Covering theorems for hyperbolic 3-manifolds

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1 Introduction

In this paper we will discuss a collection of theorems whose proofs revolve around understanding the way in which hyperbolic 3-manifolds cover one another. We will emphasize the interplay between the topology and the geometry. We will also state a number of conjectures about the geometry, topology and group theory of hyperbolic 3-manifolds and discuss the relationships between these conjectures. The main conjecture was first posed as a question by Al Marden [14].

Main Conjecture: *If N is a hyperbolic 3-manifold with finitely generated fundamental group, then N is topologically tame, i.e. homeomorphic to the interior of a compact 3-manifold.*

The best partial result in the direction of the main conjecture is due to Francis Bonahon. The main conjecture and Bonahon's theorem serve as some justification of the fact that we will often restrict to the setting of topologically tame hyperbolic 3-manifolds

Theorem 1.1 *(Bonahon [5]) If N is a hyperbolic 3-manifold with finitely generated, freely indecomposable fundamental group, then N is topologically tame.*

For simplicity we will assume throughout this paper that **our hyperbolic 3-manifolds have no cusps**, i.e. that every homotopically non-trivial closed curve is homotopic to a closed geodesic. All of the theorems we state will

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be true without this assumption, at least in some form, but the supporting definitions would have to change. For example a hyperbolic manifold with cusps is geometrically finite if it has finitely generated fundamental group and its convex core has finite volume.

We will also assume, again for simplicity of exposition, that **all our 3-manifolds are orientable.**

2 Geometrically finite hyperbolic 3-manifolds

Let $N = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold with finitely generated fundamental group. Its convex core $C(N)$ is the smallest convex submanifold such that the inclusion map is a homotopy equivalence. Explicitly, $C(N)$ is the quotient by Γ of the convex hull $CH(L_\Gamma)$ of Γ's limit set. Ahlfors' finiteness theorem [1], together with an observation of Thurston (see Epstein-Marden [10]), states that $\partial C(N)$ is a finite collection of closed hyperbolic surfaces. If R is a component of $N - C(N)$ then R is homeomorphic to $S \times (0, \infty)$ and its metric is quasi-isometric to $cosh^2 t ds_S^2 + dt^2$ where ds_S^2 is a hyperbolic metric on S and t is the real coordinate.

N is said to be *geometrically finite* if $C(N)$ is compact. We see immediately that N is topologically tame if it is geometrically finite. If $N = \mathbf{H}^3/\Gamma$ is geometrically finite then we will call Γ a *geometrically finite Kleinian group.*

Theorem 2.1 *(Thurston) If $N = \mathbf{H}^3/\Gamma$ is a geometrically finite hyperbolic 3-manifold with infinite volume and $\widehat{N}$ is a cover of N with finitely generated fundamental group, then $\widehat{N}$ is also geometrically finite.*

Proof of 2.1: Notice that we may represent $\widehat{N}$ as $\mathbf{H}^3/\widehat{\Gamma}$ where $\widehat{\Gamma} \subset \Gamma$. Since $C(N)$ is compact there exist a constant D such that if $x \in C(N)$, then $d(x, \partial C(N)) \leq D$. When we lift to the universal cover, this implies that if $\tilde{x} \in CH(L_\Gamma)$, then $d(\tilde{x}, \partial CH(L_\Gamma)) \leq D$. Now, since $\widehat{\Gamma} \subset \Gamma$, we see that $L_{\widehat{\Gamma}} \subset L_\Gamma$ and hence that $CH(L_{\widehat{\Gamma}}) \subset CH(L_\Gamma)$. Therefore, if $\tilde{x} \in CH(L_{\widehat{\Gamma}})$, then $d(\tilde{x}, \partial CH(L_{\widehat{\Gamma}})) \leq D$, which implies that if $x \in C(\widehat{N})$, then $d(x, \partial C(\widehat{N})) \leq D$. Then, since $\partial C(\widehat{N})$ is compact, $C(\widehat{N})$ is compact.

2.1

Thurston's geometrization theorem (see Morgan [17]) asserts that if M is a compact, atoroidal, irreducible 3-manifold with a non-toroidal boundary component, then the interior of M is homeomorphic to an infinite volume geometrically finite hyperbolic 3-manifold. So one obtains a strong topological theorem from this simple geometric argument.

Theorem 2.2 *(Thurston) Let M be a compact, atoroidal, irreducible 3-manifold with a non-toroidal boundary component. Then every cover $\widehat{M}$ of M with finitely generated fundamental group has a manifold compactification.*

Notice that any infinite volume topologically tame hyperbolic 3-manifold is homeomorphic to the interior of a compact, atoroidal irreducible 3-manifold with a non-toroidal boundary component. So we can now turn the argument around and obtain new information about covers of any infinite volume topologically tame hyperbolic 3-manifold.

Corollary 2.3 *([6]) If N is an infinite volume topologically tame hyperbolic 3-manifold and $\widehat{N}$ is a cover of N with finitely generated fundamental group, then $\widehat{N}$ is also topologically tame.*

Corollary 2.3 should really be regarded as a geometric consequence of the topological theorem 2.2, as topological tameness has many geometric and analytic consequences, see [6] and [7].

Remarks: Theorem 2.1 also holds for geometrically finite hyperbolic 3-manifolds with cusps (see proposition 7.1 in Morgan [17].) If M is a compact, atoroidal, irreducible 3-manifold with a toroidal boundary component, then the geometrically finite hyperbolic structure on M produced by the geometrization theorem, necessarily, has a cusp. Thus to obtain a complete proof of Theorem 2.2, one must use the more general version of theorem 2.1. Similarly, one must note that theorem 3.1 holds for geometrically finite hyperbolic 3-manifolds with cusps, in order to derive the complete version of theorem 3.2.

3 The finitely generated intersection property

A group G is said to have the *finitely generated intersection property* if whenever H and H' are finitely generated subgroups of G, then $H \cap H'$ is finitely generated.

Theorem 2.1 assures us that if we wish to establish the finitely generated intersection property for a geometrically finite Kleinian group Γ whose associated manifold $N = \mathbf{H}^3/\Gamma$ has infinite volume, then we need only consider intersections of geometrically finite subgroups.

Theorem 3.1 *(Susskind [23]) Let $N = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold and let Γ_1 and Γ_2 be geometrically finite subgroups of Γ, then $\Gamma_1 \cap \Gamma_2$ is geometrically finite.*

One may combine theorem 3.1 with Thurston's geometrization theorem and theorem 2.1 to obtain a theorem of Hempel:

Theorem 3.2 *(Hempel [12]) Let M be a compact, atoroidal, irreducible 3-manifold with a non-toroidal boundary component. Then $\pi_1(M)$ has the finitely generated intersection property.*

Again, since any infinite volume hyperbolic 3-manifold with finitely generated fundamental group is homotopy equivalent to a compact, atoroidal, irreducible 3-manifold with a non-toroidal boundary component, we derive a geometric analogue of theorem 3.2.

Theorem 3.3 *(Anderson [2]) If $N = \mathbf{H}^3/\Gamma$ is an infinite volume hyperbolic 3-manifold, then Γ has the finitely generated intersection property.*

Notice that the derivation of theorems 3.2 and 3.3 from theorem 3.1 followed the same pattern as in section 2. This pattern will reoccur in section 6.

The method of proof of theorem 3.1 involves developing information about the limit set of $\Gamma_1 \cap \Gamma_2$. (The *limit set* of a Kleinian group Γ is the smallest closed Γ-invariant subset of the sphere at infinity for $\mathbf{H}^3$, see Maskit [16].) Another consequence of this analysis has a particularly nice statement when N has no cusps.

Theorem 3.4 *(Susskind [23]) Let $N = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold with no cusps and let Γ_1 and Γ_2 be geometrically finite subgroups of Γ, then*

$$L_{\Gamma_1 \cap \Gamma_2} = L_{\Gamma_1} \cap L_{\Gamma_2}.$$

Historical Remarks: Greenberg [11] proved theorems 3.1 and 3.4 for hyperbolic surfaces, Maskit [15] proved them for component subgroups of a finitely generated Kleinian group, and Susskind and Swarup [24] proved them for hyperbolic n-manifolds. Hempel [12] proved a portion of theorem 3.1 independently in establishing theorem 3.2. One may apply Selberg's lemma to see that theorem 3.3 holds for hyperbolic 3-orbifolds (see Anderson [2].).

Anderson [3] further proved that if Γ is a Kleinian group without parabolic elements whose limit set is not the entire sphere, Γ_1 is any finitely generated subgroup and Γ_2 is any geometrically finite subgroup, then $\Gamma_1 \cap \Gamma_2$ is geometrically finite and $L_{\Gamma_1 \cap \Gamma_2} = L_{\Gamma_1} \cap L_{\Gamma_2}$.

Soma has recently claimed (in a footnote to [22]) that if Γ_1 and Γ_2 are topologically tame subgroups of a Kleinian group Γ (which has no parabolic elements), then $L_{\Gamma_1 \cap \Gamma_2} = L_{\Gamma_1} \cap L_{\Gamma_2}$. (See also Anderson [4].)

4 Geometrically Infinite Hyperbolic 3-manifolds

Let N be a topologically tame, infinite volume hyperbolic 3-manifold which is *geometrically infinite*, i.e. not geometrically finite. One might naturally wonder which covers of N are geometrically finite and which are geometrically infinite.

Before we answer this question, we must introduce some terminology. An end E of a hyperbolic 3-manifold is said to be *geometrically finite* if it has a neighborhood which does not intersect the convex core. Otherwise, it is said to be *geometrically infinite*. Notice that a hyperbolic 3-manifold with finitely generated fundamental group is geometrically finite if and only if all its ends are geometrically finite. The following theorem generalizes work of Thurston [25].

Covering Theorem: ([8]) *Let $\widehat{N}$ be a topologically tame hyperbolic 3-manifold which covers another hyperbolic 3-manifold N by a local isometry $p : \widehat{N} \to N$. If $\widehat{E}$ is a geometrically infinite end of $\widehat{N}$ then either*

a) $\widehat{E}$ has a neighborhood $\widehat{U}$ such that p is finite-to-one on $\widehat{U}$, or

b) N has finite volume and has a finite cover N' which fibers over the circle such that if N_S denotes the cover of N' associated to the fiber subgroup then $\widehat{N}$ is finitely covered by N_S. Moreover, if $\widehat{N} \neq N_S$, then $\widehat{N}$ is homeomorphic to the interior of a twisted I-bundle which is doubly covered by N_S.

The main tool in the proof of the covering theorem is a rough description of the geometry of geometrically infinite ends of topologically tame hyperbolic 3-manifolds. We recall that a *simplicial hyperbolic surface* is a map $f : S \to N$ from a surface S into a hyperbolic 3-manifold N such that there exists a triangulation T of S, such that f maps each face of T into a non-degenerate, totally geodesic triangle in N. Moreover, the sum of the angles of these triangles about each vertex must be at least 2π. The key point is that the intrinsic geometry of $f(S)$ has curvature ≤ -1.

The Filling Theorem: ([8]) *Let N be a topologically tame hyperbolic 3-manifold and let E be a geometrically infinite end of N. Then E has a neighborhood U homeomorphic to $S \times [0, \infty)$ (where S is a closed surface), such that every point in some subneighborhood U' of U is in the image of some simplicial hyperbolic surface $f_x : S \to U$ which is homotopic (within U) to $S \times \{0\}$. Moreover, given any $A > 0$ we may choose a subneighborhood U^A such that if $x \in U^A$ and γ is any compressible curve on S, then $f_x(\gamma)$ has length at least A.*

We can now give a short outline of the derivation of the covering theorem from the filling theorem. If the covering $p : \widehat{N} \to N$ is infinite-to-one on a geometrically infinite end $\widehat{E}$, then some point in N is in the image of an infinite sequence of simplicial hyperbolic surfaces in $\widehat{N}$. Moreover, any two of these surfaces bound an immersed $S \times I$. Since there is a limit to how congested the geometry can get locally we see that there really must be only finitely many of these surfaces up to local homotopy. We may then construct a map of a 3-manifold Q which fibers over the circle into N such that $i_* : \pi_1(Q) \to \pi_1(N)$ is injective, which allows us to conclude that possibility (b) occurs.

One may now use the covering theorem to give a topological characterization of which covers of a topologically tame hyperbolic 3-manifold N are geometrically finite. We notice that the covering theorem is only useful in characterizing which covers of N are geometrically finite because we know (theorem 2.3) that every cover of N with finitely generated fundamental group is topologically tame.

Corollary 4.1 *([8]) Let $N = \mathbf{H}^3/\Gamma$ be an infinite volume topologically tame hyperbolic 3-manifold. Then if $\widehat{\Gamma}$ is a finitely generated subgroup of Γ either*

a) $\widehat{N} = \mathbf{H}^3/\widehat{\Gamma}$ is geometrically finite, or

b) $\widehat{N}$ has a geometrically infinite end $\widehat{E}$ such that $p : \widehat{N} \to N$ is finite-to-one on some neighborhood $\widehat{U}$ of $\widehat{E}$.

If E is a geometrically infinite end of N with a neighborhood homeomorphic to $S \times [0, \infty)$, then we will call the image, under the homomorphism induced by the inclusion map, of $\pi_1(S)$ into $\pi_1(N)$ a *geometrically infinite (maximal) peripheral subgroup.* Notice that if N is topologically tame, then it has only finitely many geometrically infinite (maximal) peripheral subgroups (up to conjugacy). We may now give a group-theoretic characterization of which covers are geometrically finite.

Corollary 4.2 *([8]) Let $N = \mathbf{H}^3/\Gamma$ be an infinite volume topologically tame hyperbolic 3-manifold. Then if $\widehat{\Gamma}$ is a finitely generated subgroup of Γ either*

a) $\widehat{N} = \mathbf{H}^3/\widehat{\Gamma}$ is geometrically finite, or

b) $\widehat{\Gamma}$ contains a (conjugate of a) finite index subgroup of a geometrically infinite (maximal) peripheral subgroup.

One, nearly immediate, corollary of the filling theorem is:

Corollary 4.3 *([8]) If N is a topologically tame hyperbolic 3-manifold, then there exists K such that $inj_N(x) \leq K$ for all points $x \in C(N)$.*

Curt McMullen has proposed the following related conjecture:

Conjecture A: (McMullen) *Given an integer n, there exists K_n such that if N is a hyperbolic 3-manifold such that $\pi_1(N)$ is generated by $\leq n$ elements, than $C(N)$ contains no embedded ball of radius $\geq K_n$.*

Historical Remarks: Bill Thurston [25] proved the covering theorem for geometrically tame hyperbolic 3-manifolds with freely indecomposable fundamental group. Francis Bonahon [5] proved that all hyperbolic 3-manifolds with finitely generated freely indecomposable fundamental group are both topologically and geometrically tame. Combining these two results it would have been possible to prove analogues of corollaries 4.1 and 4.2 for covers of N with freely indecomposable fundamental groups. Canary [6] used Bonahon's work [5] to prove that all topologically tame hyperbolic 3-manifolds are geometrically tame. Scott and Swarup [19] previously proved corollary 4.2 for the fiber subgroups of 3-manifolds which fiber over the circle.

5 Closed hyperbolic 3-manifolds

The main barrier to extending many of the results in the previous section to closed hyperbolic 3-manifolds and their covers is the following conjecture which is a special case of the main conjecture.

Conjecture B: (Simon [18]) *If N is a closed hyperbolic 3-manifold and $\widehat{N}$ is a cover of N with finitely generated fundamental group, then $\widehat{N}$ is topologically tame.*

One, very partial, result in this direction is the following:

Proposition 5.1 *([8]) Let $N = \mathbf{H}^3/\Gamma$ be a closed hyperbolic 3-manifold. If $\widehat{\Gamma}$ is a finitely generated group in the kernel of a surjection $\rho : \Gamma \to \mathbf{Z}$, then $\widehat{N} = \mathbf{H}^3/\widehat{\Gamma}$ is topologically tame.*

Let $N = \mathbf{H}^3/\Gamma$ be a closed hyperbolic 3-manifold and $\widehat{N} = \mathbf{H}^3/\widehat{\Gamma}$ be a topologically tame cover of N. Then the covering theorem guarantees that $\widehat{N}$ is geometrically finite unless $\widehat{\Gamma}$ contains a virtual fiber subgroup with index at most two. (A subgroup $\widehat{\Gamma}$ of Γ is said to be a *virtual fiber subgroup* if there exists a finite index subgroup Γ' of Γ such that $N' = \mathbf{H}^3/\Gamma'$ fibers over the circle and $\widehat{\Gamma}$ corresponds to the fiber subgroup. See Soma [20] for a discussion of virtual fiber subgroups.) The following conjecture is thus equivalent to conjecture B.

Conjecture C: *If $N = \mathbf{H}^3/\Gamma$ is a closed hyperbolic 3-manifold and $\widehat{\Gamma}$ is a finitely generated subgroup of Γ, then $\widehat{\Gamma}$ is either geometrically finite or contains a virtual fiber subgroup of Γ of index at most two.*

One may apply proposition 5.1 to obtain the following, very partial, case of conjecture C.

Corollary 5.2 *([8]) Let $N = \mathbf{H}^3/\Gamma$ be a closed hyperbolic 3-manifold and $\widehat{\Gamma}$ be a finitely generated group in the kernel of a surjection $\rho : \Gamma \to \mathbf{Z}$. Then $\widehat{\Gamma}$ is geometrically finite if and only if $\widehat{\Gamma}$ does not contain a virtual fiber subgroup of Γ of index at most two.*

Jaco (see example V.19.d in [13]) showed that if a compact 3-manifold has a finite cover which fibers over the circle then its fundamental group does not have the finitely generated intersection property. There is no known example of a closed hyperbolic 3-manifold which does not have a finite cover which fibers over the circle, and hence no example of a closed hyperbolic 3-manifold whose fundamental group has the finitely generated intersection property. This led Thurston to pose the following conjecture.

Conjecture D: (Thurston) *Every closed hyperbolic 3-manifold has a finite cover which fibers over the circle.*

One consequence of Conjecture D would be:

Conjecture E: *The fundamental group of a hyperbolic 3-manifold N has the finitely generated intersection property if and only if N has infinite volume.*

In turn, Conjecture D would be a consequence of conjecture E together with either Conjecture B or C.

An immediate consequence of either Conjecture B or C and the covering theorem would be a weaker form of Conjecture E:

Conjecture F: *Let N be a closed hyperbolic 3-manifold. $\pi_1(N)$ has the finitely generated intersection property if and only if N does not have a finite cover which fibers over the circle.*

Historical Remarks: Simon [18] originally conjectured that all covers (with finitely generated fundamental group) of a closed, irreducible 3-manifold are topologically tame. Proposition 5.1 is a generalization of proposition 10.2 in Culler-Shalen [9]. Soma [21] previously noted that Conjecture D implied a form of Conjecture E. In this same paper, he investigates the finitely generated intersection property for more general classes of geometric 3-manifolds.

6 A curious group-theoretic property of Kleinian groups

The first consequence of $\widehat{\Gamma}$ being a finite index subgroup of Γ is that for all $g \in \Gamma$, there exists $n(g) \neq 0$, such that $g^{n(g)} \in \widehat{\Gamma}$. In the proof of corollary 4.2 it seemed that the converse of this would be useful. Surprisingly, the converse holds for all fundamental groups of infinite volume hyperbolic 3-manifolds. The following simple, but elegant, proof was provided by Jim Anderson.

Theorem 6.1 *(Anderson) Let $N = \mathbf{H}^3/\Gamma$ be an infinite volume geometrically finite hyperbolic 3-manifold and $\widehat{\Gamma}$ a finitely generated subgroup of Γ. If for all $g \in \Gamma$ there exists $n(g) \neq 0$ such that $g^{n(g)} \in \widehat{\Gamma}$, then $\widehat{\Gamma}$ has finite index in Γ.*

Proof of 6.1: If γ is a hyperbolic element of Γ with fixed points x and y, then $\gamma^{n(\gamma)}$ is a hyperbolic element of $\widehat{\Gamma}$ with fixed points x and y. Since fixed points of hyperbolic elements are dense in the limit set (see Maskit [16]) and $L_{\widehat{\Gamma}} \subset L_\Gamma$, we see that $L_\Gamma = L_{\widehat{\Gamma}}$. Since N is geometrically finite and has infinite volume, D_Γ is non-empty. (Here D_Γ denotes the complement, in the sphere at infinity for $\mathbf{H}^3$, of L_Γ.) Ahlfors' finiteness theorem [1] asserts that both $S_\Gamma = D_\Gamma/\Gamma$ and $S_{\widehat{\Gamma}} = D_{\widehat{\Gamma}}/\widehat{\Gamma}$ are finite area hyperbolic surfaces. Since $D_\Gamma = D_{\widehat{\Gamma}}$ and $\widehat{\Gamma} \subset \Gamma$, we see that $S_{\widehat{\Gamma}}$ covers S_Γ. Since both surfaces have finite area, the covering is finite-to-one. Hence $\widehat{\Gamma}$ is a finite index subgroup of Γ.

6.1

Again, combining this with Thurston's geometrization theorem, we get a theorem with topological assumptions.

Theorem 6.2 *Let M be a compact, atoroidal, irreducible 3-manifold with a non-toroidal boundary component and let G be a finitely generated subgroup of $\pi_1(M)$. If for all $g \in \pi_1(M)$ there exists $n(g) \neq 0$ such that $g^{n(g)} \in G$, then G has finite index in $\pi_1(M)$.*

We again may return this to a more general theorem with geometric assumptions.

Theorem 6.3 *Let Γ be a discrete, finitely generated subgroup of $Isom_+(\mathbf{H}^3)$ such that $\mathbf{H}^3/\Gamma$ has infinite volume and let $\widehat{\Gamma}$ be a finitely generated subgroup of Γ. If for all $g \in \Gamma$ there exists $n(g) \neq 0$ such that $g^{n(g)} \in \widehat{\Gamma}$, then $\widehat{\Gamma}$ has finite index in Γ.*

The following conjecture would follow from either Conjecture B or Conjecture C.

Conjecture G: *Let Γ be a discrete, finitely generated subgroup of $Isom_+(\mathbf{H}^3)$ and let $\widehat{\Gamma}$ a finitely generated subgroup. If for all $g \in \Gamma$ there exists $n(g) \neq 0$ such that $g^{n(g)} \in \widehat{\Gamma}$, then $\widehat{\Gamma}$ has finite index in Γ.*

We finish by asking which classes of 3-manifold groups or, more generally, groups does this property hold for? Do all hyperbolic groups, in the sense of Gromov, have this property? Do all finitely presented hyperbolic groups have this property?

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