, σ) with alphabet $\mathcal{A}$ which encodes the non-wandering set of the geodesic flow on $T^1(\mathbb{H}^2/\Gamma)$. In this section, we will sketch the construction of this coding and recall its crucial properties.

They begin with the classical coding of a free group, as described by Bowen and Series [7]. One begins with a fundamental domain D_0 for a convex cocompact Fuchsian Γ, containing the origin 0 in the Poincaré disk model, all of whose vertices lie in $\partial\mathbb{H}^2$, so that the set $\mathcal{S}$ of face pairings of D_0 is a minimal symmetric generating set for Γ. One then labels any translate $\gamma(D_0)$ by the group element γ. Any geodesic ray r_z beginning at the origin and ending at $z \in \Lambda(\Gamma)$ passes through an infinite sequence of translates, so we get a sequence $c(z) = (\gamma_k)_{k\in\mathbb{N}}$. One may then turn this into an infinite sequence in $\mathcal{S}$ by considering $b(z) = (\gamma_k\gamma_{k-1}^{-1})_{k\in\mathbb{N}}$ (where we adopt the convention that $\gamma_0 = id$.) If Γ is convex cocompact, this produces a well behaved one-sided Markov shift (Σ^+_{BS}, σ) with finite alphabet $\mathcal{S}$. The obvious map $\omega : \Sigma^+_{BS} \to \Lambda(\Gamma)$ which takes $b(z)$ to z is then a bi-Hölder and (Σ^+_{BS}, σ) encodes the non-wandering portion of the geodesic flow of $\mathbb{H}^2/\Gamma$.

If one attempts to implement this procedure when Γ is not convex cocompact, then one must omit all geodesic rays which end at a parabolic fixed point and there is no natural way to do this from a coding perspective. Moreover, if one simply restricts ω to the allowable words then ω will not be Hölder in this case. (To see that ω will not be Hölder, choose $x, y \in \Sigma^+_{BS}$, so that $x_i = y_i = \alpha$ for all $1 \le i \le n$, where α is a parabolic face-pairing, and $x_{n+1} \neq y_{n+1}$, then $d_{\Sigma^+_{BS}}(x,y) = e^{-n}$, while $d_{\partial\mathbb{H}^2}(\omega(x), \omega(y))$ is comparable to $\frac{1}{n^2}$.)

Roughly, the Stadlbauer-Ledrappier-Sarig begins with $c(z) = (\gamma_k)$ and clumps together all terms in $b(z) = (\gamma_k\gamma_{k-1}^{-1})$ which lie in a subword which is a high power of a parabolic element. One must then append to our alphabet all powers of minimal word length parabolic elements and and disallow infinite words beginning or ending in infinitely repeating parabolic elements. When Γ is geometrically finite, but not co-finite area, Dal'bo and Peigné [16] implemented this process to powerful effect for geometrically finite Fuchsian groups with infinite area quotients. However, when Γ is co-finite area, the actual description is more intricate. The states Stadlbauer-Ledrappier-Sarig use record a finite amount of information about both the past and the future of the trajectory.

Let $\mathcal{C}$ be the collection of all freely reduced words in $\mathcal{S}$ which have minimal word length in their conjugacy class and generate a maximal parabolic subgroup of Γ. Notice that minimal word length representative of a conjugacy class of α is unique up to cyclic permutation. (One may in fact choose D_0 so that all but one pair of parabolic elements of $\mathcal{C}$ is conjugate to a face-pairing.) Since there are only finitely many conjugacy classes of maximal parabolic subgroups of Γ, $\mathcal{C}$ is finite. They then choose a sufficiently large even number $2N$ so that the length of every element of $\mathcal{C}$ divides $2N$ and let $\mathcal{C}^*$ be the collection of powers of elements of $\mathcal{C}$ of length exactly $2N$. (One may assume that two elements of $\mathcal{C}^*$ share a subword of length at least 2 if and only if they are cyclic permutations of one another.)

Let $\mathcal{A}_1$ be the set of all strings $(b_0, b_1, \ldots, b_{2N})$ in $\mathcal{S}$ so that $b_0 b_1 \cdots b_{2N}$ is freely reduced in $\mathcal{S}$ and so that neither $b_1 b_2 \cdots b_{2N}$ or $b_0 b_1 \cdots b_{2N-1}$ lies in $\mathcal{C}^*$. Let $\mathcal{A}_2$ be the set of all freely reduced strings of the form $(b, w^s, w_1, \cdots, w_{k-1}, c)$ where $w = w_1 \ldots w_{2N} \in \mathcal{C}^*$, $b \in \mathcal{S} - \{w_{2N}\}$, $1 \leq k \leq 2N$, $s \geq 1$ and $c \in \mathcal{S} - \{w_k\}$.

Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and define functions

$$r : \mathcal{A} \to \mathbb{N} \qquad \text{and} \qquad G : \mathcal{A} \to \Gamma$$

by letting $r(a) = 1$ if $a \in \mathcal{A}_1$ and $r(b, w^s, w_1, \ldots, w_{k-1}, c) = s+1$ otherwise. If $a = (b_0, b_1, \ldots, b_{2N}) \in \mathcal{A}_1$, then $G(a) = b_1$. If $a = (b, w^s, w_1 \cdots w_{k-1}, c)$, then let $G(a) = w^{s-1} w_1 \cdots w_{k+1}$. Notice that, by construction, if $n \in \mathbb{N}$, then

$$\#(r^{-1}(n)) \leq \#(\mathcal{C}^*) \left(\#(\mathcal{S})^2\right) (2N).$$

So, $r^{-1}(n)$ is always non-empty and there exists D so that $r^{-1}(n)$ has size at most D for all $n \in \mathbb{N}$, i.e. there are at most D states associated to each positive integer.

Given a geodesic ray r_z beginning at the origin and ending at a point z in the set $\Lambda_c(\Gamma)$ of points in the limit set which are not parabolic fixed points, let $c(z) = (\gamma_k)_{k \in \mathbb{N}}$ be the sequence of elements of Γ which record the translates of D_0 which r_z passes through. Let $b(z) = (b_k(z)) = (\gamma_k \gamma_{k-1}^{-1}) \in \mathcal{S}^{\mathbb{N}}$. We then associate to r_z a finite collection of infinite words in $\mathcal{S}^{\mathbb{N} \cup \{0\}}$, by allowing b_0 to be any element of $\mathcal{S}$, so that $b_0 b_1 \cdots b_{2N}$ does not lie in $\mathcal{C}^*$.

Suppose we have a word $(b_k)_{k \in \mathbb{N} \cup \{0\}}$ arising from the previous construction. If $(b_0, b_1, \ldots, b_{2N}) \in \mathcal{A}_1$, then let $x_1 = (b_0, b_1, \ldots, b_{2N})$ and shift (b_i) rightward by 1 to compute x_2. If not, let x_1 be the unique sub-string of $b_0 b_1 \ldots b_k \cdots$ which begins at b_0 and is an element of $\mathcal{A}_2$. Then, $x_1 = (b_0, w^s, w_1 \cdots w_{k-1}, b_v)$ for some $w \in \mathcal{C}^*$, $s \in \mathbb{N}$ and $v = 2Ns + k - 1$. In this case, we shift (b_i) rightward by $2N(s-1) + k + 1$ to compute x_2. One then simply proceeds iteratively. By construction, if $x_i \in \mathcal{A}_2$, then x_{i+1} must lie in $\mathcal{A}_1$.

Examples: If Γ uniformizes a once-punctured torus, then $\mathcal{S} = \{\alpha, \alpha^{-1}, \beta, \beta^{-1}\}$ is a mimimal symmetric generating set for Γ and

$$\mathcal{C} = \{\alpha\beta\alpha^{-1}\beta^{-1}, \beta\alpha^{-1}\beta^{-1}\alpha, \alpha^{-1}\beta^{-1}\alpha\beta, \beta^{-1}\alpha\beta\alpha^{-1}, \beta\alpha\beta^{-1}\alpha^{-1}, \alpha\beta^{-1}\alpha^{-1}\beta, \beta^{-1}\alpha^{-1}\beta\alpha, \alpha^{-1}\beta\alpha\beta^{-1}\}.$$

If Γ uniformizes a four times punctured sphere, then one may choose D_0 so that $\mathcal{S} = \{\alpha, \alpha^{-1}, \beta, \beta^{-1}, \gamma, \gamma^{-1}\}$ and

$$\mathcal{C} = \{\alpha, \alpha^{-1}, \beta, \beta^{-1}, \gamma, \gamma^{-1}, \alpha\beta\gamma, \beta\gamma\alpha, \gamma\alpha\beta, \gamma^{-1}\beta^{-1}\alpha^{-1}, \beta^{-1}\alpha^{-1}\gamma^{-1}, \alpha^{-1}\gamma^{-1}\beta^{-1}\}.$$

The following proposition encodes crucial properties of the coding.

Proposition 2.3. (Ledrappier-Sarig [27, Lemma 2.1], Stadlbauer [44]) *Suppose that $\mathbb{H}^2/\Gamma$ is a finite area hyperbolic surface, then (Σ^+, σ) is topologically mixing, has the big images and pre-images property (BIP), and there exists a locally Hölder continuous finite-to-one map*

$$\omega : \Sigma^+ \to \Lambda(\Gamma)$$

so that $\omega(x) = \lim(G(x_1) \cdots G(x_n))(0)$ and $\omega(x) = G(x_1)\omega(\sigma(x))$. Moreover, if γ is a hyperbolic element of Γ, then there exists $x \in \mathrm{Fix}^n$, for some $n \in \mathbb{N}$, unique up to cyclic permutation, so that γ is conjugate to $G(x_1) \cdots G(x_n)$.

Notice that every element of $\mathcal{A}$ can be preceded and succeeded by some element of $\mathcal{A}_1$, so (Σ^+, σ) clearly has (BIP). The topological mixing property is similarly easy to see directly from the definition, so the main claim of this proposition is that ω is locally Hölder continuous.

Another crucial property of the coding is that the translates of the origin associated to the Stadbauer-Ledrappier-Sarig coding approach points in the limit set conically (see property (1) on page 15 in Ledrappier-Sarig [27]).

Lemma 2.4. (Ledrappier-Sarig [27, Property (1) on page 15]) *There exists $L > 0$ so that if $x \in \Sigma^+$ and $n \in \mathbb{N}$, then*

$$d(G(x_1)G(x_2)\cdots G(x_n)(0), \overrightarrow{0\omega(x)}) \le L.$$

Since the proof of Lemma 2.4 appears in the middle of a rather technical discussion in [27], we will sketch a proof in our language. Choose a compact subset $\hat{K}$ of $\mathbb{H}^2/\Gamma$ so that its complement is a collection of cusp regions bounded by curves which are images of horocycles in $\mathbb{H}^2$. Notice that if the portion of $\overrightarrow{0\omega(x)}$ between $\gamma_s(D_0)$ and $\gamma_{s+t}(D_0)$ lies entirely in the pre-image of $\hat{K}$, and $t > s$, then $\gamma_{s+t}\gamma_s^{-1}$ is a subword of a power of an element in $\mathcal{C}$. Let K be the intersection of the pre-image of $\hat{K}$ with D_0. Notice that we may assume that $0 \in K$ (by perhaps enlarging $\hat{K}$). Suppose the last $2N+1$ letters of x_n are $b_r \cdots b_{r+2N}$, then $\overrightarrow{0\omega(x)}$ intersects one of $\gamma_r(K), \ \ldots, \ \gamma_{r+2N}(K)$ (since otherwise $b_r \cdots b_{r+2N-1}$ or $b_{r+1} \cdots b_{r+2N+1}$ would lie in $\mathcal{C}^*$, which is disallowed). But then

$$d\big(G(x_1)\cdots G(x_n)(0), \overrightarrow{0\omega(x)}\big) \le R + \operatorname{diam}(K)$$

where

$$R = \max\Big\{ d(0, (s_1 \ldots s_p)(0)) \mid s_i \in \mathcal{S}, \ p \in \{1, \ldots, 2N\}\Big\}.$$

3. Roof functions for quasifuchsian groups

If $\rho \in QC(\Gamma)$, we define a *roof function* $\tau_\rho : \Sigma^+ \to \mathbb{R}$ by setting

$$\tau_\rho(x) = B_{\xi_\rho(\omega(x))}(b_0, \rho(G(x_1))(b_0))$$

where $b_0 = (0,0,1)$ and $B_z(x,y)$ is the Busemann function based at $z \in \partial\mathbb{H}^3$ which measures the signed distance between the horoballs based at z through x and y. In the Poincaré upper half space model, we write the Busemann function explicitly as

$$\hat{B}_z(p,q) = \log\left(\frac{|p-z|^2 h(p)}{|q-z|^2 h(q)}\right)$$

where $z \in \mathbb{C} \subset \partial\mathbb{H}^3$, $p, q \in \mathbb{H}^3$ and $h(p)$ is the Euclidean height of p above the complex plane and $\hat{B}_\infty(p,q) = \frac{h(p)}{h(q)}$.

It follows from the cocycle property of the Busemann function that

$$S_m\tau_\rho(x) = \sum_{i=0}^{m-1} \tau_\rho(\sigma^i(x)) = B_{\xi_\rho(\omega(x))}(b_0, \rho(G(x_1)\cdots G(x_m))(b_0)).$$

In particular, if $x = (\overline{x_1, \ldots, x_m}) \in \Sigma^+$, then

$$S_m\tau_\rho(x) = \ell_\rho(G(x_1)\cdots G(x_m)).$$

We say that the roof function τ_ρ is *eventually positive* if there exists $C > 0$ and $N \in \mathbb{N}$ so that if $n \ge N$ and $x \in \Sigma^+$, then $S_n\tau_\rho(x) \ge C$.

The following lemma records crucial properties of our roof functions. It generalizes similar results of Ledrappier-Sarig [27, Lemma 2.2 and 3.1] in the Fuchsian setting.

Proposition 3.1. *The family* $\{\tau_\rho\}_{\rho\in QC(\Gamma)}$ *of roof functions is a real analytic family of locally Hölder continuous, eventually positive functions.*

Moreover, if $\rho \in QC(\Gamma)$*, then there exists* $C_\rho > 0$ *and* $R_\rho > 0$ *so that*

$$2\log r(x_1) - C_\rho \le \tau_\rho(x) \le 2\log r(x_1) + C_\rho$$

and

$$\Big| S_n\tau_\rho(x) - d(b_0, G(x_1)\cdots G(x_n))(b_0))\Big| \le R_\rho$$

for all $x \in \Sigma^+$ *and* $n \in \mathbb{N}$.

Proof. Since $\xi_\rho(w)$ varies complex analytically in ρ for all $w \in \Lambda(\Gamma)$, by Lemma 2.1, and $B_z(b_0, y)$ is real analytic in $z \in \widehat{\mathbb{C}}$ and $y \in \mathbb{H}^3$, we see that $\tau_\rho(x)$ varies analytically over $QC(\Gamma)$ for all $x \in \Sigma^+$.

We next obtain our claimed bounds on the roof function. If $x \in \Sigma^+$, then

$$|\tau_\rho(x)| \le d(\rho(G(x_1)(b_0), b_0)$$

so if $a \in \mathcal{A}$, there exists C_a so that if $x_1 = a$, then $|\tau_\rho(x)| \le C_a$. Since our alphabet is infinite, our work is not done.

If $w \in \mathcal{C}^*$, we may normalize so that $\rho(w)(z) = z+1$ and $b_0 = (0, 0, b_w)$ in the upper half-space model for $\mathbb{H}^3$. If $z \in \mathbb{C} \subset \partial\mathbb{H}^3$ and $r > 0$, we let $B(z, r)$ denote the Euclidean ball of radius r about z in $\mathbb{C}$. Let

$$c_w = \sup\{|g_a(b_0)| \mid G(a) = w^s g_a \text{ for some } a \in \mathcal{A}_2\}.$$

Since g_a has length at most $2N + 1$ in the alphabet $\mathcal{S}$, c_w is finite. Suppose that $x \in \Sigma^+$, $r(x_1) \ge 2$ and $G(x_1) = w^s g_a$ where $s = r(a) - 2$. Then $\xi_\rho(x) \in \rho(w^s)(B(0, e^L c_w)) = B(s, e^L c_w)$, where L is the constant from Lemma 2.4, and

$$\begin{aligned}
\tau_\rho(x) &= \log\left(\frac{|b_0 - \xi_\rho(\omega(x))|^2 h(\rho(w^s g_a)(b_0))}{|\rho(w^s g_a)(b_0) - \xi_\rho(\omega(x))|^2 h(b_0)}\right)\\
&\le \log\left(\frac{(b_w^2 + (s + e^L c_w)^2) h(\rho(g_a)(b_0))}{h(\rho(g_a)(b_0))^2 b_w}\right) = \log\left(\frac{(b_w^2 + (s + e^L c_w)^2)}{h(\rho(g_a)(b_0)) b_w}\right).
\end{aligned}$$

Similarly,

$$\tau_\rho(x) \ge \log\left(\frac{(b_w^2 + (s - e^L c_w)^2) h(\rho(g_a)(b_0))}{\big(h(\rho(g_a)(b_0))^2 + e^{2L} c_w^2\big) b_w}\right).$$

Since there are only finitely many choices of g_a, it is easy to see that there exists C_w so that

$$2\log(r(a)) - C_w \le \tau_\rho(x) \le 2\log(r(a)) + C_w$$

whenever $x \in \Sigma^+$, $r(x_1) \ge 2$ and $G(x_1) = w^s g_a$. Since there are only finitely many w in $\mathcal{C}^*$ and only finitely many words a with $r(a) < 2$, we see that there exists C_ρ so that

$$2\log(r(x_1)) - C_\rho \le \tau_\rho(x) \le 2\log(r(x_1)) + C_\rho$$

for all $x \in \Sigma^+$.

Since ω is locally Hölder continuous, there exists A and $\alpha > 0$ so that if $x, y \in \Sigma^+$ and $x_i = y_i$ for $1 \le i \le n$, then

$$d(\omega(x), \omega(y)) \le A e^{-\alpha n}.$$

Since ξ_ρ is Hölder, there exist C and $\beta > 0$ so that $d(\xi_\rho(z), \xi_\rho(w)) \le C d(z, w)^\beta$ for all $z, w \in \Lambda(\Gamma)$, so

$$d(\xi_\rho(\omega(x)), \xi_\rho(\omega(y)) \le C A^\beta e^{-\alpha\beta n}.$$

If $a \in \mathcal{A}$, then let

$$D_a = \sup\left\{ \left| \frac{\partial}{\partial z}(B_z(b_0, \rho(G(a))(b_0)) \right| \;:\; z = \xi_\rho(\omega(x)) \text{ and } x_1 = a \right\},$$

so

$$\sup\{|\tau_\rho(x) - \tau_\rho(y)| \mid x, y \in [a, x_2, \ldots, x_n]\} \leq D_a C A^\beta e^{-\alpha\beta n}.$$

However, the best general estimate one can have on on D_a is $O(r(a))$, so we will have to dig a little deeper.

We again work in the upper half-space model, and assume that $r(a) \geq 3$, $G(a) = w^s g_a$ where $s = r(a) - 2$ and normalize as before so that $\rho(w)(z) = z + 1$. We then map the limit set into the boundary of the upper-half space model by setting $\hat{\xi}_\rho = \tau \circ \xi_\rho$ where τ takes the Poincaré ball model to the upper half-space model and takes the fixed point of $\rho(w)$ to ∞. Notice that τ is K_w-bilipschitz on $\tau^{-1}(B(0, e^L c_w))$. Therefore, if $x, y \in [a, x_2, \ldots, x_n]$, then

$$|\hat{\xi}_\rho(x) - \hat{\xi}_\rho(y)| \leq K_w C A^\beta e^{-\alpha\beta(n-1)}$$

Moreover, if we work in the ball model, there exists D_w so that

$$\left| \frac{\partial}{\partial z}(\hat{B}_z(b_0, \rho(G(a))(b_0)) \right| \leq D_w$$

if $z \in \rho(w)^s(B(0, e^L c_w))$, so

$$\sup\{|\tau_\rho(x) - \tau_\rho(y)| \;\big|\; x, y \in [a, x_2, \ldots, x_n]\} \leq K_w D_w C A^\beta e^{-\alpha\beta(n-1)}.$$

Since there are only finitely many a where $r(a) \leq 2$ and only finitely many choices of w, our bounds are uniform over $\mathcal{A}$ and so τ_ρ is locally Hölder continuous.

It remains to check that τ_ρ is eventually positive. Let L be the constant provided by Lemma 2.4. Given $x \in \Sigma^+$, let $\gamma_n = G(x_1) \cdots G(x_n)$, so $d(\gamma_n(0), \overrightarrow{0\omega(z)}) \leq L$ for all n. The limit map ξ_ρ extends to a ρ-equivariant K-bilipschitz embedding $\Xi_\rho : \mathbb{H}^2 \to \mathbb{H}^3$ so that $\Xi_\rho(0) = b_0$ (see Douady-Earle [17]). It follows, from the fellow traveller property for quasi-geodesics in $\mathbb{H}^3$, that there exists a constant M (depending only on K) so that $\Xi_\rho(\overrightarrow{0\omega(x)})$ lies within M of the geodesic joining $\overrightarrow{b_0\xi_\rho(\omega(x))}$. Therefore,

$$d(\rho(\gamma_n)(b_0), \overrightarrow{b_0\xi_\rho(\omega(x))}) \leq KL + M$$

for all $n \in \mathbb{N}$, so

$$\Big| S_n\tau_\rho(x) - d(b_0, G(x_1) \cdots G(x_n))(b_0)) \Big| \leq 2(KL + M) = R_\rho$$

Since the set

$$\mathcal{B} = \{\gamma \in \Gamma \mid d(\rho(\gamma)(b_0), b_0) \leq 4(KL + M)\}$$

is finite, there exists $\hat{N}$ so that if γ has word length at least $\hat{N}$ (in the generators given $\mathcal{S}$), then γ does not lie in $\mathcal{B}$. Therefore, if $n \geq \hat{N}$ and $x \in \Sigma^+$, then $S_n\tau_\rho(x) > R_\rho > 0$. Thus, τ_ρ is eventually positive and our proof is complete. □

It is a standard feature of the Thermodynamic Formalism that one may replace an eventually positive roof function by a roof function which is strictly positive and cohomologous to the original roof function. (For a statement and proof which includes the current situation, see [8, Lemma 3.3].)

Corollary 3.2. *If $\rho \in QC(\Gamma)$, there exists a locally Hölder continuous function $\hat{\tau}_\rho$ and $c > 0$ so that $\hat{\tau}_\rho(x) \geq c$ for all $x \in \Sigma^+$ and $\hat{\tau}_\rho$ is cohomologous to τ_ρ.*

4. Phase transition analysis

We begin by extending Kao's phase transition analysis, see Kao [23, Thm. 4.1], which characterizes which linear combinations of a pair of roof functions have finite pressure. The primary use of this analysis will be in the case of a single roof function, i.e. when $a = 1$ and $b = 0$. However, we will use the full force of this result in the proof of our Manhattan curve theorem, see Theorem 6.1.

Theorem 4.1. *If $\rho, \eta \in QC(\Gamma)$, $t \in \mathbb{R}$ and $a + b > 0$, then $P(-t(a\tau_\rho + b\tau_\eta))$ is finite if and only if $t > \frac{1}{2(a+b)}$. Moreover, $P(-t(a\tau_\rho + b\tau_\eta))$ is monotone decreasing and analytic in t on $(\frac{1}{2(a+b)}, \infty)$, and*

$$\lim_{t \to \frac{1}{2(a+b)}^+} P(-t(a\tau_\rho + b\tau_\eta)) = +\infty.$$

If, in addition $a, b \geq 0$, then

$$\lim_{t \to \infty} P(-t(a\tau_\rho + b\tau_\eta)) = -\infty.$$

Riquelme and Velozo [35, Thm. 1.4] previously established results closely related to Theorem 4.1 in the more general setting of negatively curved manifolds with bounded geometry.

Proof. Mauldin and Urbanski [30, Thm 2.1.9] proved that in our setting $P(f)$ is finite if and only if

$$Z_1(f) = \sum_{s \in \mathcal{A}} e^{\sup\{f(x) \ \mid \ x_1 = s\}}$$

converges. Proposition 3.1 implies that

$$Z_1(-t(a\tau_\rho + b\tau_\eta)) \leq D \sum_{n=0}^{\infty} e^{-t(a+b)(2\log n - \max\{C_\rho, C_\eta\})}$$

so $P(-t(a\tau_\rho + b\tau_\eta))$ converges if $t > \frac{1}{2(a+b)}$. Similarly, since $r^{-1}(n)$ is non-empty if $n \geq 1$, we see that

$$Z_1(-t(a\tau_\rho + b\tau_\eta)) \geq \sum_{n=1}^{\infty} e^{-t(a+b)(2\log n + \max\{C_\rho, C_\eta\})}$$

so $P(-t(a\tau_\rho + b\tau_\eta))$ does not converge if $t \leq \frac{1}{2(a+b)}$ and

$$\lim_{t \to \frac{1}{2(a+b)}^+} Z_1(-t(a\tau_\rho + b\tau_\eta)) = +\infty.$$

It follows from the definition that $P(-t(a\tau_\rho + b\tau_\eta))$ is monotone decreasing in t and Theorem 2.2 implies that it is analytic in t on $(\frac{1}{2(a+b)}, \infty)$. In the proof of [30, Thm. 2.1.9], Mauldin and Urbanski show that given a locally Hölder continuous function f, there exist constants $q, s, M, m > 0$ so that for any $n \in \mathbb{N}$, we have

$$\sum_{i=n}^{n+s(n-1)} Z_i(f) \geqslant \frac{e^{-M+(M-m)n}}{q^{n-1}} Z_1(f)^n.$$

where if E^n is the set of allowable words of length n in $\mathcal{A}$, then

$$Z_n(f) = \sum_{w \in E^n} e^{\sup\{S_n f(x) \ \mid \ x_i = w_i \ \forall 1 \leq i \leq n\}} \quad \text{and} \quad \lim \frac{1}{n} \log Z_n(f) = P(f).$$

It follows that for all n, there exist $A > 0$ and $\hat{n} \in [n, n + s(n-1)]$ such that $Z_{\hat{n}} \geq A^n Z_1(f)^n$, so $P(f) \geq \frac{1}{1+s} Z_1(f) - \log A$. Therefore,

$$\lim_{t \to \frac{1}{2(a+b)}^+} P(-t(a\tau_\rho + b\tau_\eta)) = +\infty.$$

If $a, b \geq 0$ and $x \in \mathrm{Fix}^n$, then $S_n(a\tau_\rho + b\tau_\eta)(x) > 0$, so if $t > 1$, then

$$\sum_{x \in \mathrm{Fix}^n \ | \ x_1 = a} e^{S_n(-t(a\tau_\rho + b\tau_\eta))(x)} \leq \frac{1}{t} \sum_{x \in \mathrm{Fix}^n \ | \ x_1 = a} e^{S_n(-a\tau_\rho - b\tau_\eta)(x)}$$

since $c^t \leq \frac{1}{t} c$ if $0 \leq c \leq 1$ and $t > 1$. Therefore, $P(-t(a\tau_\rho + b\tau_\eta)) \leq P(-a\tau_\rho - b\tau_\eta) - \log t$, so $\lim_{t\to\infty} P(-t(a\tau_\rho + b\tau_\eta)) = -\infty$. $\square$

5. Entropy and Hausdorff dimension

Theorem 4.1 implies that if $\rho \in QC(\Gamma)$ then there is a unique solution $h(\rho) > \frac{1}{2}$ to $P(-h(\rho)\tau_\rho) = 0$. This unique solution $h(\rho)$ is the *topological entropy* of ρ, see the discussion in Kao [23, Section 5]. Theorem 2.2 and the implicit function theorem then imply that $h(\rho)$ varies analytically over $QC(\Gamma)$, generalizing a result of Ruelle [37] in the convex cocompact case. Since the entropy $h(\rho)$ is invariant under conjugation, we obtain analyticity of entropy over $QF(S)$. We recall that Schapira and Tapie [41, Thm. 6.2] previously established that the entropy is C^1 on $QF(S)$.

Theorem 5.1. *If S is a compact hyperbolic surface with non-empty boundary, then the topological entropy varies analytically over $QF(S)$.*

Sullivan [47] showed that the topological entropy $h(\rho)$ agrees with the Hausdorff dimension of the limit set $\Lambda(\rho(\Gamma))$, so we obtain the following corollary.

Theorem 5.2. (Sullivan [47, 48]) *If $\rho \in QC(\Gamma)$, then its topological entropy $h(\rho)$ is the exponential growth rate of the number of closed geodesics of length less than T in $N_\rho = \mathbb{H}^3/\rho(\Gamma)$. Moreover, $h(\rho)$ is the Hausdorff dimension of the limit set $\Lambda(\rho(\Gamma))$ and the critical exponent of the Poincaré series $Q_\rho(s)$.*

Theorems 5.1 and 5.2 together imply that the Hausdorff dimension of the limit set varies analytically.

Corollary 5.3. *The Hausdorff dimension of $\Lambda(\rho(\Gamma))$ varies analytically over $QC(\Gamma)$.*

Remarks: 1) Sullivan [48] also showed that $h(\rho)$ is the critical exponent of the Poincaré series

$$Q_\rho(s) = \sum_{\gamma \in \Gamma} e^{-sd(b_0, \rho(\gamma)(b_0))},$$

i.e. $Q_\rho(s)$ diverges if $s < h(\rho)$ and converges if $s > h(\rho)$.

2) Bowen [6] showed that if $\rho \in QF(S)$ and S is a closed surface, then $h(\rho) \geq 1$ with equality if and only if ρ is Fuchsian. Sullivan [46, p. 66], see also Xie [50], observed that Bowen's rigidity result extends to the case when $\mathbb{H}^2/\Gamma$ has finite area.

6. Manhattan curves

If $\rho, \eta \in QC(\Gamma)$, we define, following Burger [12], the *Manhattan curve*

$$\mathcal{C}(\rho, \eta) = \{(a, b) \in D \mid P(-a\tau_\rho - b\tau_\eta) = 0\}$$

where $D = \{(a,b) \in \mathbb{R}^2 \mid a,b \geq 0 \text{ and } (a,b) \neq (0,0)\}$. Notice that, since the Gurevich pressure is defined in terms of lengths of closed geodesics, if $\hat\rho$ is conjugate (or complex conjugate) to ρ and $\hat\eta$ is conjugate (or complex conjugate) to η, then $\mathcal{C}(\rho,\eta) = \mathcal{C}(\hat\rho,\hat\eta)$.

One may give an alternative characterization by noticing that $P(-ab_\rho - b\tau_\eta) = 0$ if and only if

$$h^{a,b}(\rho,\eta) = \lim \frac{1}{T} \log \#\{\ [\gamma] \in [\Gamma] \mid 0 < a\ell_\rho(\gamma) + b\ell_\eta(\gamma) \leq T\} = 1$$

where $[\Gamma]$ is the collection of conjugacy classes in Γ. Moreover, $h^{a,b}(\rho,\eta)$ is also the critical exponent of

$$Q^{a,b}_{\rho,\eta}(s) = \sum_{\gamma\in\Gamma} e^{-s(ad(0,\rho(\gamma)(0))+bd(0,\eta(\gamma)(0)))}.$$

(see Theorem 4.8, Remark 4.9 and Lemma 4.10 in Kao [22]).

Theorem 6.1. *If $\rho,\eta \in QC(\Gamma)$, then $\mathcal{C}(\rho,\eta)$*

(1) *is a closed subsegment of an analytic curve,*
(2) *has endpoints $(h(\rho),0)$ and $(0,h(\eta))$,*
(3) *and is strictly convex, unless ρ and η are conjugate in $\mathrm{Isom}(\mathbb{H}^3)$.*

Moreover, the tangent line to $\mathcal{C}(\rho,\eta)$ at $(h(\rho),0)$ has slope

$$-\frac{\int \tau_\eta dm_{-h(\rho)\tau_\rho}}{\int \tau_\rho dm_{-h(\rho)\tau_\rho}}.$$

Burger [12] established Theorem 6.1 for convex cocompact Fuchsian groups, with the exception of the analyticity of the Manhattan curve, which was established by Sharp [42].

Notice that if ρ and η are conjugate in $\mathrm{Isom}(\mathbb{H}^3)$, then $\tau_\rho = \tau_\eta$ so $\mathcal{C}(\rho,\eta)$ is a straight line. We will need the following technical result in the proof of Theorem 6.1.

Lemma 6.2. *If $\rho,\eta,\theta \in QC(\Gamma)$, $2(a+b) > 1$ and $P(-a\tau_\rho - b\tau_\eta) = 0$, then there exists a unique equlibrium state $m_{-a\tau_\rho-b\tau_\eta}$ for $-a\tau_\rho - b\tau_\eta$ and*

$$0 < \int_{\Sigma^+} \tau_\theta dm_{-a\tau_\rho-b\tau_\eta} < +\infty.$$

Proof. Notice that $P(-a\tau_\rho - b\tau_\eta) = 0$ and $\sup(-a\tau_\rho - b\tau_\eta) \leq |a|C_\rho + |b|C_\eta$, so there exists a unique shift-invariant Gibbs state $m_{-a\tau_\rho-b\tau_\eta}$ for $-a\tau_\rho - b\tau_\eta$, see Sarig [40, Thm. 4.9]. However, by [30, Lemma 2.2.8],

$$\int_{\Sigma^+} a\tau_\rho + b\tau_\eta \ dm_{-a\tau_\rho-b\tau_\eta} < +\infty$$

if and only if

$$\sum_{a\in\mathcal{A}} \inf(a\tau_\rho + b\tau_\eta|_{[a]}) e^{\inf(-a\tau_\rho-b\tau_\eta|_{[a]})} < \infty.$$

But, by Proposition 3.1,

$$\begin{aligned}\sum_{a\in\mathcal{A}} \inf(a\tau_\rho + b\tau_\eta|_{[a]}) e^{\inf(-a\tau_\rho-b\tau_\eta|_{[a]})} &\leq D\sum_{n\in\mathbb{N}}(|a|C_\rho + |b|C_\eta + 2(a+b)\log n)e^{|a|C_\rho+|b|C_\eta-2(a+b)\log n} \\ &= De^{|a|C_\rho+|b|C_\eta}\sum_{n\in\mathbb{N}} \frac{(|a|C_\rho + |b|C_\eta + 2(a+b)\log n)}{n^{2(a+b)}}\end{aligned}$$

which converges, since $2(a+b) > 1$. But a result of Mauldin and Urbanski [30, Thm. 2.2.9] then implies that $dm_{-a\tau_\rho-b\tau_\eta}$ is an equilibrium state for $-a\tau_\rho - b\tau_\eta$.

Proposition 3.1 implies that there exists $B > 1$ so that if n is large enough, then

$$\frac{1}{B} \leq \frac{\tau_\theta(x)}{a\tau_\rho(x) + b\tau_\eta(x)} \leq B$$

for all $x \in \Sigma^+$ so that $r(x_1) > n$. (For example, if $\log n > 4\max\{aC_\rho + bC_\eta, C_\theta, 1\}$, then we may choose $B = 8(a+b)$.) Since τ_θ is locally Hölder continuous, it is bounded on the remainder of Σ^+. Therefore,

$$\int_{\Sigma^+} \tau_\theta \; dm_{-a\tau_\rho - b\tau_\eta} < +\infty.$$

Now notice that, since τ_θ is cohomologous to a positive function $\hat{\tau}_\theta$, by Lemma 3.2,

$$\int_{\Sigma^+} \tau_\theta dm_{-a\tau_\rho - b\tau_\eta} = \int_{\Sigma^+} \hat{\tau}_\theta dm_{-a\tau_\rho - b\tau_\eta} > 0.$$

$\square$

Proof of Theorem 6.1: Corollary 5.2 implies that $(h(\rho), 0)$ and $(0, h(\eta))$ are the intersection of the Manhattan curve with the boundary of D.

Let

$$\hat{D} = \{(a,b) \in \mathbb{R}^2 \; | a + b > \frac{1}{2}\}.$$

Theorem 4.1 implies that P is finite on $\hat{D}$. Lemma 6.2 implies that if $a, b \in \hat{D}$ and $P(-a\tau_\rho - b\tau_\eta) = 0$, then there is an equilibrium state $m_{-a\tau_\rho - b\tau_\eta}$ for $-a\tau_\rho - b\tau_\eta$ and that $\int_{\Sigma^+} \tau_\theta \; dm_{-a\tau_\rho - b\tau_\eta}$ is finite for all $\theta \in QC(\Gamma)$. Theorem 2.2 then implies that

$$\frac{\partial}{\partial a} P(-a\tau_\rho - b\tau_\eta) = \int_{\Sigma^+} -\tau_\rho \; dm_{-a\tau_\rho - b\tau_\eta}$$

and

$$\frac{\partial}{\partial b} P(-a\tau_\rho - b\tau_\eta) = \int_{\Sigma^+} -\tau_\eta \; dm_{-a\tau_\rho - b\tau_\eta}.$$

Since $\int_{\Sigma^+} -\tau_\rho \; dm_{-a\tau_\rho - b\tau_\eta}$ and $\int_{\Sigma^+} -\tau_\eta \; dm_{-a\tau_\rho - b\tau_\eta}$ are both non-zero, P is a submersion on $\hat{D}$. Since P is analytic on $\hat{D}$, the implicit function theorem then implies that

$$\widehat{\mathcal{C}}(\rho, \eta) = \{(a,b) \in \hat{D} \mid P(-a\tau_\rho - b\tau_\eta) = 0\}$$

is an analytic curve and that if $(a,b) \in \mathcal{C}(\rho,\eta)$ then the slope of the tangent line to $\mathcal{C}(\rho,\eta)$ at (a,b) is given by

$$c(a,b) = -\frac{\int_{\Sigma^+} \tau_\eta \; dm_{-a\tau_\rho - b\tau_\eta}}{\int_{\Sigma^+} \tau_\rho \; dm_{-a\tau_\rho - b\tau_\eta}}.$$

Notice that $\mathcal{C}(\rho,\eta)$ is the lower boundary of the region

$$\widehat{\mathcal{C}}(\rho,\eta) = \{(a,b) \mid Q^{a,b}_{\rho,\eta}(1) < \infty\}$$

The Hölder inequality implies that if $(a,b), (c,d) \in \widehat{\mathcal{C}}(\rho,\eta)$ and $t \in [0,1]$, then

$$Q^{ta+(1-t)c, tb+(1-t)d}_{\rho,\eta} \leq Q(a,b)^t Q(c,d)^{1-t}$$

so $\widehat{\mathcal{C}}(\rho,\eta)$ is convex. Therefore, $\mathcal{C}(\rho,\eta)$ is convex.

A convex analytic curve is strictly convex if and only if it is not a line, so it remains to show that ρ and η are conjugate in $\mathrm{Isom}(\mathbb{H}^3)$ if $\mathcal{C}(\rho,\eta)$ is a straight line. So suppose that $\mathcal{C}(\rho,\eta)$ is a straight line with slope $c=-\frac{h(\rho)}{h(\eta)}$. In particular,

$$\frac{h(\rho)}{h(\eta)} = -c = -c(h(\rho),0) = \frac{\int_{\Sigma^+}\tau_\eta dm_{-h(\rho)\tau_\rho}}{\int_{\Sigma^+}\tau_\rho dm_{-h(\rho)\tau_\rho}} = -c(0,h(\eta)) = \frac{\int_{\Sigma^+}\tau_\eta dm_{-h(\eta)\tau_\eta}}{\int_{\Sigma^+}\tau_\rho dm_{-h(\eta)\tau_\eta}}. \tag{1}$$

By definition,

$$h(m_{-h(\eta)\tau_\eta}) - h(\eta)\int_{\Sigma^+}\tau_\eta\ dm_{-h(\eta)\tau_\eta} = 0$$

so, applying equation (1), we see that

$$h(m_{-h(\eta)\tau_\eta}) - h(\rho)\int_{\Sigma^+}\tau_\rho\ dm_{-h(\eta)\tau_\eta} = h(\eta)\int_{\Sigma^+}\tau_\eta\ dm_{-h(\eta)\tau_\eta} - h(\rho)\int_{\Sigma^+}\tau_\rho\ dm_{-h(\eta)\tau_\eta} = 0.$$

Since $P(-h(\rho)\tau_\rho)=0$, this implies that $m_{-h(\eta)\tau_\eta}$ is an equilibrium measure for $-h(\rho)\tau_\rho$. Therefore, by uniqueness of equilibrium measures we see that $m_{-h(\eta)\tau_\eta} = m_{-h(\eta)\tau_\rho}$. Sarig [40, Thm. 4.8] showed that this only happens when $-h(\rho)\tau_\rho$ and $-h(\eta)\tau_\eta$ are cohomologous, so the Livsic Theorem [40, Thm. 1.1] (see also Mauldin-Urbanski [30, Thm. 2.2.7]) implies that

$$\ell_\rho(\gamma) = \frac{h(\eta)}{h(\rho)}\ell_\eta(\gamma)$$

for all $\gamma\in\Gamma$. Kim [25, Th, 3] proved that if $\ell_\rho(\gamma) = c\ell_\eta(\gamma)$ for all $\gamma\in\Gamma$, then ρ and η are conjugate in $\mathrm{Isom}(\mathbb{H}^3)$. So, we have completed the proof. □

As a nearly immediate corollary one obtains a generalization of the rigidity results of Bishop-Steger [3] and Burger [12].

Corollary 6.3. *If $\rho,\eta\in QC(\Gamma)$ and $a,b\in D$, then*

$$h^{a,b}(\rho,\eta) \le \frac{h(\rho)h(\eta)}{bh(\rho)+ah(\eta)}$$

with equality if and only if ρ and η are conjugate in $\mathrm{Isom}(\mathbb{H}^3)$.

7. Pressure intersection

We define the *pressure intersection* on $QC(\Gamma)\times QC(\Gamma)$ given by

$$I(\rho,\eta) = \frac{\int_{\Sigma^+}\tau_\eta\ dm_{-h(\rho)\tau_\rho}}{\int_{\Sigma^+}\tau_\rho\ dm_{-h(\rho)\tau_\rho}}.$$

It follows from Lemma 6.2 that $I(\rho,\eta)$ is well-defined. We also define a *renormalized pressure intersection*

$$J(\rho,\eta) = \frac{h(\eta)}{h(\rho)}I(\rho,\eta).$$

We notice that the pressure intersection and renormalized pressure intersection vary analytically in ρ and η.

Proposition 7.1. *Both $I(\rho,\eta)$ and $J(\rho,\eta)$ vary analytically over $QC(\Gamma)\times QC(\Gamma)$.*

Proof. Notice that, by Theorem 4.1, Proposition 3.1 and Theorem 2.2, $P = P(-a\tau_\rho - b\tau_\eta)$ is analytic on

$$R = \{(\rho,\eta,(a,b),t)\in QC(\Gamma)\times QC(\Gamma)\times\hat{D}\}.$$

Since we observed, in the proof of Theorem 6.1, that the restriction of P to $\{\rho\} \times \{\eta\} \times \hat{D}$ is a submersion for all $\rho, \eta \in QC(\Gamma)$, P itself is a submersion, and $V = P^{-1}(0) \cap R$ is an analytic submanifold of R of codimension one. Then $-I(\rho, \eta)$ is the slope of the tangent line to $V \cap \{(\rho, \eta) \times \hat{D}\}$ at the point $(\rho, \eta, (h(\rho), 0))$, so $I(\rho, \eta)$ is analytic. Theorem 5.1 then implies that $J(\rho, \eta)$ is analytic. $\square$

We obtain the following rigidity theorem as a consequence of Theorem 6.1. The inequality portion of this result was previously established by Schapira and Tapie [41, Cor. 3.17].

Corollary 7.2. *If $\rho, \eta \in QC(\Gamma)$, then*

$$J(\rho, \eta) \geq 1$$

with equality if and only if ρ and η are conjugate in $\mathrm{Isom}(\mathbb{H}^3)$.

Proof. Recall that the slope $c = c(h(\rho), 0)$ of $\mathcal{C}(\rho, \eta)$ at $(h(\rho), 0)$ is given by

$$c = -\frac{\int_{\Sigma^+} \tau_\eta \; dm_{-h(\rho)\tau_\rho}}{\int_{\Sigma^+} \tau_\rho \; dm_{-h(\rho)\tau_\rho}} = -I(\rho, \eta).$$

However, by Theorem 6.1,

$$c \leq -\frac{h(\rho)}{h(\eta)}$$

with equality if and only if ρ and η are conjugate in $\mathrm{Isom}(\mathbb{H}^3)$. Our corollary follows immediately. $\square$

8. The pressure form

We may define an analytic section $s : QF(S) \to QC(\Gamma)$ so that $s([\rho])$ is an element of the conjugacy class of ρ. Choose co-prime hyperbolic elements α and β in Γ and let $s(\rho)$ be the unique element of $[\rho]$ so that $s(\rho)(\alpha)$ has attracting fixed point 0 and repelling fixed point ∞ and $s(\rho)(\beta)$ has attracting fixed point 1. This will allow us to abuse notation and regard $QF(S)$ as a subset of $QC(\Gamma)$.

Following Bridgeman [9] and McMullen [31], we define an analytic pressure form $\mathbb{P}$ on the tangent bundle $TQF(S)$ of $QF(S)$, by letting

$$\mathbb{P}_{T_{[\rho]}QF(S)} = s^*\Big(\mathrm{Hess}\big(J(s(\rho), \cdot)\big)|_{T_{s(\rho)}s(QF(S))}\Big)$$

which we rewrite with our abuse of notation as:

$$\mathbb{P}_{T_\rho QF(S)} = \mathrm{Hess}(J(\rho), \cdot))$$

Corollary 7.2 implies that $\mathbb{P}$ is non-negative, i.e. $\mathbb{P}(v, v) \geq 0$ for all $v \in TQF(S)$.

Since $\mathbb{P}$ is non-negative, we can define a path pseudo-metric on $QF(S)$ by setting

$$d_{\mathbb{P}}(\rho, \eta) = \inf \left\{ \int_0^1 \sqrt{\mathbb{P}(\gamma'(t), \gamma'(t))} dt \right\}$$

where the infimum is taken over all smooth paths in $QF(S)$ joining ρ to η.

We now derive a standard criterion for when a tangent vector is degenerate with respect to $\mathbb{P}$, see also [11, Cor. 2.5] and [10, Lemma 9.3].

Lemma 8.1. *If $v \in T_\rho QF(S)$, then $\mathbb{P}(v, v) = 0$ if and only if*

$$D_v\left(h\ell_\gamma\right) = 0$$

for all $\gamma \in \Gamma$.

Proof. Let $\mathcal{H}_0$ denote the space of pressure zero locally Hölder continuous functions on Σ^+. We have a well-defined Thermodynamic mapping $\psi : QF(S) \to \mathcal{H}_0$ given by $\psi(\rho) = -h(s(\rho))\tau_{s(\rho)}$. Notice that, by Proposition 3.1 and Theorem 5.1, $\psi(QF(S))$ is a real analytic family.

Suppose that $\{\rho_t\}_{t\in(-\epsilon,\epsilon)}$ is an one-parameter analytic family in $QF(S)$ and $v = \dot\rho_0$. Then

$$\frac{d^2}{dt^2}J(\rho_0,\rho_t)\Big|_{t=0} = \frac{d^2}{dt^2}\left(\frac{\int_{\Sigma^+}\psi(\rho_t)\ dm_{\psi(\rho_0)}}{\int_{\Sigma^+}\psi(\rho_0)\ dm_{\psi(\rho_0)}}\right) = \frac{\int_{\Sigma^+}\ddot\psi_0\ dm_{\psi(\rho_0)}}{\int_{\Sigma^+}\psi(\rho_0)\ dm_{\psi(\rho_0)}}$$

where

$$\ddot\psi_0 = \frac{d^2}{dt^2}\Big|_{t=0}\psi(\rho_t).$$

Theorem 2.2 implies that

$$0 = \frac{d^2}{dt^2}\Big|_{t=0}P(\psi(t)) = \mathrm{Var}(\dot\psi_0, m_{\psi(0)}) + \int_{\Sigma^+}\ddot\psi_0\ dm_{\psi(\rho_0)}$$

where

$$\dot\psi_0 = \frac{d}{dt}\Big|_{t=0}\psi(\rho_t),$$

so

$$\frac{d^2}{dt^2}J(\rho_0,\rho_t)\Big|_{t=0} = -\frac{\mathrm{Var}(\dot\psi_0, m_{\psi(0)})}{\int_{\Sigma^+}\psi(\rho_0)\ dm_{\psi(\rho_0)}}.$$

Recall, see Sarig [40, Thm. 5.12], that $\mathrm{Var}(\dot\psi_0, m_{\psi(0)}) = 0$ if and only if $\dot\psi_0$ is cohomologous to a constant function C. On the other hand, since $P(\psi_t) = 0$ for all t, the formula for the derivative of the pressure function gives that

$$0 = \frac{d}{dt}\Big|_{t=0}P(\psi_t) = \int_{\Sigma^+}\dot\psi_0\ dm_{\psi(\rho_0)}$$

so C must equal 0. However, $\dot\psi_0$ is cohomologous to 0 if and only if for all $x \in \mathrm{Fix}^n$, and all n,

$$0 = S_n\dot\psi_0(x) = \frac{d}{dt}\Big|_{t=0}S_n\psi_t(x) = \frac{d}{dt}\Big|_{t=0}\Big(h(\rho_t)\ell_{G(x_1)\dots G(x_n)}(\rho_t)\Big)$$

(see [40, Theorem 1.1]). Moreover, for every hyperbolic element $\gamma \in \Gamma$, there exists $x \in \mathrm{Fix}^n$ (for some n) so that γ is conjugate to $G(x_1)\cdots G(x_n)$, so $\ell_\gamma(\rho_t) = \ell_{G(x_1)\cdots G(x_n)}(\rho_t)$ for all t. If $\gamma \in \Gamma$ is not hyperbolic, then $\ell_\gamma(\rho_t) = 0$ for all t, so

$$\frac{d}{dt}\Big|_{t=0}\Big(h(\rho_t)\ell_\gamma(\rho_t)\Big) = 0$$

in every case. Therefore, $\dot\psi_0$ is cohomologous to 0 if and only if

$$\frac{d}{dt}\Big|_{t=0}\Big(h(\rho_t)\ell_\gamma(\rho_t)\Big) = 0$$

for all $\gamma \in \Gamma$. □

9. Main Theorem

We recall that a quasifuchsian representation $\rho : \Gamma \to \mathsf{PSL}(2,\mathbb{C})$ is said to be *fuchsian* if it is conjugate into $\mathsf{PSL}(2,\mathbb{R})$, i.e. there exists $A \in \mathsf{PSL}(2,\mathbb{C})$ so that $A\rho(\gamma)A^{-1} \in \mathsf{PSL}(2,\mathbb{R})$ for all $\gamma \in \Gamma$. The Fuchsian locus $F(S) \subset QF(S)$ is the set of (conjugacy classes of) fuchsian representations.

We say that $v \in T_\rho QF(S)$ is a *pure bending* vector if $v = \frac{\partial}{\partial t}\rho_t$, ρ_0 is fuchsian and ρ_{-t} is the complex conjugate of ρ_t for all t. Since the Fuchsian locus $F(S)$ is the fixed point set of the action of complex conjugation on $QF(S)$ and the collection of pure bending vectors at a point in $F(S)$ is half-dimensional, one gets a decomposition

$$T_\rho QF(S) = T_\rho F(S) \oplus B_\rho$$

where B_ρ is the space of pure bending vectors at ρ. If v is a pure bending vector at $\rho \in F(S)$, then v is tangent to a path obtained by bending ρ by a (signed) angle t along some measured lamination λ (see Bonahon [5, Section 2] for details).

We are finally ready to show that our pressure form is degenerate only along pure bending vectors.

Theorem 9.1. *If S is a compact hyperbolic surface with non-empty boundary, then the pressure form $\mathbb{P}$ defines an $\mathrm{Mod}(S)$-invariant path metric $d_{\mathbb{P}}$ on $QF(S)$ which is an analytic Riemannian metric except on the Fuchsian locus.*

Moreover, if $v \in T_\rho(QF(S))$, then $\mathbb{P}(v,v) = 0$ if and only if ρ is Fuchsian and v is a pure bending vector.

Proof. If v is a pure bending vector, then we may write $v = \dot\rho_0$ where ρ_{-t} is the complex conjugate of ρ_t for all t, so $h\ell_\gamma(\rho_t)$ is an even function for all $\gamma \in \Gamma$. Therefore, $D_v h\ell_\gamma = 0$ for all $\gamma \in \Gamma$, so Lemma 8.1 implies that $\mathbb{P}(v,v) = 0$.

Our main work is the following converse:

Proposition 9.2. *Suppose that $v \in T_\rho QF(S)$. If $\mathbb{P}(v,v) = 0$ and $v \neq 0$, then v is a pure bending vector.*

Recall, see [10, Lemma 13.1], that if a Riemannian metric on a manifold M is non-degenerate on the complement of a submanifold N of codimension at least one and the restriction of the Riemannian metric to TN is non-degenerate, then the associated path pseudo-metric is a metric. We will see in Corollary 10.4 that the pressure metric is mapping class group invariant. Our theorem then follows from Proposition 9.2 and the fact, established by Kao [23], that $\mathbb{P}$ is non-degenerate on the tangent space to the Fuchsian locus. □

Proof of Proposition 9.2. Now suppose that $v \in T_\rho QF(S)$ and $\mathbb{P}(v,v) = 0$. One first observes, following Bridgeman [9], that since, by Lemma 8.1, $D_v\,(h\ell_\gamma) = 0$ for all $\gamma \in \Gamma$,

$$D_v \ell_\gamma = k\ell_\gamma(\rho) \tag{2}$$

for all $\gamma \in \Gamma$, where $k = -\frac{D_v h}{h(\rho)}$.

If $\gamma \in \Gamma$, then one can locally define analytic functions $tr_\gamma(\rho)$ and $\lambda_\gamma(\rho)$ which are the trace and eigenvalue of largest modulus of (some lift of) $\rho(\gamma)$. Notice that $\ell_\gamma(\rho) = 2\log|\lambda_\gamma(\rho)|$, so we can express our degeneracy criterion (2) as

$$D_v \log|\lambda_\gamma| = k \log|\lambda_\gamma(\rho)| \tag{3}$$

for all $\gamma \in \Gamma$.

We observe that Bridgeman's Lemma 7.4 [9] goes through nearly immediately in our setting. We state the portion of his lemma we will need and provide a brief sketch of the proof for the reader's convenience.

Lemma 9.3. (Bridgeman [9, Lemma 7.4]) *If $\mathbb{P}(v,v) = 0$, $v \in T_\rho QF(S)$, $v \neq 0$ and $\gamma \in \Gamma$, then $\lambda_\gamma(\rho)^2$ and $tr_\gamma(\rho)^2$ are both real.*

Moreover, if $D_v tr_\alpha \neq 0$, then $Re\left(\frac{D_v\lambda_\alpha}{\lambda_\alpha(\rho)}\right) = 0$.

Proof. Suppose first that $D_v tr_\alpha \neq 0$. Since

$$D_v(tr_\alpha) = D_v\lambda_\alpha\left(\frac{\lambda_\alpha^2 - 1}{\lambda_\alpha^2}\right)$$

we may conclude that $D_v\lambda_\alpha \neq 0$. Choose $\gamma \in \Gamma$, so that γ is hyperbolic and does not commute with α. He then normalizes so that (the lift of) $\rho(\alpha) = \begin{bmatrix} \lambda_\alpha & 0 \\ 0 & \lambda_\alpha^{-1} \end{bmatrix}$ and (the lift of) $\rho(\gamma) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c, d are all functions defined on a neighborhood of ρ, such that a and d are non-zero. He then computes that

$$\log|\lambda_{\alpha^n\gamma}| = n\log|\lambda_\gamma| + \log|a| + \mathrm{Re}\left(\lambda_\alpha^{-2n}\left(\frac{ad-1}{a^2}\right)\right) + O(|\lambda_\alpha^{-4n}|).$$

He differentiates this equation and applies equation (3) to conclude that

$$\mathrm{Re}\left(\frac{D_v\lambda_\alpha}{\lambda_\alpha(\rho)}\left(\frac{a(\rho)d(\rho)-1}{a(\rho)^2}\right)\right) = 0. \tag{4}$$

A final analysis, which breaks down into the consideration of the cases where the argument of $\lambda_\alpha^2(\rho)$ is rational or irrational, yields that $\lambda_\alpha(\rho)^2$ is real. Since $tr_\alpha^2 = \lambda_\alpha^2 + 2 + \lambda_\alpha^{-2}$, we conclude that $tr_\alpha^2(\rho)$ is real.

One may further differentiate the equation

$$tr_{\alpha^n\gamma} = a\lambda_\alpha^n + d\lambda_a^{-n}$$

to conclude that

$$\lim\left(\frac{D_v tr_{\alpha^n\gamma}}{n\lambda_\alpha(\rho)^n}\right) = \frac{a(\rho)D_v\lambda_\alpha}{\lambda_\alpha(\rho)}$$

so $D_v tr_{\alpha^n\gamma} \neq 0$ is non-zero for all large enough n. Therefore, by the above paragraph,

$$tr^2_{\alpha^n\gamma}(\rho) = a(\rho)^2\lambda_\alpha(\rho)^{2n} + 2ad(\rho) + d(\rho)^2\lambda_\alpha(\rho)^{-2n}$$

is real for all large enough n. Taking limits allows one to conclude that $a(\rho)^2$, $d(\rho)^2$ and $a(\rho)d(\rho)$ are real. Equation (4) then yields that $Re\left(\frac{D_v\lambda_\alpha}{\lambda_\alpha(\rho)}\right) = 0$. This completes the proof when $D_v tr_\alpha \neq 0$.

Now suppose that $D_v tr_\gamma = 0$. If γ is parabolic, $\lambda_\gamma(\rho)^2 = 1$ and $tr_\gamma^2(\rho) = 4$ which are both real, so we may suppose that γ is hyperbolic. Since there are finitely many elements $\{\alpha_1, \ldots, \alpha_n\}$ of Γ so that $\rho \in QF(S)$ is determined by $\{tr_{\alpha_1}(\rho)^2, \ldots, tr_{\alpha_n}(\rho)^2\}$, see [13, Lemma 2.5], and trace functions are analytic, there exists $\alpha \in \Gamma$, so that $D_v tr_\alpha \neq 0$. The above analysis then yields that $a(\rho)^2$, $d(\rho)^2$ and $a(\rho)d(\rho)$ are all real. Therefore,

$$tr_\gamma(\rho)^2 = a(\rho)^2 + 2a(\rho)d(\rho) + d(\rho)^2 = \lambda_\gamma(\rho)^2 + 2 + \lambda_\gamma(\rho)^{-2}$$

is real. So, we may conclude that $\lambda_\gamma(\rho)^2$ is real in this case as well, which completes the proof. $\square$

Since $v \neq 0$, there exists $\alpha \in \Gamma$ so that $D_v tr_\alpha \neq 0$ and

$$Re\left(\frac{D_v\lambda_\alpha}{\lambda_\alpha(\rho)}\right) = \frac{D_v|\lambda_\alpha|}{|\lambda_\alpha(\rho)|} = D_v\log|\lambda_\alpha|,$$

equation (3) and Lemma 9.3 imply that

$$k = \frac{D_v\log|\lambda_\alpha|}{\log|\lambda_\alpha(\rho)|} = 0.$$

Therefore, $D_v\ell_\gamma = 0$ for all $\gamma \in \Gamma$.

Notice that since $tr_\gamma(\rho)^2$ is real for all $\gamma \in \Gamma$, $\rho(\Gamma)$ lies in a proper (real) Zariski closed subset of $\mathsf{PSL}(2,\mathbb{C})$, so is not Zariski dense. However, since the Zariski closure of $\rho(\Gamma)$ is a Lie subgroup, it must be conjugate to a subgroup of either $\mathsf{PSL}(2,\mathbb{R})$ or to the index two extension of $\mathsf{PSL}(2,\mathbb{R})$ obtained by appending $z \to -z$. Since ρ is quasifuchsian, its limit set $\Lambda(\rho(\Gamma))$ is a Jordan curve and no element of $\rho(\Gamma)$ can exchange the two components of its complement. Therefore, ρ is Fuchsian. (We note that this is the only place where our argument differs significantly from Bridgeman's. It replaces his rather technical [9, Lemma 15].)

We can then write $v = v_1 + v_2$ where $v_1 \in T_\rho F(S)$ and v_2 is a pure bending vector. Since v_2 is a pure bending vector,

$$0 = D_v \ell_\gamma = D_{v_1}\ell_\gamma + D_{v_2}\ell_\gamma = D_{v_1}\ell_\gamma$$

for all $\gamma \in \Gamma$. But since $v_1 \in T_\rho F(S)$ and there are finitely many curves whose length functions provide analytic parameters for $F(S)$, this implies that $v_1 = 0$. Therefore, $v = v_2$ is a pure bending vector. $\square$

10. Patterson-Sullivan measures

In this section, we observe that the equilibrium state $m_{-h(\rho)\tau_\rho}$ is a normalized pull-back of the Patterson-Sullivan measure on $\Lambda(\rho(\Gamma))$. We use this to give a more geometric interpretation of the pressure intersection of two quasifuchsian representations, and hence a geometric formulation of the pressure form.

Sullivan [45, 47] generalized Patterson's construction [32] for Fuchsian groups to define a probability measure μ_ρ supported on $\Lambda(\rho(\Gamma))$, called the *Patterson-Sullivan measure*. This measure satisfies the quasi-invariance property:

$$d\mu(\rho(\gamma)(z)) = e^{h(\rho)B_z(b_0,\rho(\gamma)^{-1}(b_0))}d\mu_\rho(z) \tag{5}$$

for all $z \in \Lambda(\rho(\Gamma))$ and $\gamma \in \Gamma$. Sullivan showed that μ_ρ is a scalar multiple of the $h(\rho)$-dimensional Hausdorff measure on $\partial\mathbb{H}^3$ (with respect to the metric obtained from its identification with $T^1_{b_0}(\mathbb{H}^3)$).

Let $\hat\mu_\rho = (\xi_\rho \circ \omega)^*\mu_\rho$ be the pull-back of the Patterson-Sullivan measure to Σ^+. Our normalization will involve the Gromov product, which is defined to be

$$\langle z, w\rangle_{b_0} = \frac{1}{2}\big(B_z(b_0,p) + B_w(b_0,p)\big) \tag{6}$$

for any pair z and w of distinct points in $\partial\mathbb{H}^3$, where p is some (any) point on the geodesic joining z to w. One may check that for all $\alpha \in \rho(\Gamma)$ and $z, w \in \Lambda(\rho(\Gamma))$ we have

$$\langle \alpha(z), \alpha(w)\rangle_{b_0} = \langle z, w\rangle_{b_0} - \frac{1}{2}\Big(B_z(b_0,\alpha^{-1}(b_0)) + B_w(b_0,\alpha^{-1}(b_0))\Big).$$

If $x \in \Sigma^+$, let

$$\Lambda(\rho(\Gamma))_x = \{\xi_\rho(\omega(y^-))\ |y \in \Sigma,\ y^+ = x\},$$

where Σ is the two-sided Markov shift associated to Σ^+ and $y^- = (y^{-1}_{1-i})_{i\in\mathbb{N}}$. Notice that each $\Lambda(\rho(\Gamma))_x$ is open in $\Lambda(\rho(\Gamma))$. Furthermore, there are only finitely many different sets which arise as $\Lambda(\rho(\Gamma))_x$ for some $x \in \Sigma^+$, since $\Lambda(\rho(\Gamma))$ depends only on x_1 and if $r(x_1) \geq 3$ and $x_1 = (b_0, w^s, w_1, \ldots, w_{k-1})$ then $\Lambda(\rho(\Gamma))$ depends only on b_0 and w. Let $H_\rho : \Sigma^+ \to (0,\infty)$ be defined by

$$H_\rho(x) = \int_{\Lambda(\rho(\Gamma))_x} e^{2h(\rho)\langle \xi_\rho(\omega(x)), z\rangle_{b_0}}\ d\mu_\rho(z).$$

Notice that $\Lambda(\rho(\Gamma))_x$ is disjoint from $\xi_\rho(I_x)$ where I_x is the component of $\partial\mathbb{H}^2 - \partial D_0$ containing $\omega(x)$, so $e^{2h(\rho)\langle \xi_\rho(\omega(x)),z\rangle_{b_0}}$ is bounded on $\Lambda(\rho(\Gamma))_x$. In particular, $H_\rho(x)$ is finite for all x. Since ω is locally Hölder continuous and ξ_ρ is Hölder, H_ρ is locally Hölder continuous.

We now show that H_ρ is the normalization of the pull-back $\hat\mu_\rho$ of Patterson-Sullivan measure which gives the equilibrium measure for $-h(\rho)\tau_\rho$. Dal'bo and Peigné [16, Prop. V.3] obtain an analogous result for negatively curved manifolds whose fundamental groups "act like" geometrically finite Fuchsian groups of co-infinite area (see also Dal'bo-Peigné [15, Cor. II.5]).

Proposition 10.1. *If S is a compact surface with non-empty boundary and $\rho \in QF(S)$, then the equilibrium state of $-h(\rho)\tau_\rho$ on Σ^+ is a scalar multiple of $H_\rho\ \hat\mu_\rho$.*

Proof. Let $\alpha(\rho,x) = \rho(G(x_1))^{-1}$ and notice that

$$\alpha(\rho,x)(\xi_\rho(\omega(x))) = \xi_\rho(\omega(\sigma(x))) \qquad \text{and} \qquad \alpha(\rho,x)(\Lambda(\rho(\Gamma))_x) = \Lambda(\rho(\Gamma))_{\sigma(x)}.$$

The quasi-invariance of Patterson-Sullivan measure implies that

$$\frac{d\hat\mu(\sigma(y))}{d\hat\mu(y)} = \frac{d\mu_\rho\left(\alpha(\rho,x)(\xi_\rho(\omega(y))\right)}{d\mu_\rho(\xi_\rho(\omega(y))} = e^{h(\rho)B_{(\xi_\rho(\omega)(y))}(b_0,\alpha(\rho,x)^{-1}(b_0))}.$$

We first check that $H_\rho\ \hat\mu_\rho$ is shift invariant.

$$\begin{aligned}
H_\rho(\sigma(x))d\hat\mu_\rho(\sigma(x)) &= \left(\int_{\Lambda(\rho(\Gamma))_{\sigma(x)}} e^{2h(\rho)<\xi_\rho(\omega(\sigma(x))),w>}d\mu_\rho(w)\right)d\mu_\rho(\xi_\rho(\omega(\sigma(x)))\\
&= \left(\int_{\Lambda(\rho(\Gamma))_{\sigma(x)}} e^{2h(\rho)<\alpha(\rho,x)(\xi_\rho(\omega(x)),\alpha(\rho,x)(v)>}d\mu_\rho(\alpha(\rho,x)(v))\right)d\mu_\rho(\alpha(\rho,x)(\xi_\rho(\omega(x)))\\
&= \left(\int_{\Lambda(\rho(\Gamma))_{x}} e^{2h(\rho)<\xi_\rho(\omega(x)),v>}e^{-h(\rho)\left(B_{\xi_\rho(\omega(x))}(b_0,\alpha(\rho,x)^{-1}(b_0)+B_v(b_0,\alpha(\rho,x)^{-1}(b_0))\right)}e^{h(\rho)B_v(b_0,\alpha(\rho,x)^{-1}(b_0))}d\mu_\rho(v)\right)\cdot\\
&\quad e^{h(\rho)B_{\xi_\rho(\omega(x))}(b_0,\alpha(\rho,x)^{-1}(b_0))}d\mu_\rho(\xi_\rho(\omega(x)))\\
&= \left(\int_{\Lambda(\rho(\Gamma))_{x}} e^{2h(\rho)<\xi_\rho(\omega(x)),v>}d\mu_\rho(v)\right)d\mu_\rho(\xi_\rho(\omega(x)))\\
&= H_\rho(x)d\hat\mu_\rho(x)
\end{aligned}$$

So $H_\rho\ \hat\mu_\rho$ is shift invariant.

Now we check that $\hat\mu_\rho$ is a (scalar multiple of a) Gibbs state for $-h(\rho)\tau_\rho$. We recall, from [30, Theorem 2.3.3], that it suffices to check that $\hat\mu_\rho$ is an eigenmeasure for the dual of the transfer operator $\mathcal{L}_{-h(\rho)\tau_\rho}$. If $g:\Sigma^+\to\mathbb{R}$ is bounded and continuous, then

$$\begin{aligned}
\int_{\Sigma^+}\mathcal{L}_{-h(\rho)\tau_\rho}(g)(x)\ d\hat\mu_\rho(x) &= \int_{\Sigma^+}\left(\sum_{y\in\sigma^{-1}(x)} e^{-h(\rho)\tau_\rho(y)}g(y)\right)d\hat\mu_\rho(x)\\
&= \int_{\Sigma^+}\left(e^{-h(\rho)\tau_\rho(y)}g(y)\right)\ d\hat\mu_\rho(\sigma(y))\\
&= \int_{\Sigma^+} g(y)\ d\hat\mu_\rho(y)
\end{aligned}$$

Therefore, $\hat\mu_\rho$ is a (scalar multiple of a) Gibbs state for $-h(\rho)\tau_\rho$.

Finally, we observe that H_ρ is bounded above. If p is a vertex of D_0, then, by construction, there exists a neighborhood U_p of p, so that if $\omega(x)\in U_p$, then there exists $w\in\mathcal{C}^*$, so that $x_1 = (b,\omega^s,w_1,\ldots,w_{k-1},c)$ for some $s\geq 2$. Recall that we require that $b\neq w_{2N}$ and $c\neq w_k$. Observe that w_1 is the face pairing of the edge of D_0 associated to I_x and that w_{2N} is the inverse of the face-pairing associated to the other edge E of ∂D_0 which ends at p. So, if I is the interval in $\partial\mathbb{H}^2 - \partial D_0$ bounded by E, then $\Lambda(\rho(\Gamma))_x$ is disjoint from $\xi_\rho(I_x \cup I)$. Therefore, H_ρ is uniformly bounded on $\omega^{-1}(U_p)$ (since $e^{2h(\rho)\langle \xi_\rho(\omega(x)),z\rangle_{b_0}}$ is uniformly bounded for all $z \in \Lambda(\rho(\Gamma))_x \subset \Lambda(\rho(\Gamma)) - \xi_\rho(I \cup I_x)$). However, D_0 has finitely many vertices $\{p_1, \ldots, p_n\}$ and H_ρ is clearly bounded above if $\omega(x) \in \partial\mathbb{H}^2 - \bigcup U_{p_i}$ (since again $e^{2h(\rho)\langle \xi_\rho(\omega(x)),z\rangle_{b_0}}$ is uniformly bounded for all $z \in \Lambda(\rho(\Gamma))_x \subset \Lambda(\rho(\Gamma)) - I_x$). Therefore, H_ρ is bounded above on Σ^+.

Since every multiple of a Gibbs state for $-h(\rho)\tau_\rho$ by a continuous function which is bounded between positive constants is also a (scalar multiple of a) Gibbs state for $-h(\rho)\tau_\rho$ (see [30, Remark 2.2.1]), we see that $H_\rho\ \hat{\mu}_\rho$ is a shift invariant Gibbs state and hence an equilibrium measure for $-h(\rho)\tau_\rho$ (see [30, Theorem 2.2.9]). □

If $\rho \in QC(\Gamma)$, let $N_\rho = \mathbb{H}^3/\rho(\Gamma)$ be the quasifuchsian 3-manifold and let $T^1(N_\rho)^{nw}$ denote the non-wandering portion of its geodesic flow. The Hopf parameterization provides a homeomorphism

$$\mathcal{H} : T^1(N_\rho)^{nw} \to \Omega = \Big(\big(\Lambda(\rho(\Gamma)) \times \Lambda(\rho(\Gamma)) - \Delta\big) \times \mathbb{R}\Big)/\Gamma$$

Let

$$\Sigma^{\hat{\tau}_\rho} = \{(x,t) :\ x \in \Sigma,\ 0 \le t \leqslant \hat{\tau}_\rho(x^+)\}/\sim$$

(where $(x, \tau_\rho(x^+)) \sim (\sigma(x), 0))$ be the suspension flow over Σ with roof function $\hat{\tau}_\rho$. Recall that $\hat{\tau}_\rho : \Sigma^+ \to (0,\infty)$ is a positive function cohomologous to τ_ρ.

The Stadlbauer-Ledrappier-Sarig coding map ω for Σ^+ extends to a continous injective coding map

$$\hat{\omega} : \Sigma \to \Lambda(\Gamma) \times \Lambda(\Gamma)$$

given by $\hat{\omega}(x) = (\omega(x^+), \omega(x^-))$ where $x^+ = (x_i)_{i\in\mathbb{N}}$ and $x^- = (x^{-1}_{1-i})_{i\in\mathbb{N}}$. One then has a continuous injective map

$$\kappa : \Sigma^{\hat{\tau}_\rho} \to \Omega$$

which is the quotient of the map $\tilde{\kappa} : \Sigma \times \mathbb{R} \to \big(\Lambda(\rho(\Gamma)) \times \Lambda(\rho(\Gamma)) - \Delta\big) \times \mathbb{R}$ given by

$$\tilde{\kappa}(x,t) = \big((\xi_\rho \times \xi_\rho)\hat{\omega}(x), t\big).$$

(The image of κ is the complement of all flow lines which do not exit cusps of N_ρ and has full measure in Ω.) The map κ conjugates the suspension flow to the geodesic flow on its image i.e. $\kappa \circ \phi_t = \phi_t \circ \kappa$ for all $t \in \mathbb{R}$ on $\kappa(\Sigma^{\hat{\tau}_\rho})$.

The Bowen-Margulis-Sullivan measure m^ρ_{BM} on Ω can be described by its lift to $\widetilde{\Omega}$ which is given by

$$\widetilde{m^\rho_{BM}}(z,w,t) = e^{2h(\rho)\langle z,w\rangle_{b_0}} d\mu_\rho(z) d\mu_\rho(w) dt.$$

The Bowen-Margulis-Sullivan measure m^ρ_{BM} is finite and ergodic (see Sullivan [47, Theorem 3]) and equidistributed on closed geodesics (see Roblin [36, Théorème 5.1.1] or Paulin-Pollicott-Schapira [33, Theorem 9.11].)

Corollary 10.2. *Suppose that $F : (\Sigma^+)^{\hat{\tau}_\rho} \to \mathbb{R}$ is a bounded continuous function and $\widehat{F} : \Sigma^{\hat{\tau}_\rho} \to \mathbb{R}$ is given by $\widehat{F}(x,t) = F(x^+,t)$. Then*

$$\frac{\int_\Omega \widehat{F} \circ \kappa^{-1}\ dm^\rho_{BM}}{\int_\Omega\ dm^\rho_{BM}} = \frac{\int_{\Sigma^+} \Big(\int_0^{\hat{\tau}_\rho(x^+)} F(x,t)\ dt\Big)\, dm_{-h(\rho)\tau_\rho}}{\int_{\Sigma^+} \tau_\rho(x^+)\ dm_{-h(\rho)\hat{\tau}_\rho}}.$$

Proof. Let

$$\widehat{R} = \{(\hat{\omega}(x), t) \in \Lambda(\rho(\Gamma)) \times \Lambda(\rho(\Gamma)) \times \mathbb{R} \mid x \in \Sigma,\ t \in [0, \hat{\tau}_\rho(x^+)]\}$$

be a fundamental domain for the action of Γ on $\big(\Lambda(\rho(\Gamma)) \times \Lambda(\rho(\Gamma)) - \Delta\big) \times \mathbb{R}$ and let

$$R = \{(\omega(x^+), t) \in \Lambda(\rho(\Gamma)) \times \mathbb{R} \mid x^+ \in \Sigma^+ \in [0, \hat{\tau}_\rho(x^+)]\}.$$

By Proposition 10.1, we have

$$\begin{aligned}
\int_\Omega \widehat{F} \circ \kappa^{-1}\, dm^\rho_{BM} &= \int_{\widehat{R}} \widehat{F} \circ \kappa^{-1} e^{h(\rho)2\langle z,w\rangle_{b_0}} d\mu_\rho(z) d\mu_\rho(w) dt \\
&= \int_R F(\omega^{-1}(z), t) \left(\int_{\Lambda(\rho(\Gamma))} e^{h(\rho)2\langle z,w\rangle_{b_0}} d\mu_\rho(w) \right) d\mu_\rho(z) dt \\
&= \int_R F(\omega^{-1}(z), t)) H_\rho(z) d\mu_\rho(z) dt \\
&= \int_{\Lambda(\rho(\Gamma))} \left(\int_0^{\hat{\tau}_\rho(\omega^{-1}(z))} F(\omega^{-1}(z), t)) dt \right) H_\rho(z) d\mu_\rho(z) \\
&= \int_{\Sigma^+} \left(\int_0^{\hat{\tau}_\rho(x_+)} F(x^+, t) dt \right) dm_{-h(\rho)\tau_\rho}(x_+)
\end{aligned}$$

In particular, if we consider $F \equiv 1$, then we see that

$$||dm^\rho_{BM}|| = \int_\Omega dm^\rho_{BM} = \int_{\Sigma^+} \left(\int_0^{\hat{\tau}_\rho(x_+)} dt \right) dm_{-h(\rho)\tau_\rho}(x_+) = \int_{\Sigma^+} \tau_\rho(x^+)\, dm_{-h(\rho)\tau_\rho}$$

so our result follows. □

Let

$$\mu_T(\rho) = \frac{1}{|R_T(\rho)|} \sum_{[\gamma] \in R_T(\rho)} \frac{\delta_{[\gamma]}}{\ell_\rho(\gamma)}$$

where $\delta_{[\gamma]}$ is the Dirac measure on the closed orbit associated to $[\gamma]$ and

$$R_T(\rho) = \{[\gamma] \in [\pi_1(S)] \mid \ell_\rho(\gamma) \le T\}.$$

(If $\gamma = \beta^n$ for $n > 1$ and β is indivisible, then $\frac{\delta_{[\gamma]}}{\ell_\rho(\gamma)} = \frac{n\delta_{[\beta]}}{\ell_\rho(\beta^n)} = \frac{\delta_{[\beta]}}{\ell_\rho(\beta)}$.) Since the Bowen-Margulis measure m^ρ_{BM} is equidistributed on closed geodesics, $\{\mu_T(\rho)\}$ converges to $\frac{m^\rho_{BM}}{||m^\rho_{BM}||}$ weakly (in the dual to the space of bounded continuous functions) as $T \to \infty$.

We finally obtain the promised geometric form for the pressure intersection. We may thus think of the pressure intersection, in the spirit of Thurston, as the Hessian of the length of a random geodesic.

Theorem 10.3. *Suppose that S is a compact surface with non-empty boundary, $X = \mathbb{H}^2/\Gamma$ is a finite area surface homeomorphic to the interior of S and $\rho \in QF(S)$. If $\{\gamma_n\} \subset \Gamma$ and $\left\{\frac{\delta_{\rho(\gamma_n)}}{\ell_\rho(\gamma_n)}\right\}$ converges weakly to $\frac{m^\rho_{BM}}{||m^\rho_{BM}||}$, then*

$$\mathrm{I}(\rho, \eta) = \lim_{n\to\infty} \frac{\ell_\eta(\gamma_n)}{\ell_\rho(\gamma_n)}.$$

Moreover,

$$\mathrm{I}(\rho, \eta) = \lim_{T\to\infty} \frac{1}{|R_T(\rho)|} \sum_{[\gamma] \in R_T(\rho)} \frac{\ell_\eta(\gamma)}{\ell_\rho(\gamma)}.$$

Proof. Let $\{\Gamma_n\}$ be a sequence of finite collections of elements of $[\Gamma]$ so that $\left\{\mu(\Gamma_n) = \frac{1}{|\Gamma_n|}\sum_{[\gamma]\in\Gamma_n}\frac{\delta_{[\gamma]}}{\ell_\rho(\gamma)}\right\}$ converges weakly to $\frac{m^\rho_{BM}}{||m^\rho_{BM}||}$. As in [23, Definition 3.9], consider the bounded continuous function $\psi : \Sigma^{\hat\tau_\rho} \to \mathbb{R}$ given by

$$\psi(x,t) \longmapsto \frac{\hat\tau_\eta(x)}{\hat\tau_\rho(x)} f\left(\frac{t}{\hat\tau_\rho(x)}\right) \text{ for all } \ t\in[0,\hat\tau_\rho(x)]$$

where $f : [0,1] \to \mathbb{R}$ is a smooth function such that $f(0) = f(1) = 0$, $f(t) > 0$ for $0 < t < 1$ and $\int_0^1 f(t)dt = 1$. Then,

$$\int_\Omega \widehat\psi\circ\kappa^{-1}d\mu(\Gamma_n) = \frac{1}{|\Gamma_n|}\sum_{[\gamma]\in\Gamma_n}\frac{\ell_\eta(\gamma)}{\ell_\rho(\gamma)}$$

where $\widehat\psi(x,t) = \psi(x^+,t)$ for all $x\in\Sigma$. So, by Corollary 10.2, $\left\{\frac{\ell_\eta(\gamma_n)}{\ell_\rho(\gamma_n)}\right\}$ converges to

$$\frac{\int_\Omega \widehat\psi\circ\kappa^{-1}\ dm^\rho_{BM}}{||m^\rho_{BM}||} = \frac{\int_{\Sigma^+}\frac{\hat\tau_\eta(x)}{\hat\tau_\rho(x)}\left(\int_0^{\hat\tau_\rho(x)} f\left(\frac{t}{\hat\tau_\rho(x)}\right)\ dt\right)dm_{-h(\rho)\tau_\rho}}{\int_{\Sigma^+}\hat\tau_\rho(x)\ dm_{-h(\rho)\tau_\rho}} = \frac{\int_{\Sigma^+}\hat\tau_\eta\ dm_{-h(\rho)\tau_\rho}}{\int_{\Sigma^+}\hat\tau_\rho\ dm_{-h(\rho)\tau_\rho}} = \frac{\int_{\Sigma^+}\tau_\eta\ dm_{-h(\rho)\tau_\rho}}{\int_{\Sigma^+}\tau_\rho\ dm_{-h(\rho)\tau_\rho}}$$

which completes the proof. $\square$

As a consequence, we obtain a geometric presentation of the pressure form which allows us to easily see that the pressure metric is mapping class group invariant.

Corollary 10.4. *If S is a compact surface with non-empty boundary and $\rho_0\in QF(S)$, then*

$$\mathbb{P}|_{T_{\rho_0}QF(S)} = \mathrm{Hess}(J(\rho_0,\rho)) = \mathrm{Hess}\left(\frac{h(\rho)}{h(\rho_0)}\lim_{T\to\infty}\frac{1}{|R_T(\rho_0)|}\sum_{[\gamma]\in R_T(\rho_0)}\frac{\ell_\rho(\gamma)}{\ell_{\rho_0}(\gamma)}\right).$$

Moreover, the pressure metric is mapping class group invariant.

Proof. The expression for the pressure form follows immediately from the definition and Theorem 10.3. Now observe that if $\phi\in\mathrm{Mod}(S)$ and $\rho\in QF(S)$, then $\phi(\rho) = \rho\circ\phi_*$, so $\ell_\rho(\gamma) = \ell_{\phi(\rho)}(\phi_*(\gamma))$. Therefore, $R_T(\phi(\rho)) = \phi_*(R_T(\rho))$, so $|R_T(\rho)| = |R_T(\phi(\rho))|$ for all T which implies that $h(\rho) = h(\phi(\rho))$. We can also check that

$$\begin{aligned} I(\rho_0,\rho) &= \lim_{T\to\infty}\frac{1}{|R_T(\rho_0)|}\sum_{[\gamma]\in R_T(\rho_0)}\frac{\ell_\rho(\gamma)}{\ell_{\rho_0}(\gamma)} \\ &= \lim_{T\to\infty}\frac{1}{|R_T(\rho_0)|}\sum_{[\gamma]\in R_T(\rho)}\frac{\ell_{\phi(\rho)}(\phi_*(\gamma))}{\ell_{\phi(\rho_0)}(\phi_*(\gamma))} \\ &= \lim_{T\to\infty}\frac{1}{|R_T(\phi(\rho_0))|}\sum_{[\gamma]\in R_T(\phi(\rho_0))}\frac{\ell_{\phi(\rho)}(\gamma)}{\ell_{\phi(\rho_0)}(\gamma)} \\ &= I(\phi(\rho_0),\phi(\rho)) \end{aligned}$$

Therefore, $J(\rho_0,\rho) = J(\phi(\rho_0),\phi(\rho))$ for all $\phi\in\mathrm{Mod}(S)$ and $\rho_0,\rho\in QF(S)$, so the renormlized pressure intersection is mapping class group invariant, so the pressure metric is mapping class group invariant. $\square$

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