THE THURSTON METRIC ON HYPERBOLIC DOMAINS AND BOUNDARIES OF CONVEX HULLS

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Abstract. We show that the nearest point retraction is a uniform quasi-isometry from the Thurston metric on a hyperbolic domain $\Omega \subset \hat{\mathbb{C}}$ to the boundary $\mathrm{Dome}(\Omega)$ of the convex hull of its complement. As a corollary, one obtains explicit bounds on the quasi-isometry constant of the nearest point retraction with respect to the Poincaré metric when Ω is uniformly perfect. We also establish Marden and Markovic's conjecture that Ω is uniformly perfect if and only if the nearest point retraction is Lipschitz with respect to the Poincaré metric on Ω.

1. Introduction

In this paper we continue the investigation of the relationship between the geometry of a hyperbolic domain Ω in $\hat{\mathbb{C}}$ and the geometry of the boundary $\mathrm{Dome}(\Omega)$ of the convex core of its complement. The surface $\mathrm{Dome}(\Omega)$ is hyperbolic in its intrinsic metric ([14, 29]) and the nearest point retraction $r : \Omega \to \mathrm{Dome}(\Omega)$ is a conformally natural homotopy equivalence. Marden and Markovic [24] showed that, if Ω is uniformly perfect, the nearest point retraction is a quasi-isometry with respect to the Poincaré metric on Ω and the quasi-isometry constants depend only on the lower bound for the injectivity radius of Ω in the Poincaré metric.

In this paper, we show that the nearest point retraction is a quasi-isometry with respect to the Thurston metric for *any* hyperbolic domain Ω with quasi-isometry constants which do not depend on the domain. We recover, as corollaries, many of the previous results obtained on the relationship between a domain Ω and $\mathrm{Dome}(\Omega)$ and obtain explicit constants for the first time in several cases. We also see that the nearest point retraction is Lipschitz if and only if the domain is uniformly perfect, which was originally conjectured by Marden and Markovic [24].

We will recall the definition of the Thurston metric in section 3. It is 2-bilipschitz (see Kulkarni-Pinkall [22]) to the perhaps more familiar quasihyperbolic metric originally defined by Gehring and Palka [19]. Recall that if $\Omega \subset \mathbb{C}$ and $\delta(z)$ denotes the Euclidean distance from $z \in \Omega$ to $\partial\Omega$, then the quasihyperbolic metric is simply $q(z) = \frac{1}{\delta(z)}dz$. The Thurston metric has the advantage of being conformally natural, i.e. invariant under conformal automorphisms of Ω.

Theorem 1.1. *Let $\Omega \subset \hat{\mathbb{C}}$ be a hyperbolic domain. Then the nearest point retraction $r : \Omega \to \mathrm{Dome}(\Omega)$ is 1-Lipschitz and a (K, K_0)-quasi-isometry with respect to the Thurston metric τ on Ω and the intrinsic hyperbolic metric on $\mathrm{Dome}(\Omega)$, where $K \approx 8.49$ and $K_0 \approx 7.12$. In particular,*

$$d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) \le d_\tau(z, w) \le K\ d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) + K_0.$$

Furthermore, if Ω is simply connected, then

$$d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) \le d_\tau(z, w) \le K'\ d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) + K_0'.$$

where $K' \approx 4.56$ and $K_0' \approx 8.05$.

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Marden and Markovic's [24] proof that the nearest point retraction is a quasi-isometry in the Poincaré metric if Ω is uniformly perfect does not yield explicit bounds, as it makes crucial use of a compactness argument. Combining Theorem 1.1 with work of Beardon and Pommerenke [2], which explains the relationship between the Poincaré metric and the quasihyperbolic metric, we obtain explicit bounds on the quasi-isometry constants. We recall that Ω is *uniformly perfect* if and only if there is a lower bound for the injectivity radius of Ω in the Poincaré metric.

Corollary 1.2. *Let Ω be a uniformly perfect domain in $\hat{\mathbb{C}}$ and let $\nu > 0$ be a lower bound for its injectivity radius in the Poincaré metric, then $r : \Omega \to \mathrm{Dome}(\Omega)$ is $2\sqrt{2}(k+\frac{\pi^2}{2\nu})$-Lipschitz and a $(2\sqrt{2}(k+\frac{\pi^2}{2\nu}), K_0)$-quasi-isometry with respect to the Poincaré metric. In particular,*

$$\frac{1}{2\sqrt{2}(k+\frac{\pi^2}{2\nu})}\ d_{\mathrm{Dome}(\Omega)}(r(z),r(w)) \le d_\rho(z,w) \le K\ d_{\mathrm{Dome}(\Omega)}(r(z),r(w)) + K_0$$

for all $z, w \in \Omega$ where $k = 4 + \log(2+\sqrt{2})$.

Examples in Section 10, show that the Lipschitz constant cannot have asymptotic form better than $O(\frac{1}{|\log \nu|\ \nu})$ as $\nu \to 0$.

When Ω is simply connected, we obtain:

Corollary 1.3. *Let Ω be a simply connected domain in $\hat{\mathbb{C}}$. Then*

$$\frac{1}{2}\ d_{\mathrm{Dome}(\Omega)}(r(z),r(w)) \le d_\rho(z,w) \le K'\ d_{\mathrm{Dome}(\Omega)}(r(z),r(w)) + K_0'$$

for all $z, w \in \Omega$.

In particular, this gives a simple reproof of Epstein, Marden and Markovic's [15] result that the nearest point retraction is 2-Lipschitz. It only relies on the elementary fact that r is 1-Lipschitz (see Lemma 5.2) with respect to the Thurston metric and a result of Herron, Ma and Minda [20] concerning the relationship between the Thurston metric and the Poincaré metric in the simply connected setting.

Marden and Markovic [24] also showed that the nearest point retraction is a quasi-isometry (with respect to the Poincaré metric) if and only if Ω is uniformly perfect. As another corollary of Theorem 1.1 we obtain the following alternate characterization of uniformly perfect domains which was conjectured by Marden and Markovic. Corollaries 1.2 and 1.4 together give an alternate proof of Marden and Markovic's original characterization.

Corollary 1.4. *The nearest point retraction $r : \Omega \to \mathrm{Dome}(\Omega)$ is Lipschitz, with respect to the Poincaré metric on Ω and the intrinsic hyperbolic metric on $\mathrm{Dome}(\Omega)$ if and only if the set Ω is uniformly perfect. Moreover, if Ω is not uniformly perfect, then r is not a quasi-isometry with respect to the Poincaré metric.*

Sullivan [27, 14] showed that there exists a uniform M such that if Ω is simply connected, then there is a conformally natural M-quasiconformal map from Ω to $\mathrm{Dome}(\Omega)$ which admits a bounded homotopy to the nearest point retraction. (We say that a map from Ω to $\mathrm{Dome}(\Omega)$ is *conformally natural* if it is Γ-equivariant whenever Γ is a group of Möbius transformations preserving Ω.) Epstein, Marden and Markovic [15, 16] showed that the best bound on M must lie strictly between 2.1 and 13.88. While, combining work of Bishop [5] and Epstein-Markovic [17], the best bound of M lies between 2.1 and 7.82 if we do not require the quasiconformal map to be conformally natural.

Marden and Markovic [24] showed that a domain is uniformly perfect if and only if there is a quasiconformal map from Ω to $\mathrm{Dome}(\Omega)$ which is homotopic to the nearest point retraction by a bounded homotopy. Moreover, they show that there are bounds on the quasiconformal constants which depend only on a lower bound for the injectivity radius of Ω. We note that again these bounds are not concrete.

In order to use the nearest point retraction to obtain a conformally natural quasiconformal map between Ω and $\mathrm{Dome}(\Omega)$ we show that r lifts to a quasi-isometry between the universal cover $\tilde{\Omega}$ of Ω and the universal cover $\mathrm{Dome}(\tilde{\Omega})$ of $\mathrm{Dome}(\Omega)$. Although they do not make this explicit in their paper, this is essentially the same approach taken by Marden and Markovic [24]. We recall that quasi-isometric surfaces need not even be homeomorphic and that even homeomorphic quasi-isometric surfaces need not be quasiconformal to one another. Notice that the quasi-isometry constants for the lift of the nearest point retraction are, necessarily, not as good as those for the nearest point retraction. Examples in Section 10 show that the first quasi-isometry constant of $\tilde{r}$ cannot have form better than $O(\nu e^{\frac{\pi^2}{2\nu}})$ as $\nu \to 0$.

Theorem 1.5. *Suppose that Ω is a uniformly perfect hyperbolic domain and $\nu > 0$ is a lower bound for its injectivity radius in the Poincaré metric. Then the nearest point retraction lifts to a quasi-isometry*

$$\tilde{r} : \tilde{\Omega} \to \mathrm{Dome}(\tilde{\Omega})$$

between the universal cover of Ω (with the Poincaré metric) and the universal cover of $\mathrm{Dome}(\Omega)$ *with quasi-isometry constants depending only on ν. In particular,*

$$\frac{1}{2\sqrt{2}(k+\frac{\pi^2}{2\nu})}\, d_{\mathrm{Dome}(\tilde{\Omega})}(\tilde{r}(z),\tilde{r}(w)) \le d_{\tilde{\Omega}}(z,w) \le L(\nu) d_{\mathrm{Dome}(\tilde{\Omega})}(\tilde{r}(z),\tilde{r}(w)) + L_0$$

for all $z, w \in \tilde{\Omega}$ where $L(\nu) = O(e^{\frac{\pi^2}{2\nu}})$ as $\nu \to 0$ and $L_0 \approx 8.05$.

We then apply work of Douady and Earle [13] to obtain explicit bound on the quasiconformal map homotopic to the nearest point retraction when Ω is uniformly perfect.

Corollary 1.6. *There exists an explicit function $M(\nu)$ such that if Ω is uniformly perfect and $\nu > 0$ is a lower bound for its injectivity radius in the Poincaré metric, then there is a conformally natural $M(\nu)$-quasiconformal map $\phi : \Omega \to \mathrm{Dome}(\Omega)$ which admits a bounded homotopy to r. Moreover, if Ω is not uniformly perfect, then there does not exist a bounded homotopy of r to a quasiconformal map.*

We will give explicit formulas for L and M in section 9.

The table below summarizes the main results concerning the nearest point retraction in the cases that the domain is simply connected, uniformly perfect and hyperbolic but not uniformly perfect.

	Ω simply connected	Ω uniformly perfect $\text{inj}_\Omega > \nu$	Ω hyperbolic but not uniformly perfect
Lipschitz bounds on r in Poincaré metric	r is 2-Lipschitz (see [15] and Cor. 1.3)	r is $K(\nu)$-Lipschitz where $K(\nu) = 2\sqrt{2}(k + \frac{\pi^2}{2\nu})$ $k \approx 5.76$ (Corollary 1.2)	r is not Lipschitz (Corollary 1.4)
quasi-isometry bounds on r in Poincaré metric	r is a (K', K'_0)-quasi-isometry where $K' \approx 4.56$, $K'_0 \approx 8.05$ (Corollary 1.3)	r is a $(K(\nu), K_0)$-quasi-isometry where $K_0 \approx 7.12$ (Corollary 1.2)	r is not a quasi-isometry (see [24] and Cor. 1.4)
Lipschitz bounds on r in Thurston metric	r is 1-Lipschitz (Theorem 1.1)	r is 1-Lipschitz (Theorem 1.1)	r is 1-Lipschitz (Theorem 1.1)
quasi-isometry bounds on r in Thurston metric	r is a (K', K'_0)-quasi-isometry (Theorem 1.1)	r is a (K, K_0)-quasi-isometry where $K \approx 8.49$ (Theorem 1.1)	r is a (K, K_0)-quasi-isometry (Theorem 1.1)
conformally natural quasiconformal map homotopic to r by a bounded homotopy	M-quasiconformal map where $2.1 < M < 13.88$ (see [15] and [16])	$M(\nu)$-quasiconformal map (Corollary 1.6)	no quasiconformal map (see [24] and Cor. 1.6)
quasiconformal map homotopic to r by a bounded homotopy	M'-quasiconformal map where $2.1 < M' < 7.82$ (see [5] and [17])	$M(\nu)$-quasiconformal map (Corollary 1.6)	no quasiconformal map (see [24] and Cor. 1.6)

The only constant in the table above which is known to be sharp is the Lipschitz constant in the simply connected case. It would be desirable to know sharp bounds (or better bounds) in the remaining cases. We refer the reader to the excellent discussion of the relationship between the nearest point retraction and complex analytic problems, especially Brennan's conjecture, in Bishop [4].

One of the main motivations for the study of domains and the convex hulls of their complements comes from the study of hyperbolic 3-manifolds where one is interested in the relationship between the boundary of the convex core of a hyperbolic manifold and the boundary of its conformal bordification. If $N = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold, one considers the domain of discontinuity $\Omega(\Gamma)$ for Γ's action on $\hat{\mathbb{C}}$. The *conformal boundary* $\partial_c N$ is the quotient $\Omega(\Gamma)/\Gamma$ and $\hat{N} = N \cup \partial_c N$ is the natural conformal bordification of N. The *convex core* $C(N)$ of N is obtained as the quotient of the convex hull of $\hat{\mathbb{C}} - \Omega(\Gamma)$ and its boundary $\partial C(N)$ is the quotient of $\text{Dome}(\Omega)$. The nearest point retraction descends to a homotopy equivalence

$$\bar{r} : \partial_c N \to \partial C(N).$$

In this setting, Sullivan's theorem implies that if $\partial_c N$ is incompressible in $\hat{N}$, then there is a quasiconformal map, with uniformly bounded dilatation, from $\partial_c N$ to $\partial C(N)$ which is homotopic to the nearest point retraction.

Since the Thurston metric is Γ-invariant and the nearest point retraction is Γ-equivariant we immediately obtain a manifold version of Theorem 1.1.

Theorem 1.7. *Let $N = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold. Then the nearest point retraction $r : \partial_c N \to C(N)$ is a (K, K_0)-quasi-isometry with respect to the Thurston metric τ on $\partial_c N$ and the intrinsic hyperbolic metric on $\partial C(N)$. In particular,*

$$d_{\partial C(N)}(\bar{r}(z), \bar{r}(w)) \le d_\tau(z, w) \le K\ d_{\partial C(N)}(\bar{r}(z), \bar{r}(w)) + K_0.$$

We also obtain a version of Corollary 1.2 in the manifold setting.

Corollary 1.8. *Let $N = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold such that there is a lower bound $\nu > 0$ for the injectivity radius of $\Omega(\Gamma)$ (in the Poincaré metric), then*

$$\frac{1}{2\sqrt{2}(k+\frac{\pi^2}{2\nu})}\ d_{\partial C(N)}(\bar r(z),\bar r(w)) \le d_\rho(z,w) \le K\ d_{\partial C(N)}(\bar r(z),\bar r(w)) + K_0$$

for all $z, w \in \partial_c N$ where $k = 4 + \log(2+\sqrt{2})$.

We also obtain a version of Corollary 1.6 which guarantees the existence of a quasiconformal map from the conformal boundary to the boundary of the convex core.

Corollary 1.9. *If $N = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold and $\nu > 0$ is a lower bound for the injectivity radius of $\Omega(\Gamma)$ (in the Poincaré metric), then there is a $M(\nu)$-quasiconformal map $\phi : \partial_c N \to \partial C(N)$ which admits a bounded homotopy to the nearest point retraction $\bar r$.*

If N is a hyperbolic 3-manifold with finitely generated fundamental group, then its domain of discontinuity is uniformly perfect (see Pommerenke [26]), so Corollaries 1.8 and 1.9 immediately apply to all hyperbolic 3-manifolds with finitely generated fundamental group. More generally, analytically finite hyperbolic 3-manifolds (i.e. those whose conformal boundary has finite area) are known to be uniformly perfect [11].

We recall that it was earlier shown (see [8]) that the nearest point retraction $\bar r : \partial_c N \to \partial C(N)$ is $(2\sqrt{2}(k+\frac{\pi^2}{2\nu}))$-Lipschitz whenever N is analytically finite and has a $J(\nu)$-Lipschitz homotopy inverse where $J(\nu) = O(e^{\frac{C}{\nu}})$ (for some explicit constant $C > 0$) as $\nu \to 0$. Moreover, if $N = \mathbb{H}^3/\Gamma$ is analytically finite and $\Omega(\Gamma)$ is simply connected, then $\bar r$ has a $(1+\frac{\pi}{\sinh^{-1}(1)})$-Lipschitz homotopy inverse. Remark (2) in Section 7 of [8] indicates that one cannot improve substantially on the asymptotic form of the Lipschitz constant for $\bar r$ (see also Section 6 of [12]), while Proposition 6.2 of [8] shows that one cannot improve substantially on the asymptotic form of the Lipschitz constant of the homotopy inverse for $\bar r$.

Outline of paper: Many of the techniques of proof in this paper are similar to those in our previous papers [7] and [8]. The consideration of the Thurston metric allows us to obtain more general results, while the fact that the Thurston metric behaves well under approximation allows us to reduce many of our proofs to the consideration of the case where Ω is the complement of finitely many points in $\hat{\mathbb{C}}$ and the convex hull of its complement is a finite-sided ideal polyhedron. Therefore, our techniques are more elementary than much of the earlier work on the subject.

In Section 2 we recall basic facts and terminology concerning the nearest point retraction and the boundary of the convex core. In Section 3 we discuss the Poincaré, quasihyperbolic and Thurston metrics on a hyperbolic domain. In section 4 we establish the key approximation properties which allow us to reduce the proof of Theorems 1.1 and 1.5 to the case where $\hat{\mathbb{C}} - \Omega$ is a finite collection of points. In Section 5 we study the geometry of the finitely punctured case and introduce a notion of intersection number which records the total exterior dihedral angle along a geodesic in Dome(Ω). If α is an arc in Dome(Ω), then the difference between its length and the length of $r^{-1}(\alpha)$ in the Thurston metric is exactly this intersection number, see Lemma 5.2.

Section 6 contains the key intersection number estimates used in the proof of Theorem 1.1. The first estimate, Lemma 6.1, bounds the total intersection numbers of moderate length geodesic arcs in the thick part of Dome(Ω). (Lemma 6.1 is a mild generalization of Lemma 4.3 in [8] and we defer its proof to an appendix.) We then bound the total intersection number of a short geodesic (Lemma 6.6) and the angle of intersection of an edge of Dome(Ω) with a short geodesic (Lemma 6.7). We give a quick proof of a weaker form of Lemma 6.6 and defer the proof of the better estimate to the appendix. In a remark at the end of Section 6, we see that one could use this weaker estimate to obtain a version of Theorem 1.1 with slightly worse constants.

Section 7 contains the proof of Theorem 1.1. It follows immediately from Lemma 5.2 that the nearest point retraction is 1-Lipschitz with respect to the Thurston metric, but it remains to bound the distance between two points x and y in Ω in terms of the distance between $r(x)$ and $r(y)$ in $\text{Dome}(\Omega)$. We do so by first dividing the shortest geodesic joining $r(x)$ to $r(y)$ into segments of length roughly G (for some specific constant G). The segments in the thick part have bounded intersection number by Lemma 6.1. We use Lemmas 6.6 and 6.7 to replace geodesic segments in the thin part by bounded length arcs which consist of one segment missing every edge and another segment with bounded intersection number. We then obtain a path joining $r(x)$ to $r(y)$ with length and intersection number bounded above by a multiple of $d_{\text{Dome}(\Omega)}(r(x), r(y)) + G$. Lemma 5.2 can then be applied to complete the proof.

In Section 8 we derive consequences of Theorem 1.1, including Corollaries 1.2, 1.3 and 1.4. In Section 9 we prove Theorem 1.5, using techniques similar to those in the proof of Theorem 1.1. We then use work of Douady and Earle [13] to derive Corollary 1.6. In Section 10, we study the special case of round annuli and observe that the asymptotic form of our estimates in Corollary 1.2 and Theorem 1.5 cannot be substantially improved. The appendix, Section 11, contains the complete proofs of Lemmas 6.1 and 6.6. The proof of Lemma 6.1 is considerably simpler than the proof of Lemma 4.3 in [8] as we are working in the finitely punctured case.

2. The nearest point retraction and convex hulls

If $\Omega \subseteq \hat{\mathbb{C}}$ is a hyperbolic domain, we let $CH(\hat{\mathbb{C}} - \Omega)$ be the hyperbolic convex hull of $\hat{\mathbb{C}} - \Omega$ in $\mathbb{H}^3$. We then define

$$\text{Dome}(\Omega) = \partial CH(\hat{\mathbb{C}} - \Omega)$$

and recall that it is hyperbolic in its intrinsic metric ([14, 29]). In the special case that $\hat{\mathbb{C}} - \Omega$ lies in a round circle, $CH(\hat{\mathbb{C}} - \Omega)$ is a subset of a totally geodesic plane in $\mathbb{H}^3$. In this case, it is natural to consider $\text{Dome}(\Omega)$ to be the double of $CH(\hat{\mathbb{C}} - \Omega)$ along its boundary (as a surface in the plane).

We define the *nearest point retraction*

$$r : \Omega \to \text{Dome}(\Omega),$$

by letting $r(z)$ be the point of intersection of the smallest horosphere about z which intersect $\text{Dome}(\Omega)$. We note that $r : \Omega \to \text{Dome}(\Omega)$ is a conformally natural homotopy equivalence. The nearest point retraction extends continuously to a conformally natural map

$$r : \mathbb{H}^3 \cup \Omega \to CH(\hat{\mathbb{C}} - \Omega)$$

where $r(x)$ is the (unique) nearest point on $CH(\hat{\mathbb{C}} - \Omega)$ if $x \in \mathbb{H}^3$ (see [14]).

We recall that a *support plane* to $\text{Dome}(\Omega)$ is a plane P which intersects $\text{Dome}(\Omega)$ and bounds an open half-space H_P disjoint from $\text{Dome}(\Omega)$. The half-space H_P is associated to a maximal open disc D_P in Ω. Since D_P is maximal, ∂D_P contains at least two points in $\partial\Omega$. Therefore, P contains a geodesic $g \in \text{Dome}(\Omega)$. Notice that P intersects $\text{Dome}(\Omega)$ in either a single edge or a convex subset of P which we call a *face* of $\text{Dome}(\Omega)$. If $z \in \Omega$, then the geodesic ray starting at $r(z)$ in the direction of z is perpendicular to a support plane P_z through $r(z)$.

3. The Poincaré metric, the quasihyperbolic metric and the Thurston metric

Given a hyperbolic domain Ω in $\hat{\mathbb{C}}$ there is a unique metric ρ of constant curvature -1 which is conformally equivalent to the Euclidean metric on Ω. This metric ρ is called the *Poincaré metric* on Ω and is denoted

$$\rho(z) = \lambda_\rho(z)\ dz.$$

Alternatively, the Poincaré metric can be given by defining the length of a tangent vector $v \in T_x(\Omega)$ to be the infimum of the hyperbolic length over all vectors v' such that there exists a conformal map $f : \mathbb{H}^2 \to \hat{\mathbb{C}}$ such that $f(\mathbb{H}^2) \subset \Omega$ and $df(v') = v$.

In [2], Beardon and Pommerenke show that the quasihyperbolic metric and the Poincaré metric are closely related. We recall that, the *quasihyperbolic metric* on $\Omega \subset \mathbb{C}$ is the conformal metric

$$q(z) = \frac{1}{\delta(z)}\ dz$$

where $\delta(z)$ is the Euclidean distance from z to $\partial\Omega$. They let

$$\beta(z) = \inf\left\{ \left| \log \frac{|z-a|}{|z-b|} \right| \ : a \in \partial\Omega,\ b \in \partial\Omega,\ |z-a| = \delta(z) \right\}$$

Alternatively, one can define $\beta(z) = \pi M$ where M is the maximal modulus of an essential round annulus (i.e. one which separates $\hat{\mathbb{C}} - \Omega$) in Ω whose central (from the conformal viewpoint) circle passes through z. We recall that if A is any annulus in $\hat{\mathbb{C}}$, then it is conformal to an annulus of the form $\{z \mid \frac{1}{t} < |z| < t\}$ and we define its *modulus* to be

$$\mathrm{mod}(A) = \frac{1}{2\pi} \log(t^2).$$

The central circle of A is the pre-image of the unit circle in $\{z \mid \frac{1}{t} < |z| < t\}$.

Theorem 3.1. (Beardon-Pommerenke [2]) *If Ω is a hyperbolic domain in $\mathbb{C}$, then*

$$\frac{1}{\sqrt{2}(k+\beta(z))} q(z) \le \rho(z) \le \frac{2k+\frac{\pi}{2}}{k+\beta(z)} q(z)$$

for all $z \in \Omega$, where $k = 4 + \log(3+2\sqrt{2}) \approx 5.76$. Moreover,

$$\rho(z) \le 2q(z)$$

for all $z \in \Omega$. If Ω is simply connected, then

$$\frac{1}{2} q(z) \le \rho(z) \le 2q(z).$$

Using the fact that the core curves of large modulus essential annuli are short in the Poincaré metric, one sees that a lower bound on the injectivity radius of Ω gives an explicit bilipschitz equivalence between the Poincaré metric and the quasihyperbolic metric when Ω is uniformly perfect.

Corollary 3.2. *If Ω is a uniformly perfect hyperbolic domain and $\nu > 0$ is a lower bound for the injectivity radius of Ω in the Poincaré metric, then*

$$\frac{1}{\sqrt{2}(k+\frac{\pi^2}{2\nu})} q(z) \le \rho(z) \le 2q(z)$$

for all $z \in \Omega$.

Proof. Corollary 3.3 in [12] asserts that if $\beta(z) \ge M$, then $\mathrm{inj}_\Omega(z) \le \frac{\pi^2}{2M}$. Therefore, if $\mathrm{inj}_\Omega(z) \ge \nu > 0$, then $\beta(z) \le \frac{\pi^2}{2\nu}$. The result then follows from Theorem 3.1. ☐

Another metric closely related to the quasihyperbolic metric is the Thurston metric. The *Thurston metric*

$$\tau(z) = \lambda_\tau(z)\ dz$$

on Ω is defined by letting the length of a vector $v \in T_x(\Omega)$ be the infimum of the hyperbolic length of all vectors $v' \in \mathbb{H}^2$ such that there exists a Möbius transformation f such that $f(\mathbb{H}^2) \subset \Omega$ and $df(v') = v$. By definition, the Thurston metric is conformally natural and conformal to the Euclidean metric. Furthermore, by our alternate definition of the Poincaré metric, we have the inequality

$$\rho(z) \leq \tau(z) \tag{1}$$

for all $z \in \Omega$.

The Thurston metric is also known as the projective metric, as it arises from regarding Ω as a complex projective surface and giving it the metric Thurston described on such surfaces. Kulkarni and Pinkall defined and studied an analogue of this metric in all dimensions and it is also sometimes called the Kulkarni-Pinkall metric. See Herron-Ma-Minda [20], Kulkarni-Pinkall [22], McMullen [25] and Tanigawa [28] for further information on the Thurston metric.

Kulkarni and Pinkall (see Theorem 7.2 in [22]) proved that the Thurston metric and the quasi-hyperbolic metric are 2-bilipschitz.

Theorem 3.3. ([22]) *If Ω is a hyperbolic domain in $\mathbb{C}$ then*

$$\frac{1}{2}\tau(z) \leq q(z) \leq \tau(z).$$

In the simply connected setting, Herron, Ma and Minda (see Theorem 2.1 and Lemma 3.1 in [20]) prove that the Poincaré metric and the Thurston metric are also 2-bilipschitz.

Theorem 3.4. ([20]) *If Ω is a simply connected hyperbolic domain in $\hat{\mathbb{C}}$, then*

$$\frac{1}{2}\tau(z) \leq \rho(z) \leq \tau(z).$$

We will make use of the following alternate characterization of the Thurston metric. We note that it is not difficult to derive Theorem 3.3 from this characterization.

Lemma 3.5. *If Ω is a hyperbolic domain in $\mathbb{C}$, then the Thurston metric $\tau(z)$ on Ω is given by*

$$\tau(z) = \frac{1}{h(z)}\ dz$$

where $h(z)$ is the radius of the maximal horoball B_z based at z whose interior misses Dome(Ω).

Proof. Recall the disk model Δ for $\mathbb{H}^2$ with the metric

$$\rho_\Delta(z) = \frac{2}{1-|z|^2}\ dz = \lambda_\Delta(z)\ dz.$$

By normalizing, we may define the length of a vector $v \in T_x(\Omega)$ in the Thurston metric to be the infimum of the length of vectors $v' \in T_0\Delta$ such that there exists a Möbius transformation m such that $m(\Delta) \subset \Omega$ with $dm_0(v') = v$. The hyperbolic length of v' is then simply $2|v'| = \frac{2}{|m'(0)|}|v|$. Thus

$$\tau(z) = \inf_m \frac{2}{|m'(0)|}$$

where the infimum is taken over all Möbius transformations m such that $m(\Delta) \subset \Omega$ and $m(0) = z$.

Given such a Möbius transformation m, we let $D_m = m(\Delta)$ and let P_m be the hyperbolic plane with the same boundary as D_m. We further let B_m be the horoball based at z which is tangent to P_m. As $D_m \subseteq \Omega$, P_m does not intersect the interior of the convex hull $CH(\hat{\mathbb{C}} - \Omega)$. Similarly, the interiors of B_m and $CH(\hat{\mathbb{C}} - \Omega)$ are disjoint. Let h_m be the radius of the horoball B_m. Notice that $h_m \le h(z)$. Let n be a parabolic Möbius transformation with fixed point z, which sends $B_m \cap P_m$ to the point vertically above z. Then, n fixes B_m and sends D_m to a disk centered at z with radius $2h_m$. Therefore, since $n \circ m$ sends a disk of radius 1 to a disk of radius $2h_m$ and sends centers to centers,

$$|n \circ m'(0)| = |n'(z)m'(0)| = 2h_m.$$

Since n is parabolic and fixes z, $|n'(z)| = 1$, which implies that $|m'(0)| = 2h_m$. Therefore

$$\tau(z) = \inf_m \frac{2}{|m'(0)|} = \inf_m \frac{1}{h_m} \ge \frac{1}{h(z)}$$

On the other hand, let P_z be the support plane to $r(z)$ which is perpendicular to the geodesic ray joining $r(z)$ to z. Then P_z is tangent to B_z at $r(z)$. Let D_z be the (open) disk in Ω containing z whose boundary agrees with ∂P_z. If m is a Möbius transformation such that $m(0) = z$ and $m(\Delta) = D_z$, then, by the above analysis, $|m'(0)| = 2h(z)$. It follows that

$$\tau(z) = \frac{1}{h(z)}$$

as desired. □

4. Finite approximations and the geometry of domains and convex hulls

Given a hyperbolic domain Ω, let $\{x_n\}_{n=1}^{\infty}$ be a dense set of points in $\hat{\mathbb{C}} - \Omega$. If $X_n = \{x_1, \ldots, x_n\}$, then we call $\{X_n\}$ a *nested finite approximation* to $\hat{\mathbb{C}} - \Omega$. Notice that $\{X_n\}$ converges to $\hat{\mathbb{C}} - \Omega$ in the Hausdorff topology on closed subsets of $\hat{\mathbb{C}}$. We let $\Omega_n = \hat{\mathbb{C}} - X_n$, denote the quasihyperbolic metric on Ω_n by q_n, and let $r_n : \Omega_n \to \mathrm{Dome}(\Omega_n)$ be the nearest point retraction. Since $\Omega \subseteq \Omega_n$, q_n and r_n are both defined on Ω. Results of Bowditch [6] imply that $\{CH(X_n)\}$ converges to $CH(\hat{\mathbb{C}} - \Omega)$ and that $\{\mathrm{Dome}(\Omega_n)\}$ converges to $\mathrm{Dome}(\Omega)$ in the Hausdorff topology on closed subsets of $\mathbb{H}^3$.

The following approximation lemma will allow us to reduce the proof of many of our results to the case where Ω is the complement of finitely many points.

Lemma 4.1. *Let $\{X_n\}$ be a nested finite approximation to $\hat{\mathbb{C}} - \Omega$. If, for all n, $\Omega_n = \hat{\mathbb{C}} - X_n$, q_n is the quasihyperbolic metric on Ω_n, and p_n is the Thurston metric on Ω_n, then*

1. *the sequence $\{q_n\}$ of metrics converges uniformly on compact subsets of Ω to the quasihyperbolic metric q on Ω,*
2. *the sequence $\{\tau_n\}$ of metrics converges uniformly on compact subsets of Ω to the Thurston metric τ on Ω,*
3. *the associated nearest point retractions $\{r_n : \Omega_n \to \mathrm{Dome}(\Omega_n)\}$ converge to the nearest point retraction $r : \Omega \to \mathrm{Dome}(\Omega)$ uniformly on compact subsets of Ω,*
4. *if $z, w \in \Omega$, then $\{d_{\mathrm{Dome}(\Omega_n)}(r_n(z), r_n(w))\}$ converges to $d_{\mathrm{Dome}(\Omega)}(r(z), r(w))$, and*
5. *if $z \in \Omega$, then*
$$\lim \mathrm{inj}_{\mathrm{Dome}(\Omega_n)}(r_n(z)) = \mathrm{inj}_{\mathrm{Dome}(\Omega)}(r(z)).$$

Proof. Let $\delta_n(z)$ denote the distance from $z \in \Omega_n$ to X_n. It is clear that $\{\delta_n\}$ converges to δ uniformly on compact subsets of Ω. Therefore, $\{q_n\}$ converges uniformly to q on compact subsets of Ω.

If $z \in \Omega_n$, let B_z^n denote the maximal horoball based at z whose interior is disjoint from $CH(X_n)$ and whose closure intersects $\mathrm{Dome}(\Omega_n)$ at $r_n(z)$. Suppose that $\{z_n\} \subset \Omega$ converges to $z \in \Omega$. Since $\{CH(X_n)\}$ converges to $CH(\hat{\mathbb{C}} - \Omega)$, $\{B_{z_n}^n\}$ converges to a horoball B based at z whose interior is disjoint from $CH(\hat{\mathbb{C}} - \Omega)$ and whose closure intersects $\mathrm{Dome}(\Omega)$ at $\lim r_n(z_n)$. By definition, $r(z)$ is this intersection point, so $r(z) = \lim r_n(z_n)$. Therefore, $\{r_n\}$ converges uniformly to r on Ω, establishing (3). Moreover, applying Lemma 3.5, we see that $\{\tau_n(z_n)\}$ converges to $\tau(z)$, so $\{\tau_n\}$ converges uniformly to τ on Ω, which proves (2).

Suppose that $z, w \in \Omega$. If γ_n is a path in $\mathrm{Dome}(\Omega_n)$ of length

$$l_n = d_{\mathrm{Dome}(\Omega_n))}(r_n(z), r_n(w)),$$

then γ_n converges, up to subsequence, to a path in $\mathrm{Dome}(\Omega)$ of length at most $\liminf l_n$ joining $r(z)$ to $r(w)$. Therefore,

$$d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) \leq \liminf l_n.$$

On the other hand, if γ is a path in $\mathrm{Dome}(\Omega)$ of length $l = d_{\mathrm{Dome}}(\Omega)(r(z), r(w))$, then $r_n(\gamma)$ is a path of length at most l on $\mathrm{Dome}(\Omega_n)$ joining $r_n(r(z))$ to $r_n(r(w))$. Notice that $\{d_{\mathrm{Dome}(\Omega_n)}(r_n(r(z)), r_n(z))\}$ and $\{d_{\mathrm{Dome}(\Omega_n)}(r_n(r(w)), r_n(w))\}$ both converge to 0, since $\{\mathrm{Dome}(\Omega_n)\}$ converges to $\mathrm{Dome}(\Omega)$. Therefore, $\limsup l_n \leq l$. It follows that $\lim l_n = l$, establishing (4).

The proof of (5) follows much the same logic as the proof of (4).

□

5. The geometry of finitely punctured spheres

In this section we consider the situation where Ω is the complement of finitely many points in $\hat{\mathbb{C}}$. In this case, the convex core $CH(\hat{\mathbb{C}} - \Omega)$ of the complement of Ω is an ideal hyperbolic polyhedron and its boundary $\mathrm{Dome}(\Omega)$ consists of a finite number of ideal hyperbolic polygons meeting along edges. We give a precise description of the Thurston metric in this case (Lemma 5.1) and show that r is 1-Lipschitz with respect to the Thurston metric (Lemma 5.2). We also introduce an intersection number for a curve $\alpha : [0, 1] \to \mathrm{Dome}(\Omega)$ which is transverse to all edges and give a formula for the length of its pre-image $r^{-1}(\alpha)$ in the Thurston metric on Ω.

5.1. **The Thurston metric in the finitely punctured case.** Let Ω be the complement of a finite set in $\hat{\mathbb{C}}$. Each face F of the polyhedron $CH(\hat{\mathbb{C}} - \Omega)$ is contained in a unique plane P_F. Furthermore, P_F bounds a unique open half-space H_F which does not intersect $CH(\hat{\mathbb{C}} - \Omega)$. The half-space H_F defines an open disk D_F in Ω. Let F' be the ideal polygon in D_F with same vertices as F. Abusing notation, we will call F' a *face* of Ω. If two faces F_1 and F_2 meet in an edge e with exterior dihedral angle θ_e then the spherical bigon $B_e = D_{F_1} \cap D_{F_2}$ is adjacent to the polygons F_1' and F_2' and has angle θ_e. Thus we have a decomposition of Ω into faces F' and spherical bigons B corresponding to the edges of $CH(\hat{\mathbb{C}} - \Omega)$.

If F' is a face of Ω then it inherits a hyperbolic metric $p_{F'}$ which is the restriction of the Poincaré metric on D_F. Each spherical bigon B_e inherits a Euclidean metric in the following manner. Let f_e be a Möbius transformation taking B_e to the Euclidean rectangle $S_e = \mathbf{R} \times [0, \theta_e]$. We then give B_e the Euclidean metric p_e obtained by pulling back the standard Euclidean metric on S_e. Since any Möbius transformation preserving S_e is a Euclidean isometry, this metric is independent of our choice of f_e.

Lemma 5.1 shows that the Thurston metric agrees with the metrics defined above. One may view this as a special case of the discussion in section 2.1 of Tanigawa [28], see also section 8 of Kulkarni-Pinkall [22] and section 3 of McMullen [25].

Lemma 5.1. *Let Ω be the complement of finitely many points in $\hat{\mathbb{C}}$. Then*

(1) *The Thurston metric on Ω agrees with $\rho_{F'}$ on each face F' of Ω and agrees with p_e on each spherical bigon B_e.*
(2) *With respect to the Thurston metric, the nearest point retraction $r : \Omega \to \mathrm{Dome}(\Omega)$ maps each face F' isometrically to the face F of $\mathrm{Dome}(\Omega)$ and is Euclidean projection on each bigon. That is, if B_e is a bigon, there is an isometry $g_e : e \to \mathbf{R}$ such that*

$$\begin{array}{ccc} B & \xrightarrow{r} & e \\ \Big\downarrow f_e & & \Big\downarrow g_e \\ \mathbf{R} \times [0,\theta] & \xrightarrow{\pi_1} & \mathbf{R} \end{array}$$

Proof: First suppose that F' is a face of Ω. We may normalize by a Möbius transformation so that D_F is the unit disk and $z = 0$. In this case, the maximal horoball at z has height 1, so the Thurston metric for Ω at z agrees with the Poincaré metric on D_F at z. Moreover, one sees that r agrees with the hyperbolic isometry from the unit disk to P_F on F'.

Now let e be an edge between two faces F_1 and F_2. We may normalize so that e is the vertical line joining 0 to ∞ and define the isometry g_e by $g_e(0,0,t) = \log t$. We may further assume that B_e is the sector $\{se^{it} \mid s > 0,\ 0 \le t \le \theta\}$. For a point $z = se^{it} \in B$, the maximal horoball at z is tangent to e at height s. Therefore the Thurston metric on B is $\frac{1}{s}dz$, which agrees with p_e. Moreover, $r(se^{it}) = s$, so r has the form described above. $\square$

5.2. **Intersection number and the nearest point retraction.** If $\alpha : [0,1] \to \mathrm{Dome}(\Omega)$ is a curve on $\mathrm{Dome}(\Omega)$ which is transverse to the edges of $\mathrm{Dome}(\Omega)$, then we define $i(\alpha)$ to be the sum of the dihedral angles at each of its intersection points with an edge. We allow the endpoints of α to lie on edges. (In a more general setting, $i(\alpha)$ is thought of as the intersection number of α and the bending lamination on $\mathrm{Dome}(\Omega)$, see [14].) We denote by $l_h(\alpha)$ the length of α in the intrinsic metric on $\mathrm{Dome}(\Omega)$ (which is also its length in the usual hyperbolic metric on $\mathbb{H}^3$).

If $\gamma : [0,1] \to \Omega$ is a rectifiable curve in Ω, then let $l_\tau(\gamma)$ denote its length in the Thurston metric. Similarly, we can define $l_q(\gamma)$ to be its length in the quasihyperbolic metric and $l_\rho(\gamma)$ to be its length in the Poincaré metric. (Since all these metrics are complete and locally bilipschitz, a curve is rectifiable in one metric if and only if it is rectifiable in another.)

Lemma 5.2. *Suppose that Ω is the complement of finitely many points in $\hat{\mathbb{C}}$.*

(1) *If $\gamma : [0,1] \to \Omega$ then*
$$l_\tau(\gamma) \ge l_h(r \circ \gamma).$$
(2) *If $\alpha : [0,1] \to \mathrm{Dome}(\Omega)$ is a curve transverse to the edges of $\mathrm{Dome}(\Omega)$ then*
$$l_\tau(r^{-1}(\alpha)) = l_h(\alpha) + i(\alpha).$$

Proof. By Lemma 5.1, r is an isometry on the faces of $\mathrm{Dome}(\Omega)$ and Euclidean projection on the bigons. It follows that r is 1-Lipschitz. Thus, for $\gamma : [0,1] \to \Omega$,

$$l_\tau(\gamma) \ge l_h(r \circ \gamma).$$

Now let $\alpha : [0,1] \to \mathrm{Dome}(\Omega)$ be a curve transverse to the edges of $\mathrm{Dome}(\Omega)$. Let $\{x_i = \alpha(t_i)\}_{i=1}^{n-1}$ be the finite collection of intersection points of α with the edges of $\mathrm{Dome}(\Omega)$. Let $t_0 = 0$ and $t_n = 1$. (Notice that if $\alpha(0)$ lies on an edge, then $t_0 = t_1$, while $t_n = t_{n-1}$ if $\alpha(1)$ lies on an edge.)

Let α_i be the restriction of α to (t_{i-1}, t_i). Then

$$l_\tau(r^{-1}(\alpha)) = \sum_{i=1}^{n} l_\tau(r^{-1}(\alpha_i)) + \sum_{i=1}^{n-1} l_\tau(r^{-1}(x_i))$$

As r is an isometry on the interior of the faces,

$$\sum_i l_\tau(r^{-1}(\alpha_i)) = \sum_i l_h(\alpha_i) = l_h(\alpha).$$

If x_i lies on the edge e_i of $\mathrm{Dome}(\Omega)$ with exterior dihedral angle θ_i, then, by Lemma 5.1, $r^{-1}(x_i)$ has length θ_i in the Thurston metric. Therefore,

$$\sum_{i=1}^{n-1} l_\tau(r^{-1}(x_i)) = \sum_{i=1}^{n-1} \theta_i = i(\alpha).$$

Thus,

$$l_\tau(r^{-1}(\alpha)) = l_h(\alpha) + i(\alpha)$$

as claimed. □

Remark: Part (2) is closely related to Theorem 3.1 in McMullen [25] which shows that

$$l_\rho(r^{-1}(\alpha)) \leq l_h(\alpha) + i(\alpha).$$

6. Intersection number estimates

We continue to assume throughout this section that Ω is the complement of finitely many points in $\hat{\mathbb{C}}$. We obtain bounds on $i(\alpha)$ for short geodesic arcs in the thick part (Lemma 6.1) and short simple closed geodesics in $\mathrm{Dome}(\Omega)$ (Lemma 6.6). We use Lemmas 6.1 and 6.6 to bound the angle of intersection between an edge of $\mathrm{Dome}(\Omega)$ and a short simple closed geodesic (Lemma 6.7).

6.1. **Short geodesic arcs in the thick part.** We first state a mild generalization of Lemma 4.3 from [8] which gives the bound on $i(\alpha)$ for short geodesic arcs. In [8] we define a function

$$F(x) = \frac{x}{2} + \sinh^{-1}\left(\frac{\sinh\left(\frac{x}{2}\right)}{\sqrt{1-\sinh^2\left(\frac{x}{2}\right)}}\right)$$

and we let $G(x) = F^{-1}(x)$. The function F is monotonically increasing and has domain $(0, 2\sinh^{-1}(1))$. The function $G(x)$ has domain $(0,\infty)$, has asymptotic behavior $G(x) \asymp x$ as x tends to 0, and $G(x)$ approaches $2\sinh^{-1}(1)$ as x tends to ∞.

Lemma 6.1. *Suppose that Ω is the complement of finitely many points in $\hat{\mathbb{C}}$. If $\alpha : [0,1] \to \mathrm{Dome}(\Omega)$ is a geodesic path and*

$$l_h(\alpha) \leq G(\mathrm{inj}_{\mathrm{Dome}(\Omega)}(\alpha(t))),$$

for some $t \in [0,1]$, then

$$i(\alpha) \leq 2\pi.$$

In the appendix, we give a self-contained proof of Lemma 6.1. The proof uses the same basic techniques as in [8], but the arguments are much more elementary, as we need not use the general theory of bending measures.

6.2. **Cusps and collars.** In this section, we recall a version of the Collar Lemma (see Buser [10]) which gives a complete description of the portion of a hyperbolic surface with injectivity radius less than $\sinh^{-1}(1)$.

If γ is a simple closed geodesic in a hyperbolic surface S, then we define the *collar* of γ to be the set

$$N(\gamma) = \{x \mid d_S(x,\gamma) \le w(\gamma)\} \text{ where } w(\gamma) = \sinh^{-1}\left(\frac{1}{\sinh\left(\frac{l_S(\gamma)}{2}\right)}\right).$$

A *cusp* is a subsurface isometric to $S^1 \times [0,\infty)$ with the metric

$$\left(\frac{e^{-2t}}{\pi^2}\right) d\theta^2 + dt^2.$$

Collar Lemma: *Let S is a complete hyperbolic surface.*

(1) *The collar $N(\gamma)$ about a simple closed geodesic γ is isometric to $\mathbf{S}^1 \times [-w(\gamma_i), w(\gamma_i)]$ with the metric*
$$\left(\frac{l_S(\gamma_i)}{2\pi}\right)^2 \cosh^2 t \; d\theta^2 + dt^2.$$
(2) *A collar about a simple closed geodesic is disjoint from the collar about any disjoint simple closed geodesic and from any cusp of S*
(3) *Any two cusps in S are disjoint.*
(4) *If $\mathrm{inj}_S(x) \le \sinh^{-1}(1)$, then x lies in a cusp or in a collar about a simple closed geodesic of length at most $2\sinh^{-1}(1)$.*
(5) *If $x \in N(\gamma)$, then*
$$\sinh(\mathrm{inj}_S(x)) = \sinh(l_S(\gamma)/2)\cosh(d_S(x,\gamma_i)).$$

If C is a cusp or a collar about a simple closed geodesic, then we will call a curve of the form $S^1 \times \{t\}$, in the coordinates given above, a *cross-section*. The following observations concerning the intersections of cross-sections with edges will be useful in the remainder of the section.

Lemma 6.2. *Suppose that Ω is the complement of a finite collection X of points in $\hat{\mathbb{C}}$ and C is a cusp of* $\mathrm{Dome}(\Omega)$. *Then every edge which intersects C intersects every cross-section of C and does so orthogonally. Moreover, if γ is any cross-section of C, then $i(C) = 2\pi$.*

Proof. Identify the universal cover $\mathrm{Dome}(\tilde{\Omega})$ of $\mathrm{Dome}(\Omega)$ with $\mathbb{H}^2$ so that the pre-image of C is the horodisk $H = \{z \mid Im(z) \ge 1\}$, which is invariant under the covering transformation $z \to z+2$. If $\tilde{e}$ is the pre-image of an edge, then $\tilde{e}$ either terminates at ∞, or has two finite endpoints. If $\tilde{e}$ terminates at infinity then it crosses every cross-section of H orthogonally. Otherwise, since the edge e is simple, the endpoints of $\tilde{e}$ must have Euclidean distance less than 2 from one another and therefore $\tilde{e}$ does not intersect H (see also Lemma 1.24 in [9]).

Every cusp is associated to a point $x \in X$, which we may regard as an ideal vertex of $CH(X)$. If we move x to ∞, then each cross-section is a Euclidean polygon in a horosphere, so has total external angle exactly 2π. □

Lemma 6.3. *Suppose that Ω is the complement of finitely many points in $\hat{\mathbb{C}}$ and $N(\gamma)$ is collar about a simple closed geodesic γ on* $\mathrm{Dome}(\Omega)$. *If e is an edge of* $\mathrm{Dome}(\Omega)$, *then each component of $e \cap N(\gamma)$ intersects every cross-section of $N(\gamma)$ exactly once.*

Proof. Again identify the universal cover of Dome(Ω) with $\mathbb{H}^2$. Let $\tilde{\gamma}$ be a component of the pre-image of γ and let $N(\tilde{\gamma})$ be the metric neighborhood of $\tilde{\gamma}$ of radius $w(\gamma)$. Then $N(\tilde{\gamma})$ is a component of the pre-image of $N(\gamma)$. Notice that $\tilde{\gamma}$ is the axis of a hyperbolic isometry g in the conjugacy class determined by the simple closed curve γ.

Notice that every edge e of Dome(Ω) is an infinite simple geodesic which does not accumulate on γ. Let $\tilde{e}$ be the pre-image of an edge which intersects $N(\tilde{\gamma})$, then neither of its endpoints can be an endpoint of γ. If the endpoints of $\tilde{e}$ lie on the same side of $\tilde{\gamma}$, one may use the fact that they must lie in a single fundamental domain for g and the explicit description of $N(\tilde{\gamma})$ to show that $\tilde{e}$ cannot intersect $N(\tilde{\gamma})$ (see also Lemma 1.21 in [9]). If its endpoints lie on opposite sides of $\tilde{\gamma}$, then it intersects every cross-section of $N(\tilde{\gamma})$ exactly once, as desired. $\square$

We now observe that if γ is a homotopically non-trivial curve on Dome(Ω), then $i(\gamma) \geq 2\pi$.

Lemma 6.4. *If Ω is the complement of finitely many points in $\hat{\mathbb{C}}$ and $\gamma : S^1 \to \mathrm{Dome}(\Omega)$ is homotopically non-trivial in* Dome(Ω)*, then*

$$i(\gamma) \geq 2\pi.$$

Proof. If γ is a closed geodesic, then, by the Gauss-Bonnet theorem, its geodesic curvature is at least 2π. But $i(\gamma)$ is the sum of the dihedral angles of the edges γ intersects, and is therefore at least as large as the geodesic curvature of γ, so $i(\gamma) \geq 2\pi$ in this case.

If γ is any homotopically non-trivial curve on Dome(Ω), then it is homotopic to either a geodesic γ' or a (multiple of a) cross-section C of a cusp and either $i(\gamma) \geq i(\gamma')$ or $i(\gamma) \geq i(C)$. In either case, $i(\gamma) \geq 2\pi$. (Recall that Lemma 6.2 implies that $i(C) = 2\pi$.) $\square$

As an immediate corollary of Lemmas 6.4, 5.1 and 4.1 we see that every essential curve in a hyperbolic domain has length greater than 2π in the Thurston metric.

Corollary 6.5. *If $\Omega \subset \hat{\mathbb{C}}$ is a hyperbolic domain and $\alpha : S^1 \to \mathrm{Dome}(\Omega)$ is homotopically non-trivial in Ω, then*

$$l_\tau(\alpha) > 2\pi.$$

6.3. **Intersection bounds for short simple closed geodesics in** Dome(Ω)**.** We may use a variation on the construction from Lemma 6.1 to uniformly bound $i(\gamma)$ for all short geodesics.

Lemma 6.6. *Suppose that Ω is the complement of finitely many points in $\hat{\mathbb{C}}$. If γ is a simple closed geodesic of length less than $2\sinh^{-1}(1)$, then*

$$2\pi \leq i(\gamma) \leq 2\pi + 2\tan^{-1}(\sinh(l_h(\gamma)/2)) \leq \frac{5\pi}{2}$$

The lower bound $i(\gamma) \geq 2\pi$ follows immediately from Lemma 6.4. For the moment, we will give a much briefer proof that $i(\gamma) \leq 6\pi$ and defer the proof of the better upper bound to the appendix.

Proof that $i(\gamma) \leq 6\pi$: Let x be a point in $N(\gamma)$ with injectivity radius $\sinh^{-1}(1)$ and let C be the cross-section of $N(\gamma)$ which contains x. Part (5) of the Collar Lemma implies that

$$\sinh(l_h(\gamma)/2)\cosh d_h(x, \gamma) = 1,$$

and part (1) then implies that

$$l_h(C) = \frac{l_h(\gamma)}{\sinh(l_h(\gamma)/2)} \leq 2.$$

We may therefore divide γ' into $\left\lceil \frac{l_h(C)}{G(\sinh^{-1}(1))} \right\rceil = 3$ segments of length at most $G(\sinh^{-1}(1)) \approx .83862$, denoted C_1, C_2 and C_3. Each C_i is homotopic (relative to its endpoints) to a geodesic arc α_i and $i(C_i) = i(\alpha_i)$ for all i, since no edge can intersect either α_i or C_i twice. Lemma 6.1 implies that $i(\alpha_i) \leq 2\pi$. Therefore,

$$i(C) = i(C_1) + i(C_2) + i(C_3) = i(\alpha_1) + i(\alpha_2) + i(\alpha_3) \leq 3(2\pi) = 6\pi.$$

Lemma 6.3 implies that $i(\gamma) = i(C)$ completing the proof. □

6.4. **Angle bounds for edges and short closed geodesics.** The final tool we will need in the proof of Theorem 1.1 is a bound on the angle between an edge of Dome(Ω) and a short simple closed geodesic on Dome(Ω).

Lemma 6.7. *Suppose that Ω is the complement of finitely many points in $\hat{\mathbb{C}}$. If γ is a simple closed geodesic of length $l_h(\gamma) < 2\sinh^{-1}(1)$, then there is an edge e_m intersecting γ at an angle*

$$\phi_m \geq \sin^{-1}\left(\frac{4}{5}\right) \approx .9272.$$

Furthermore, any other edge e_i intersecting γ intersects in an angle

$$\phi_i \geq \Phi = \sin^{-1}\left(\frac{4}{5\sqrt{2}+3}\right) \approx 0.4084.$$

Proof. Let γ intersect edges $\{e_i\}_{i=1}^m$ at points $\{x_i\}_{i=1}^m$ and with angles $\{\phi_i\}_{i=1}^m$. We may assume that the largest value of ϕ_i is achieved at ϕ_m. Furthermore, let θ_i^b be the exterior dihedral angle of the faces which meet at e_i and let θ_i^c be the exterior angle of γ at x_i.

A simple (Euclidean) calculation yields

$$\cos(\theta_i^c) = \sin^2(\phi_i)\cos(\theta_i^b) + \cos^2(\phi_i).$$

Consider $f_y : [0, \frac{\pi}{2}] \to [0, \pi]$, given by

$$f_y(x) = \cos^{-1}(\sin^2(y)\cos(x) + \cos^2(y)).$$

Then $\theta_i^c = f_{\phi_i}(\theta_i^b)$. One may compute that

$$\frac{df_y}{dx} = \frac{\sin^2(y)\sin(x)}{\sin(f_y(x))} > 0 \qquad \text{and} \qquad \frac{d^2 f_y}{dx^2} = -\frac{\sin^2(y)\cos^2(y)(1-\cos(x))^2}{\sin^3(f_y(x))} < 0$$

Since f_y is an increasing concave down function, $f_y(0) = 0$ and

$$\lim_{x \to 0^+} f_y'(x) = \sin(y)$$

we see that

$$\theta_i^c \leq \sin(\phi_i)\ \theta_i^b.$$

Let $c(\gamma)$ be the geodesic curvature of γ. Then

$$c(\gamma) = \sum \theta_i^c \leq \sin(\phi_m) \sum \theta_i^b = \sin(\phi_m)\ i(\gamma).$$

Lemma 6.6, and the fact that $2\pi \leq c(\gamma) \leq i(\gamma)$, imply that

$$2\pi \leq c(\gamma) \leq \sin(\phi_m)(2\pi + 2\tan^{-1}(\sinh(l_h(\gamma)/2)) \leq \frac{5\pi}{2}\sin(\phi_m).$$

Therefore,

$$\phi_m \geq \sin^{-1}\left(\frac{\pi}{\pi + \tan^{-1}(\sinh(l_h(\gamma)/2))}\right) \geq \sin^{-1}\left(\frac{4}{5}\right) \approx .9272.$$

We now use the bound on ϕ_m to bound the other angles of intersection. Let e_i be an edge, where $i < m$, intersecting γ. Identify the universal cover of Dome(Ω) with $\mathbb{H}^2$ and let g be a geodesic in $\mathbb{H}^2$ covering γ. There is a lift h_m of e_m which is a geodesic intersecting g at point y_1 with angle ϕ_m. Translating along g we have another lift h'_m of e_m intersecting g at point y_2 a distance $l_h(\gamma)$ from y_1. Let h_i be a lift of e_i whose intersection z with g, lies between y_1 and y_2. Without loss of generality, we may assume that $d(z, y_1) \leq l_h(\gamma)/2$. As edges do not intersect, h_i lies between h_m and h'_m. Let p and q be the endpoints of h_m and let T_p (respectively T_q) be the triangle with vertices y_1, z, and p (respectively y_1, z and q). Let ψ_p and ψ_q be the internal angles of T_p and T_q at z. We may assume that the internal angle of T_p at y_1 is $\pi - \phi_m$ and hence that the internal angle of T_q at y_1 is ϕ_m. By construction,

$$\phi_i \geq \min\{\psi_p, \psi_q\} = \psi_p.$$

Basic calculations in hyperbolic trigonometry (see [1, Theorem 7.10.1]), give that

$$\cosh(d(z, y_1)) = \frac{1 - \cos\psi_p \cos\phi_m}{\sin\psi_p \sin\phi_m}$$

and

$$\sinh(d(z, y_1)) = \frac{\cos\psi_p - \cos\phi_m}{\sin\psi_p \sin\phi_m}.$$

One may then check that

$$\cosh(d(z, y_1)) + \cos\phi_m \sinh(d(z, y_1)) = \frac{\sin\phi_m}{\sin\psi_p}.$$

Therefore,

$$\psi_p = \sin^{-1}\left(\frac{\sin(\phi_m)}{\cosh(d(z, y_1)) + \cos(\phi_m)\sinh(d(z, y_1))}\right).$$

Since, $d(z, y_1) \leq l_h(\gamma)/2 \leq \sinh^{-1}(1)$ and $\sin(\phi_m) \geq 4/5$,

$$\phi_i \geq \psi_p \geq \Phi = \sin^{-1}\left(\frac{\frac{4}{5}}{\sqrt{2} + \frac{3}{5}}\right) \approx .4084.$$

□

Remark: Using the elementary estimate $i(\gamma) \leq 6\pi$ in Lemma 6.7 we obtain an angle bound of $\Phi \geq .141$. Using this we can prove a weaker form of Theorem 1.1 where K_0 is replaced by $\hat{K}_0 \approx 14.09$ and K is replaced with $\hat{K} \approx 16.81$.

7. The proof of Theorem 1.1

We are now prepared to prove Theorem 1.1 which asserts that the nearest point retraction is a uniform quasi-isometry in the Thurston metric. Our approximation result, Lemma 4.1, allows us to reduce to the finitely punctured case. We begin by producing an upper bound on the distance between points in Ω whose images on Dome(Ω) are close.

Proposition 7.1. *Suppose that Ω is the complement of finitely many points in $\hat{\mathbb{C}}$. If $z, w \in \Omega$ and $d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) \leq G(\sinh^{-1}(1))$ then*

$$d_\tau(z, w) \leq K_0 = G(\sinh^{-1}(1)) + 2\pi \approx 7.1219.$$

Proof. Let $\alpha : [0,1] \to \text{Dome}(\Omega)$ be a shortest path joining $r(z)$ to $r(w)$ and let $G = G(\sinh^{-1}(1)) \approx .838682$. By assumption,

$$l_h(\alpha) = d_{\text{Dome}(\Omega)}(r(z), r(w)) \le G.$$

If α contains a point of injectivity radius greater than $\sinh^{-1}(1)$, then Lemma 6.1 implies that $i(\alpha) \le 2\pi$. Lemma 5.2 then gives that

$$d_\tau(z,w) \le l_\tau(r^{-1}(\alpha)) = l_h(\alpha) + i(\alpha) \le G + 2\pi = K_0 \approx 7.1219$$

If α does not contain a point of injectivity radius greater than $\sinh^{-1}(1)$, then, by the Collar Lemma, α is either entirely contained in a cusp or in the collar $N(\gamma)$ about a simple closed geodesic γ with $l_h(\gamma) < 2\sinh^{-1}(1)$.

Let $\alpha(1)$ lie on cross-section C of a cusp or collar. Every cross-section of a cusp has length at most 2, and we may argue, exactly as in Section 6.3, that if C is a cross-section of a collar, then

$$l_h(C) = \sinh(\text{inj}_{\text{Dome}(\Omega)}(\alpha(1)))\left(\frac{l_h(\gamma)}{\sinh(l_h(\gamma)/2)}\right) \le 2.$$

In either case, $l_h(C) \le 2$. If C is a cross-section of a cusp, then $i(C) = 2\pi$, by Lemma 6.2, while if C is a cross-section of a collar, then, by Lemmas 6.3 and 6.6, $i(C) = i(\gamma) \le 5\pi/2$. Therefore, Lemma 5.2 implies that

$$l_\tau(r^{-1}(C)) = l_h(C) + i(C) \le 2 + \frac{5\pi}{2} \approx 9.8540.$$

If α is entirely contained in a cusp, then we may join $\alpha(0)$ to a point p on the cross-section C by an arc B orthogonal to each cross-section such that $l_h(B) \le l_h(\alpha) \le G$. Also, since each edge is orthogonal to every cross-section, B either misses every edge of $\text{Dome}(\Omega)$ or is a subarc of an edge. If B misses every edge, then $i(B) = 0$ and $B' = r^{-1}(B)$ is an arc joining z to a point x on $r^{-1}(C)$ of length $l_\tau(B') = l_h(B)$. If B is a subarc of an edge e, then $r^{-1}(B)$ contains a vertical (in the Euclidean coordinates described in Lemma 5.1) arc B' which joins z to a point p on $r^{-1}(C)$ such that $l_\tau(B') = l_h(B)$. In both cases, since $l_\tau(r^{-1}(C)) \le 2 + \frac{5\pi}{2}$, p can be joined to $w \in r^{-1}(C)$ by an arc of length at most $1 + \frac{5\pi}{4}$ in the Thurston metric. Therefore,

$$d_\tau(z,w) \le d_\tau(z,p) + d_\tau(p,w) \le l_\tau(B') + 1 + \frac{5\pi}{4} \le G + 1 + \frac{5\pi}{4} \approx 5.7657.$$

Finally, suppose that α is contained in the collar $N(\gamma)$ about a simple closed geodesic γ with $l_h(\gamma) < 2\sinh^{-1}(1)$. We may assume that $\alpha(0)$ lies on cross-section $S^1 \times \{s\}$, $\alpha(1)$ lies on the cross-section C given by $S^1 \times \{t\}$, $s > 0$, $s > |t|$ and $|s - t| \le G$. As every edge that intersects the collar $N(\gamma)$ must intersect the core geodesic γ (see Lemma 6.3), we can decompose $N(\gamma)$ along edges to obtain a collection of quadrilaterals with two opposite sides corresponding to edges. As the edges intersect γ in an angle at least Φ, we can foliate each quadrilateral by geodesics which also intersect γ in an angle greater than Φ. Thus, we obtain a foliation of $N(\gamma)$ by such geodesics. We let B be the subarc of a leaf in the foliation joining $\alpha(0)$ to a point p on the cross-section C. Then, either B does not intersect any edges of $\text{Dome}(\Omega)$ or is a subset of an edge. In either case, B belongs to geodesic g which intersects γ at an angle $\phi \ge \Phi$.

We now use the angle bound to obtain an upper bound on the length of B, which will allow us to complete the argument much as in the cusp case. The hyperbolic law of sines implies that

$$l_h(B) = \sinh^{-1}\left(\frac{\sinh(s)}{\sin(\phi)}\right) - \sinh^{-1}\left(\frac{\sinh(t)}{\sin(\phi)}\right)$$

If we let $f(s) = \sinh^{-1}\left(\frac{\sinh(s)}{\sin(\phi)}\right)$, then $l_h(B) = f(s) - f(t)$. Since f is odd and increasing and $|s-t| \le G$,

$$l_h(B) \le \sup\left\{ f(s) - f(s-G) \mid s \ge \frac{G}{2} \right\}.$$

Since $f(s) - f(s-G)$ is decreasing on $[\frac{G}{2}, \infty)$ (since $f''(t) \le 0$ for all t) it achieves its maximum value when $s = \frac{G}{2}$, which implies that

$$l_h(B) \le 2\sinh^{-1}\left(\frac{\sinh(G/2)}{\sin(\Phi)}\right) \approx 1.8831.$$

Since B is either disjoint from all edges of Dome(Ω) or is a subset of an edge, we argue as before to show that $r^{-1}(B)$ contains an arc B' joining z to a point p in $r^{-1}(C)$ such that

$$l_\tau(B') = l_h(B) \le 2\sinh^{-1}\left(\frac{\sinh(G/2)}{\sin(\Phi)}\right) \approx 1.8831.$$

Therefore, as before,

$$\begin{aligned} d_\tau(z,w) &\le d_\tau(z,p) + d_\tau(p,w) \\ &\le l_\tau(B') + 1 + \frac{5\pi}{4} \\ &\le 2\sinh^{-1}\left(\frac{\sinh(G/2)}{\sin(\Phi)}\right) + 1 + \frac{5\pi}{4} \approx 6.8101 \end{aligned}$$

Therefore, we see that in all cases

$$d_\tau(z,w) \le K_0 \approx 7.1219.$$

□

As a nearly immediate corollary, we show that the nearest point retraction is a uniform quasi-isometry in the finitely punctured case.

Corollary 7.2. *Suppose that Ω is the complement of finitely many points in $\hat{\mathbb{C}}$. Then,*

$$d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) \le d_\tau(z,w) \le K\ d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) + K_0$$

where $K_0 \approx 7.12$ and $K = \frac{K_0}{G(\sinh^{-1}(1))} \approx 8.49$.

Proof. The lower bound follows from the fact that r is 1-Lipschitz in the Thurston metric (see Lemma 5.2). To prove the upper bound we consider a shortest path α in Dome(Ω) from $r(z)$ to $r(w)$. We decompose α into $\left\lceil \frac{d_{\mathrm{Dome}(\Omega)}(r(z),r(w))}{G(\sinh^{-1}(1))} \right\rceil$ geodesic segments of length at most $G(\sinh^{-1}(1))$. Then, applying Lemma 7.1,

$$d_\tau(z,w) \le K_0 \left\lceil \frac{d_{\mathrm{Dome}(\Omega)}(r(z), r(w))}{G(\sinh^{-1}(1))} \right\rceil \le K d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) + K_0$$

where $K = \frac{K_0}{G(\sinh^{-1}(1))}$. □

We may now combine Corollary 7.2 and Lemma 4.1 to prove Theorem 1.1 in the general case.

Theorem 1.1. *Let Ω be a hyperbolic domain. Then the nearest point retraction $r : \Omega \to \mathrm{Dome}(\Omega)$ is a (K, K_0)-quasi-isometry with respect to the Thurston metric τ on Ω and the intrinsic hyperbolic metric on* $\mathrm{Dome}(\Omega)$ *where $K \approx 8.49$ and $K_0 \approx 7.12$. In particular,*

$$d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) \leq d_\tau(z, w) \leq K\ d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) + K_0.$$

Furthermore, if Ω is simply connected, then

$$d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) \leq d_\tau(z, w) \leq K'\ d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) + K'_0.$$

where $K' \approx 4.56$ and $K'_0 \approx 8.05$.

Proof. Let $\{X_n\}$ be a nested finite approximation for $\hat{\mathbb{C}} - \Omega$, let $\Omega_n = \hat{\mathbb{C}} - X_n$ with projective metric τ_n and let $r_n : \Omega_n \to \mathrm{Dome}(\Omega_n)$ be the associated nearest point retraction. Then, by Lemma 4.1, $\{d_{\mathrm{Dome}(\Omega_n)}(r_n(z), r_n(w))\}$ converges to $d_{\mathrm{Dome}(\Omega)}(r(z), r(w))$ and $\{d_{\tau_n}(z, w)\}$ converges to $d_\tau(z, w)$ for all $z, w \in \Omega$. Corollary 7.2 implies that

$$d_{\mathrm{Dome}(\Omega_n)}(r_n(z), r_n(w)) \leq d_{\tau_n}(z, w) \leq K\ d_{\mathrm{Dome}(\Omega_n)}(r_n(z), r_n(w)) + K_0$$

for all n and all $z, w \in \Omega$, so the result follows.

Now suppose that is Ω simply connected and let k_n be the minimum injectivity radius in X_n along the shortest curve α_n joining $r_n(z)$ and $r_n(w)$. Lemma 4.1 implies that $k_n \to \infty$ as $n \to \infty$. As in the proof of Corollary 7.2, we decompose α_n into segments of length $G(k_n)$ and conclude that

$$d_{\tau_n}(z, w) \leq \left(\frac{2\pi + G(k_n)}{G(k_n)}\right) d_{\mathrm{Dome}(\Omega_n)}(r_n(z), r_n(w)) + (2\pi + G(k_n))$$

As $\lim_{n\to\infty} G(k_n) = 2\sinh^{-1}(1)$, we again apply Lemma 4.1 to obtain

$$d_\tau(z, w) \leq K' d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) + K'_0$$

where $K' = \frac{2\pi + 2\sinh^{-1}(1)}{2\sinh^{-1}(1)} \approx 4.56$ and $K'_0 = 2\pi + 2\sinh^{-1}(1) \approx 8.05$. □

8. Consequences of Theorem 1.1

In this section, we derive a series of corollaries of Theorem 1.1. We begin by combining Theorem 1.1 with Beardon and Pommerenke's work [2] and Theorem 3.3 to obtain explicit quasi-isometry bounds for the nearest point retraction with respect to the Poincaré metric on a uniformly perfect domain.

Corollary 1.2. *Let Ω be a uniformly perfect domain in $\hat{\mathbb{C}}$ and let $\nu > 0$ be a lower bound for its injectivity radius in the Poincaré metric, then*

$$\frac{1}{2\sqrt{2}(k + \frac{\pi^2}{2\nu})}\ d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) \leq d_\rho(z, w) \leq K\ d_{\mathrm{Dome}(\Omega)}(r(z), r(w)) + K_0$$

for all $z, w \in \Omega$ where $k = 4 + \log(2 + \sqrt{2})$.

Proof. Combining Corollary 3.2, Theorem 3.3 and the fact that $\rho(z) \leq \tau(z)$, we see that

$$\frac{1}{2\sqrt{2}(k + \frac{\pi^2}{2\nu})} \tau(z) \leq \rho(z) \leq \tau(z)$$

for all $z \in \Omega$. It follows that

$$\frac{1}{2\sqrt{2}(k + \frac{\pi^2}{2\nu})}\ d_\tau(z, w) \leq d_\rho(z, w) \leq d_\tau(z, w)$$

for all $z, w \in \Omega$. The result then follows by applying Theorem 1.1. □

If Ω is simply connected, then we may apply Theorem 3.4 to obtain better quasi-isometry constants. Bishop (see Lemma 8 in [3]) first showed that the nearest point retraction is a quasi-isometry in the simply connected case, but did not obtain explicit constants.

Corollary 1.3. *If Ω is a simply connected domain in $\hat{\mathbb{C}}$, then*

$$\frac{1}{2}\ d_{\mathrm{Dome}(\Omega)}(r(z),r(w)) \le d_\rho(z,w) \le K'\ d_{\mathrm{Dome}(\Omega)}(r(z),r(w)) + K'_0$$

for all $z,w\in\Omega$.

We now establish Marden and Markovic's conjecture that the nearest point retraction is Lipschitz (with respect to the Poincaré metric on Ω) if and only if Ω is uniformly perfect.

Corollary 1.4. *If Ω is a hyperbolic domain, then the nearest point retraction is Lipschitz (where we give Ω the Poincaré metric) if and only if Ω is uniformly perfect. Moreover, if Ω is not uniformly perfect, then r is not a quasi-isometry with respect to the Poincaré metric.*

Proof. If Ω is uniformly perfect, then Corollary 1.2 implies that r is Lipschitz.

If Ω is not uniformly perfect, then there exist essential round annuli in Ω with arbitrarily large moduli. For each n, let A_n be a separating round annulus in Ω of modulus $\frac{4n}{2\pi}$. We may normalize so that $\infty\notin\Omega$ and $A_n=\{z \mid e^{-2n}\le|z|\le e^{2n}\}$ and choose $x_n=e^{-n}$ and $y_n=e^n$. If z lies in the subannulus $B_n=\{z \mid e^{-n}\le|z|\le e^n\}$, then

$$|z|-e^{-2n}\le\delta(z)\le|z|+e^{-2n},$$

so

$$\frac{1}{2}|z|\le(1-e^{-n})|z|\le\delta(z)\le(1+e^{-n})|z|\le 2|z|.$$

As every path from x_n to y_n contains an arc joining the boundary components of B_n, this implies that

$$n\le d_q(x_n,y_n)\le 4n.$$

Therefore, $d_q(x_n,y_n)\to\infty$, so, by Theorem 3.3, $d_\tau(x_n,y_n)\to\infty$. Theorem 1.1 then implies that

$$d_h(r(x_n),r(y_n))\to\infty.$$

On the other hand, if γ is the segment of the real line joining x_n to y_n, then $\beta(z)>n$ for all $z\in\gamma$, so by Theorem 3.1,

$$l_\rho(\gamma)\le\left(\frac{2k+\frac{\pi}{2}}{k+n}\right)l_q(\gamma)\le\left(\frac{2k+\frac{\pi}{2}}{k+n}\right)4n\le C$$

for all n and some constant $C>0$. Therefore,

$$d_\rho(x_n,y_n)\le C$$

for all n. It follows that r is not Lipschitz with respect to the Poincaré metric on Ω. Moreover, r is not even a quasi-isometry. □

9. The lift of the nearest point retraction

In this section we show that if Ω is uniformly perfect, then the nearest point retraction lifts to a quasi-isometry, with quasi-isometry constants depending only on the geometry of the domain. The proof requires only Lemma 6.1 and does not make use of the analysis of the thin part obtained in Lemmas 6.6 and 6.7. We then use this result to show that r is homotopic to a quasiconformal homeomorphism if Ω is uniformly perfect.

Theorem 1.5. *Suppose that Ω is a uniformly perfect hyperbolic domain and $\nu > 0$ is a lower bound for its injectivity radius in the Poincaré metric. Then the nearest point retraction lifts to a quasi-isometry*

$$\tilde{r} : \tilde{\Omega} \to \mathrm{Dome}(\tilde{\Omega})$$

between the universal cover of Ω (with the Poincaré metric) and the universal cover of $\mathrm{Dome}(\Omega)$ *where the quasi-isometry constants depend only on ν. In particular,*

$$\frac{1}{2\sqrt{2}(k+\frac{\pi^2}{2\nu})}\, d_{\mathrm{Dome}(\tilde{\Omega})}(\tilde{r}(z),\tilde{r}(w)) \le d_{\tilde{\Omega}}(z,w) \le L(\nu) d_{\mathrm{Dome}(\tilde{\Omega})}(\tilde{r}(z),\tilde{r}(w)) + L_0(\nu)$$

for all $z, w \in \tilde{\Omega}$, where $m = \cosh^{-1}(e^2) \approx 2.69$,

$$g(\nu) = \frac{1}{2} e^{-m} e^{\frac{-\pi^2}{2\nu}}, \qquad L(\nu) = \frac{(2\pi + G(g(\nu))}{G(g(\nu))}, \text{ and}$$

$$L_0(\nu) = (2\pi + G(g(\nu))) \le L_0 = 2\pi + 2\sinh^{-1}(1) \approx 8.05.$$

Proof. Corollary 1.2 implies that r is $(2\sqrt{2}(k+\frac{\pi^2}{2\nu}))$-Lipschitz, so therefore $\tilde{r}$ is also $(2\sqrt{2}(k+\frac{\pi^2}{2\nu}))$-Lipschitz.

Given $z, w \in \tilde{\Omega}$, let $\tilde{\beta} : [0,1] \to \tilde{\Omega}$ be the geodesic arc in $\tilde{\Omega}$ joining them and let $\beta : [0,1] \to \Omega$ be its projection to Ω (i.e. $\beta = \pi_\Omega \circ \tilde{\beta}$ where $\pi_\Omega : \tilde{\Omega} \to \Omega$ is the universal covering map.) Then $d_{\mathrm{Dome}(\tilde{\Omega})}(\tilde{r}(z),\tilde{r}(w))$ is the length of the geodesic arc α on $\mathrm{Dome}(\Omega)$ joining $r(\beta(0))$ and $r(\beta(0))$ in the proper homotopy class of $r \circ \beta$.

Let $\{X_n\}$ be a nested finite approximation to $\hat{\mathbb{C}} - \Omega$, let $\Omega_n = \hat{\mathbb{C}} - X_n$, let τ_n be the Thurston metric on Ω_n, and let $r_n : \Omega_n \to \mathrm{Dome}(\Omega_n)$ be the nearest point retraction. Let α_n be the geodesic arc in $\mathrm{Dome}(\Omega_n)$ joining $r_n(\beta(0))$ and $r_n(\beta(1))$ in the proper homotopy class of $r_n \circ \beta$. Let k_n be the minimum of the injectivity radius of $\mathrm{Dome}(\Omega_n)$ at points in α_n. If $a_n \le k_n$, then we may divide α_n into segments of length $G(a_n)$ and apply Lemmas 5.2 and 6.1, as in the proof of Corollary 7.2, to obtain the bound,

$$\begin{aligned} l_\rho(r_n^{-1}(\alpha_n)) &\le (2\pi + G(a_n))) \left\lceil \frac{l_h(\alpha_n)}{G(\nu_n)} \right\rceil \\ &\le \frac{(2\pi + G(a_n)}{G(a_n)} l_h(\alpha_n) + (2\pi + G(a_n)). \end{aligned}$$

Since $\{\mathrm{Dome}(\Omega_n)\}$ converges to $\mathrm{Dome}(\Omega)$, $\{r_n\}$ converges to r, $l_h(\alpha_n) \le l_{\tau_n}(\beta)$ for all n, and $\{\tau_n\}$ converges to τ (see Lemma 4.1), $\{\alpha_n\}$ converges, up to subsequence, to a geodesic path α_∞ on $\mathrm{Dome}(\Omega)$ joining $r(\beta(0))$ to $r(\beta(1))$. Since $r_n(\alpha_\infty)$ lies arbitrarily close to α_n (for large enough n in the subsequence) which is properly homotopic to $r_n \circ \beta$, we see that α_∞ is homotopic to $r \circ \beta$, and hence agrees with α.

Lemma 9.1 in [8] gives that if $\nu > 0$ is a lower bound for the injectivity radius of Ω, in its Poincaré metric, then $g(\nu)$ is a lower bound for the injectivity radius of $\mathrm{Dome}(\Omega)$. Since $\{\alpha_n\}$ converges to α, up to subsequence, $\lim \mathrm{inj}_{\mathrm{Dome}(\Omega_n)}(z) = \mathrm{inj}_{\mathrm{Dome}(\Omega)}(z)$ for all $z \in \Omega$ (see Lemma 4.1 again), and the injectivity radius function is 1-Lipschitz on any hyperbolic surface, we see that $\liminf k_n \geq g(\nu)$. So, we may choose a_n so that $\lim a_n = g(\nu)$, Therefore, by taking limits, we see that

$$l_\rho(r^{-1}(\alpha)) \leq \frac{(2\pi + G(g(\nu))}{G(g(\nu))} l_h(\alpha) + (2\pi + G(g(\nu))).$$

The curve $r^{-1}(\alpha)$ contains an arc joining $r(\beta(0))$ to $r(\beta(1))$ in the homotopy class of β, so

$$l_\rho(r^{-1}(\alpha)) \geq l_\rho(\beta) = d_{\tilde{\Omega}}(z, w).$$

Therefore,

$$d_{\tilde{\Omega}}(z, w) \leq \frac{(2\pi + G(g(\nu))}{G(g(\nu))} d_{\mathrm{Dome}(\tilde{\Omega})}(\tilde{r}(z), \tilde{r}(w)) + (2\pi + G(g(\nu))).$$

□

Remark: The proof of Corollary 1.4 also implies that $\tilde{r}$ is not a quasi-isometry with respect to the Poincaré metric if Ω is not uniformly perfect. Since any homotopically non-trivial curve in Ω has length greater than 2π in the Thurston metric (by Lemma 6.5), $\tilde{r}$ is not even a quasi-isometry with respect to the (lift of the) Thurston metric on $\tilde{\Omega}$ if Ω is not uniformly perfect.

We now use the Douady-Earle extension theorem to obtain a quasiconformal map homotopic to the nearest point retraction whenever Ω is uniformly perfect. The constants obtained in this proof are explicit, but clearly far from optimal.

Corollary 1.6. *If Ω is uniformly perfect and $\nu > 0$ is a lower bound for its injectivity radius in the Poincaré metric, then there is a conformally natural $M(\nu)$-quasiconformal map $\phi : \Omega \to \mathrm{Dome}(\Omega)$ which admits a bounded homotopy to r, where*

$$M(\nu) = 4(10)^8 e^{70N(\max\{2\sqrt{2}(k+\frac{\pi^2}{2\nu}), L(\nu)\}, L_0)}$$

and $N(K, C) = e^{1546K^4 \max\{C,1\}}$. Moreover, if Ω is not uniformly perfect, then there does not exist a bounded homotopy of r to a quasiconformal map.

Proof. A close examination of the standard proof that quasi-isometries of $\mathbb{H}^2$ extend to quasisymmetries of $S^1 = \partial_\infty \mathbb{H}^2$, see, e.g., Lemma 3.43 of [21] and Lemma 5.9.4 of [29], yields that a (K, C)-quasi-isometry of $\mathbb{H}^2$ extends to a $N(K, C)$-quasisymmetry of S^1. The Beurling-Ahlfors extension of a k-quasisymmetry of S^1 is a $2k$-quasiconformal map (see [23]). Proposition 7 of Douady-Earle [13] then implies that every k-quasisymmetry admits a conformally natural quasiconformal extension with dilatation at most $4(10)^8 e^{70k}$.

First suppose that Ω is uniformly perfect and identify both $\tilde{\Omega}$ and $\mathrm{Dome}(\tilde{\Omega})$ with $\mathbb{H}^2$. Then $\tilde{r}$ lifts to a $(\max\{J(\nu), L(\nu)\}, L_0)$-quasi-isometry of $\tilde{\Omega}$ and so extends to a $N(\max\{J(\nu), L(\nu)\}, L_0)$-quasisymmetry of S^1, which itself extends to a conformally natural $M(\nu)$-quasiconformal map $\tilde{g} : \tilde{\Omega} \to \mathrm{Dome}(\tilde{\Omega})$. It follows from Proposition 4.3.1 and Theorem 4.3.2 in Fletcher-Markovic [18] that the straight-line homotopy between $\tilde{r}$ and $\tilde{g}$ is bounded. Since $\tilde{r}$ is the lift of a conformally natural map and the Douady-Earle extension is conformally natural, we see that $\tilde{g}$ descends to a conformally natural $M(\nu)$-quasiconformal map $g : \Omega \to \mathrm{Dome}(\Omega)$ and that the straight-line homotopy between $\tilde{r}$ and $\tilde{g}$ descends to a bounded homotopy.

A quasiconformal homeomorphism between hyperbolic surfaces is a quasi-isometry (see [18, Theorem 4.3.2]). Thus, if r admits a bounded homotopy to a quasiconformal map, it must itself be a quasi-isometry. Therefore, by Corollary 1.4, if r admits a bounded homotopy to a quasiconformal map, Ω must be uniformly perfect.

□

10. Round annuli

In this section, we will consider the special case of round annuli. A hyperbolic domain contains points with small injectivity radius (in the Poincaré metric) if and only if it contains round annuli with large modulus, so round annuli are natural test cases for the constants we obtain. Moreover, hyperbolic domains which are not uniformly perfect contain round annuli of arbitrarily large modulus.

Let $\Omega(s)$ denote the round annulus lying between concentric circles of radius 1 and $e^s > 1$ about the origin. One may calculate that in the Poincaré metric $\Omega(s)$ is a complete hyperbolic annulus with core curve of length $\frac{2\pi^2}{s}$ and that $\mathrm{Dome}(\Omega(s))$ is a complete hyperbolic annulus with core curve of length $\frac{2\pi}{\sinh(\frac{s}{2})}$ (see Example 3.A in Herron-Minda-Ma [20] and Theorem 2.16.1 in Epstein-Marden [14]). More explicitly, $\Omega(s)$ is homeomorphic to $S^1 \times \mathbb{R}$ with the metric

$$ds^2 = \left(\frac{\pi}{s}\right)^2 \cosh^2(t) d\theta^2 + dt^2$$

and $\mathrm{Dome}(\Omega(s))$ is homeomorphic to $S^1 \times \mathbb{R}$ with the metric

$$ds^2 = \left(\frac{1}{\sinh(\frac{s}{2})}\right)^2 \cosh^2(t) d\theta^2 + dt^2.$$

The nearest point retraction $r_s : \Omega(s) \to \mathrm{Dome}(\Omega(s))$ takes the core curve of $\Omega(s)$ to the core curve of $\mathrm{Dome}(\Omega(s))$. More generally, it takes any cross-section of $\Omega(s)$ to a cross-section of $\mathrm{Dome}(\Omega(s))$ (and restricts to a dilation) and takes any vertical geodesic perpendicular to the core curve of $\Omega(s)$ to a vertical geodesic perpendicular to the core curve of $\mathrm{Dome}(\Omega(s))$.

Let $\nu(s) = \frac{\pi^2}{s}$ be the minimal value for the injectivity radius on $\Omega(s)$. One may check that there exists a constant $C > 0$ such that if $\mathrm{inj}_{\Omega(s)}(x) = \sinh^{-1}(1)$, then $\mathrm{inj}_{\mathrm{Dome}(\Omega(s))}(r_s(x)) \geq C$. We assume from now on that $\nu(s) < \sinh^{-1}(1)$. Let

$$t_s = \cosh^{-1}\left(\frac{1}{\sinh^{-1}(\frac{\pi^2}{s})}\right) = O(\log(s))$$

as $s \to \infty$. Choose points $x_1 = (\theta_0, t_s)$ and $x_2 = (\theta_0, -t_s)$ in $\Omega(s)$ for fixed $\theta_0 \in S^1$. Notice that $\mathrm{inj}_{\Omega(s)}(x_1) = \mathrm{inj}_{\Omega(s)}(x_2) = \sinh^{-1}(1)$ and x_1 and x_2 lie on opposite sides of a geodesic perpendicular to the core curve of $\Omega(s)$. Therefore, $r_s(x_1)$ and $r_s(x_2)$ lie on opposite sites of a geodesic perpendicular to the core curve of $\mathrm{Dome}(\Omega(s))$. Since, $\mathrm{inj}_{\mathrm{Dome}(\Omega(s))}(r(x_i)) \geq C$, we see that

$$d(r_s(x_1), r_s(x_2)) \geq 2\cosh^{-1}\left(\frac{\sinh(C)}{\sinh^{-1}(\frac{\pi}{\sinh(\frac{s}{2})})}\right) = O(s)$$

as $s \to \infty$. It follows that the Lipschitz constant of r_s is at least

$$\frac{\cosh^{-1}\left(\frac{\sinh(C)}{\sinh^{-1}(\frac{\pi}{\sinh(\frac{s}{2})})}\right)}{\cosh^{-1}\left(\frac{1}{\sinh^{-1}(\frac{\pi^2}{s})}\right)} = O\left(\frac{s}{\log s}\right)$$

as $s \to \infty$. But, since $\nu(s) = \frac{\pi^2}{s}$, this expression is also $O(\frac{1}{|\log \nu(s)|\ \nu(s)})$ as $\nu(s) \to 0$. Notice that the Lipschitz constant in Corollary 1.2 is $2\sqrt{2}(k + \frac{\pi^2}{\nu}) = O(1/\nu)$ as $\nu \to 0$, suggesting that our constants are close to having the correct form. (We remark that Corollary 1.2 contradicts part (3) of Theorem 2.16.1 in [14], which is incorrect as stated.)

If the lift of the nearest point retraction to a map from $\tilde{\Omega}(s)$ to $\mathrm{Dome}(\tilde{\Omega}(s))$ is a $(K(s), C(s))$-quasi-isometry, then

$$K(s) \geq \frac{\pi \sinh(\frac{s}{2})}{s} = O\left(\frac{e^{\frac{s}{2}}}{s}\right)$$

as $s \to \infty$, since it takes the lift of the core curve of $\Omega(s)$ to the lift of the core curve of $\mathrm{Dome}(\Omega(s))$ equivariantly. Notice that this expression is also $O(\nu(s)e^{\frac{\pi^2}{2\nu(s)}})$ as $\nu(s) \to 0$, and that $L(\nu) = O(e^{\frac{\pi^2}{2\nu}})$ in Theorem 1.5, so again our constants are close to having the right form. This also illustrates the fact that the quasi-isometry constants of the lift will typically be larger than the quasi-isometry constants of the original map.

We now consider the Thurston metric on the round annulus $\Omega(s)$. In this case, the nearest point retraction is a homeomorphism. One may extend the analysis in section 5 to show that the nearest point retraction is an isometry, with respect to the Thurston metric, on lines through the origin. On a circle about the origin, with length L in the Thurston metric, the nearest point retraction is a dilation with dilation constant $\frac{L-2\pi}{L}$. In particular, each such circle has length greater than 2π and the core curve of $\Omega(s)$ has length $2\pi + \frac{2\pi}{\sinh(\frac{s}{2})}$ in the Thurston metric. (One may also derive these facts more concretely using Lemma 3.5 to compute the Thurston metric. See Example 3.A in [20] for an explicit calculation.) It follows that the nearest point retraction is a $(1, 2\pi)$-quasi-isometry for all $\Omega(s)$. It is natural to ask what the best possible general quasi-isometry constants are for the nearest point retraction.

11. Appendix: Proofs of intersection number estimates

In this appendix we give the proofs of Lemmas 6.1 and 6.6. The proof of Lemma 6.1 follows the same outline as the proof of Lemma 4.3 in [8], but is technically much simpler as we are working in the setting of finitely punctured domains. We give the proof of Lemma 6.1 both to illustrate the simplifications achieved and to motivate the proof of Lemma 6.6 which has many of the same elements.

Throughout this appendix, X will be a finite set of points in $\hat{\mathbb{C}}$, $\Omega = \hat{\mathbb{C}} - X$ and $\mathrm{Dome}(\Omega) = \partial CH(X)$.

11.1. **Families of support planes.**

We begin by associating a family of support planes to a geodesic arc α in $\mathrm{Dome}(\Omega)$. If these support planes all intersect, we will obtain a bound on $i(\alpha)$. The proof of this bound is much easier in our simpler setting than in [8]. Our lemma is essentially a special case of Lemma 4.1 in [8].

Let $\alpha : [0,1] \to \mathrm{Dome}(\Omega)$ be a geodesic arc transverse to the edges of $\mathrm{Dome}(\Omega)$. Let $\{x_i = \alpha(t_i)\}_{i=1}^{n-1}$ be the finite collection of intersection points of the interior $\alpha((0,1))$ of α with edges of $\mathrm{Dome}(\Omega)$. If the initial point of α lies on an edge, we denote the point by x_0 and the edge by e_0, while if the endpoint of α lies on an edge, we denote the point x_n and the edge by e_n. Suppose that x_i lies on edge e_i and that e_i has exterior dihedral angle θ_i. We let F_i be the face containing $\alpha(t_{i-1}, t_i)$. (If $\alpha(0)$ lies on an edge e_0, we let F_0 be the face abutting e_0 which does not contain $\alpha(0, t_1)$.) At each point x_i, there is a 1-parameter family of support planes $\{P^i_s \mid 0 \leq s \leq \theta_i\}$ where P^i_0 contains F_i and the dihedral angle between P^i_s and P^i_t is $|s - t|$. Concatenating these one parameter families we obtain the one parameter family of support planes $\{P_t \mid 0 \leq t \leq i(\alpha)\}$ along α, called the *full family of support planes* to α.

In general, we will allow our families of support planes to omit sub-families of support planes associated to the endpoints. We say that

$$\{P_t \mid a \leq t \leq b\} \subset \{P_t \mid 0 \leq t \leq i(\alpha)\}$$

is a *family of support planes* to α if

(1) either $a = 0$ or $\alpha(0)$ lies on an edge e_0 and $a \leq \theta_0$, and
(2) either $b = i(\alpha)$ or $\alpha(1)$ lies on an edge e_n and $i(\alpha) - b \leq \theta_n$.

Notice that a full family of support planes is a family of support planes in this definition.

Lemma 11.1. *Suppose that Ω is the complement of a finite collection X of points in $\hat{\mathbb{C}}$ and $\alpha : [0,1] \to \mathrm{Dome}(\Omega)$ is a geodesic path (which may have endpoints on edges). If $\{P_t \mid a \leq t \leq b\}$ is a family of support planes along α such that the associated half-spaces satisfy $H_t \cap H_a \neq \emptyset$ for all $t \in [a,b]$, then*

$$|b - a| \leq \psi < \pi$$

where ψ is the exterior dihedral angle between P_a and P_b. Moreover, α intersects each edge at most once.

Proof: Let $\{x_i\}$ be the intersection points of α with edges $\{e_i\}$. Let $\{P_t \mid a_i \leq t \leq b_i\}$ be the support planes to e_i in the family of support planes $\{P_t \mid a \leq t \leq b\}$. As $H_t \cap H_a \neq \emptyset$, then $g_t = P_t \cap P_a$ is a geodesic if $P_t \neq P_a$. If $e_i \subset P_a$, then $g_t = e_i$ for all $t \in [a_i, b_i]$ such that $P_t \neq P_a$. If e_i is not on P_a then $\{g_t| \ a_i \leq t \leq b_i\}$ is a continuously varying family of disjoint geodesics lying between g_{a_i} and g_{b_i} on P_a.

If every edge α intersects is contained in P_a, then the result is obvious, so we will assume that there is an edge which α intersects which does not lie in P_a. Choose d so that e_{d+1} is the first edge α intersects that does not belong to P_a.

If an edge e_j with $j > d$ lies in P_a choose e_j to be the first edge such that $j > d$ and e_j is on P_a. Otherwise, choose e_{j-1} to be the last edge which α intersects. By construction, $j > d+1$, since x_{d+1} does not lie on P_a. Notice that $g_t = e_d$ for $a_d < t \leq b_d$ and that $\{g_t \mid b_d \leq t \leq b_{j-1}\}$ is a continuous family of geodesics on P_a. If the family of geodesics $\{g_t \mid b_d \leq t \leq b_{j-1}\}$ are not all disjoint, then there must exist $i > d$, such that if $t \in (a_{i+1}, b_{i+1}]$, then g_t lies on the same side of $g_{a_{i+1}}$ as g_{a_i}. However, this implies that there exists $u \in (a_{i+1}, b_{i+1}]$ such that P_u intersects the interior of the face of $\mathrm{Dome}(\Omega)$ contained in P_{a_i}. Since P_u bounds a half-space H_u disjoint from $CH(X)$, this is a contradiction. Therefore, $\{g_t \mid b_d \leq t \leq b_{j-1}\}$ is a disjoint family of geodesics.

If e_j lies in P_a, then there is a support plane P_t separating some point on e_d from a point on e_j. This is a contradiction. Therefore no edge e_j with $j > d$ lies in P_a and $b_{j-1} = b$, so $\{g_t \mid b_d \leq t \leq b\}$ is a disjoint family of geodesics on P_a. It follows that α intersects each edge exactly once, since otherwise there would be distinct values $s, t \in [b_d, b]$ such that $P_s = P_t$ and hence $g_s = g_t$.

Let ψ_i be the exterior dihedral angle between P_a and P_{a_i} let $\theta_i = b_i - a_i$ be the dihedral angle between P_{b_i} and P_{a_i} for all $i > d$. Let m be the maximal value of i for an edge e_i and let $\psi_{m+1} = \psi$ be the exterior dihedral angle between P_a and P_b and let $\theta_m = b_m - a_m$ denote the dihedral angle between P_{a_m} and P_b.

We let $P_k = P_{a_k}$ for $k \leq m$ and $P_{m+1} = P_b$. Now for $d+1 \leq k \leq m$ we let P be the unique plane perpendicular to the planes P_a, P_k, and P_{k+1} (see figure 1). Considering the triangle T on P given by the intersection with the three planes, we see that

$$\psi_k + \theta_k \leq \psi_{k+1}.$$

Then, by induction,

$$\psi_{d+1} + \sum_{i=d+1}^{m} \theta_i \leq \psi_{m+1}$$

As $\psi_{d+1} = \theta_d$, we obtain

$$|b - a| = \sum_{i=d}^{m} \theta_i \leq \psi_{m+1} = \psi.$$

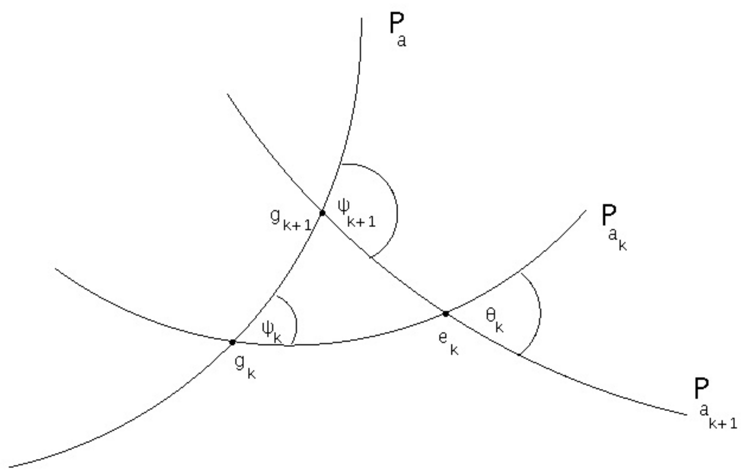

FIGURE 1. $\psi_k + \theta_k \leq \psi_{k+1}$

□

11.2. **Facts from hyperbolic trigonometry.** The following elementary facts from hyperbolic geometry, which were established in [8], will be used in the proof of Lemma 6.1.

Lemma 11.2. ([8, Lemma 3.1]) *Suppose that T is a hyperbolic triangle with a side of length C and opposite angle equal to γ. Then the triangle has perimeter at most*

$$C + 2\sinh^{-1}\left(\frac{\sinh(C/2)}{\sin(\gamma/2)}\right)$$

and this maximal perimeter is realized uniquely by an isosceles triangle.

Lemma 11.3. ([8, Lemma 3.2]) *Let P_0, P_1 and P_2 be planes in $\mathbb{H}^3$ such that ∂P_0 and ∂P_1 are tangent in $\hat{\mathbb{C}}$ and ∂P_1 and ∂P_2 are also tangent in $\hat{\mathbb{C}}$. If $L \leq 2\sinh^{-1}(1)$ and $\beta : [0,1] \to \mathbb{H}^3$ is a curve of length at most L which intersects all three planes, then P_0 and P_2 intersect with interior dihedral angle θ, where*

$$\theta \geq 2\cos^{-1}(\sinh(L/2)).$$

In the proof of Lemma 6.6 we will need the following extension of Lemma 11.3 which bounds the length of closed curves intersecting all three sides of a triangle.

Lemma 11.4. *Let P_0, P_1 and P_2 be planes in $\mathbb{H}^3$ such that ∂P_0 and ∂P_1 are tangent in $\hat{\mathbb{C}}$ and ∂P_1 and ∂P_2 are also tangent in $\hat{\mathbb{C}}$. If P_0 and P_2 intersect at an interior angle of θ and $\beta : [0,1] \to \mathbb{H}^3$ is a closed curve which intersects all three planes, then $l_h(\beta) \geq R(\theta)$, where*

$$R(\theta) = \begin{cases} 2\sinh^{-1}\left(\frac{1}{\tan(\theta/2)}\right) & \theta > \pi/2 \\ 2\sinh^{-1}\left(\frac{\sin(\theta)}{\tan(\theta/2)}\right) & \theta \leq \pi/2 \end{cases}$$

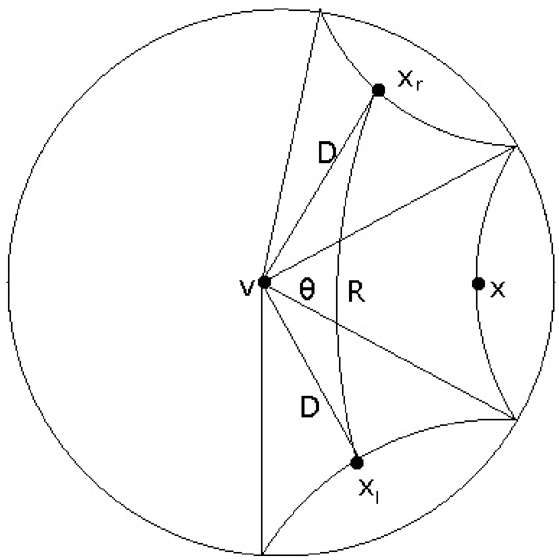

FIGURE 2. Triangle T_θ

Proof. Let P be the plane which intersects P_0, P_1 and P_2 orthogonally and let T be the triangle they bound in P. If β' is the projection of β onto P, then $l_h(\beta') \leq l_h(\beta)$ and β' intersects all three sides of T. Therefore, it suffices to bound the length of a curve in P which intersects all three sides of T. Since T is convex it suffices to consider curves which lie in T.

Let v be the non-ideal vertex of T and let D be length of the perpendicular τ which joins a point x in the opposite side of T to v. Then, by hyperbolic trigonometry, e.g. [1, Theorem 7.11.2],

$$\sinh(D) = \frac{1}{\tan(\theta/2)}.$$

In $\mathbb{H}^2$ we take two copies T_l and T_r of T and glue T_l to left edge of T containing v and T_r to the right edge containing v. We label the points in the copies corresponding to x by x_l and x_r. We can join x_l and x_r by edges of length D to v which subtend an angle 2θ at v in the union of the three triangles (see figure 2). If $\theta < \pi/2$, we join the points x_l and x_r by a geodesic g in the union of the three copies and let $R = d(x_l, x_r)$. Then we have an isoceles triangle with vertices x_l , x_r, and v and two sides of length D which make an angle 2θ. We may decompose this triangle into two right angled triangles with hypotenuse of length D and again apply [1, Theorem 7.11.2] to show that

$$\sinh(D) = \frac{\sinh(l_h(g)/2)}{\sin(\theta)}.$$

If $\theta \geq \frac{\pi}{2}$, then we join x_l to x_r by a path g of length $2D$ made from copies of τ in T_l and T_r.

In either case, g projects to a closed curve in T meeting all three sides. To complete the proof, we must show that g projects to the minimal length such curve. If g' is the minimal curve, then we may unfold it to a geodesic joining a point x_l' to the side of T_l containing x_l to a point x_r' in the side of T_r containing x_r. Moreover, if we join x_l' and x_r' to v by geodesics, the two geodesics each have length at least D and make an angle of 2θ. It follows that the length of g' is at least as great as that of g, with equality if and only if $g = g'$. □

11.3. **Proof of Lemma 6.1.** We are now prepared to prove Lemma 6.1, whose statement we recall below:

Lemma 6.1 *If Ω is the complement of a finite collection X of points in $\hat{\mathbb{C}}$ and $\alpha : [0,1] \to \mathrm{Dome}(\Omega)$ is a geodesic path with $l_h(\alpha) \leq G(\mathrm{inj}_{\mathrm{Dome}(\Omega)}(\alpha(s)))$, for some $s \in [0,1]$, then*

$$i(\alpha) \leq 2\pi.$$

Proof: Our assumptions imply that there exists $s_0 \in [0,1]$ such that $\mathrm{inj}_{\mathrm{Dome}(\Omega)}(\alpha(s_0)) \leq F(l_h(\alpha))$ (since $G = F^{-1}$). We may assume that $l_h(\alpha[0,s_0]) \leq l_h(\alpha)/2$. (If this is not the case we simply consider $\bar{\alpha} : [0,1] \to \mathrm{Dome}(\Omega)$ given by $\bar{\alpha}(s) = \alpha(1-s)$ and notice that $i(\alpha) = i(\bar{\alpha})$.)

We first handle the case where $\mathrm{Dome}(\Omega)$ is contained in a plane. In this case, if $i(\alpha) > 2\pi$, then there exists a subarc α_1 containing $\alpha(s_0)$ with $i(\alpha_1) = 2\pi$. Recall that $\mathrm{Dome}(\Omega)$ is the double of $CH(X)$ in this case. Both endpoints of α_1 must lie in the same "copy" of $CH(X)$. One may then join the endpoints of α_1 by a geodesic arc β in this copy, so that $\delta = \alpha_1 \cup \beta$ is homotopically non-trivial in $\mathrm{Dome}(\Omega)$ and $l(\delta) < 2l(\alpha)$. It follows that

$$\mathrm{inj}_{\mathrm{Dome}(\Omega)}(\alpha(s_0)) < l_h(\alpha) < F(l_h(\alpha))$$

which is a contradiction.

We now move on to the general case. Let $\{P_t \mid 0 \leq t \leq i(\alpha)\}$ be the full family of support planes to α. There are three cases.

Case 1: If $H_t \cap H_0 \neq \emptyset$ for all t, then Lemma 11.1 implies that

$$i(\alpha) \leq \pi.$$

If we are not in case 1, let P_{t_1} be the first support plane with $H_{t_1} \cap H_0 = \emptyset$

Case 2: If $H_t \cap H_{t_1}$ for all $t \geq t_1$, then by applying Lemma 11.1 to the two families $\{P_t \mid 0 \leq t < t_1\}$ and $\{P_t \mid t_1 \leq t \leq i(\alpha)\}$, we have

$$i(\alpha) \leq \pi + \pi = 2\pi.$$

Case 3: If we are not in case (1) or (2), we let P_{t_2} be the first support plane such that $H_{t_2} \cap H_{t_1} = \emptyset$. Consider the configuration of three planes P_0, P_{t_1}, and P_{t_2}. Let $\alpha_1 = \alpha|_{[0,s_1]}$ be the subarc of α of minimum length having $\{P_t \mid 0 \leq t \leq t_2\}$ as a family of support planes. Let x and y be the endpoints of α_1. Lemma 11.3 implies that P_0 and P_{t_2} intersect and have dihedral angle θ satisfying

$$\theta \geq 2\cos^{-1}(\sinh(l_h(\alpha_1)/2)).$$

We join x to y by a shortest path β on $P_0 \cup P_{t_2}$ which intersects $P_0 \cap P_{t_2}$ at a point $z \in P_0 \cap P_{t_2}$. We take the triangle T with vertices x, y, and z. Then the interior angle θ_z of T at z satisfies $\theta_z \geq \theta$. Since x and y lie in α_1, $d(x,y) \leq l_h(\alpha_1)$. Then, by Lemma 11.2,

$$d(x,z) + d(z,y) \leq 2\sinh^{-1}\left(\frac{\sinh(d(x,y)/2)}{\sin(\theta_z/2)}\right) \leq 2\sinh^{-1}\left(\frac{\sinh(l_h(\alpha_1)/2)}{\sin(\theta/2)}\right).$$

Substituting the bound for θ we get

$$d(x,z)+d(z,y) \le 2\sinh^{-1}\left(\frac{\sinh(l_h(\alpha_1)/2)}{\sqrt{1-\sinh^2(l_h(\alpha_1)/2)}}\right).$$

If $\delta = \alpha_1 \cup \beta$, then

$$l_h(\delta) \le l_h(\alpha_1) + 2\sinh^{-1}\left(\frac{\sinh\left(\frac{l_h(\alpha_1)}{2}\right)}{\sqrt{1-\sinh^2\left(\frac{l_h(\alpha_1)}{2}\right)}}\right) = 2F(l_h(\alpha_1)).$$

We next show that $r(\delta)$ is homotopically non-trivial on Dome(Ω). Since the restriction of r to $\mathbb{H}^3 - \mathrm{int}(CH(X)))$ is a homotopy equivalence onto Dome(Ω), it suffices to show that δ is homotopically non-trivial in $\mathbb{H}^3 - \mathrm{int}(CH(X))$. Let $b = \alpha(s_1)$ be the first point on α which has support plane P_{t_1}. Then b lies on an edge e of Dome(Ω) which is contained in P_{t_1}. Let Q_b be the plane through b which is perpendicular to P_{t_1} and let R_b be the intersection of Q_b with the closed half-space bounded by P_{t_1} which is disjoint from $\mathrm{int}(CH(X))$. Lemma 11.1 implies that α_1 intersects b exactly once at $\alpha(s_1)$ and the intersection is transverse. By construction, β cannot intersect R_b and α_1 cannot intersect $R_b - b$. Therefore, δ intersects R_b exactly once and does so transversely. Since R_b is a properly embedded half-plane in $\mathbb{H}^3 - \mathrm{int}(CH(X))$, we see that δ is homotopically non-trivial in $\mathbb{H}^3 - \mathrm{int}(CH(X))$ as desired.

Since r is 1-Lipschitz on $\mathbb{H}^3$,

$$l_h(r(\delta)) \le l_h(\delta) \le 2F(l_h(\alpha_1)) < 2F(l_h(\alpha)).$$

In particular, since $r(\delta)$ is homotopically non-trivial and passes through $\alpha(0)$, we see that

$$\mathrm{inj}_{\mathrm{Dome}(\Omega)}(x) \le F(l_h(\alpha))$$

for all $x \in \alpha_1$. If $\alpha(s_1) \in \alpha_1$, i.e. if $s_0 \le s_1$, then we have achieved a contradiction and we are done. If $\alpha(s_0)$ does not lie in α_1, then let $\alpha_2 = \alpha|_{[s_1,s_0]}$. Then $\gamma = \bar{\alpha}_2 * \bar{r}(\beta) * \alpha_1 * \alpha_2$ is a homotopically non-trivial loop through $\alpha(s_0)$ such that

$$l_h(\gamma) = 2l_h(\alpha_2) + l_h(r(\delta)) < l_h(\alpha) + 2F(l_h(\alpha_1)).$$

Since

$$l_h(\alpha) + 2F(l_h(\alpha_1)) < l_h(\alpha) + 2F(l_h(\alpha)/2) < 2F(l_h(\alpha))$$

we have again achieved a contradiction. □

11.4. **Proof of Lemma 6.6.** We now establish Lemma 6.6 which bounds intersection number for short simple closed geodesics.

Lemma 6.6 *Suppose that Ω is the complement of finite collection X of points in $\hat{\mathbb{C}}$. If γ is a simple closed geodesic of length less than $2\sinh^{-1}(1)$, then*

$$2\pi \le i(\gamma) \le 2\pi + 2\tan^{-1}(\sinh(l_h(\gamma)/2)) \le \frac{5\pi}{2}$$

Proof. We first note that if Dome(Ω) lies in a plane and γ is a simple closed geodesic on Dome(Ω), then γ is the double of a simple geodesic arc in $CH(X)$, so $i(\gamma) = 2\pi$.

For the remainder of the proof, we assume that $CH(X)$ does not lie in a plane. We choose a parameterization $\alpha : [0,1] \to \text{Dome}(\Omega)$ of γ which is an embedding on $(0,1)$ and such that $\alpha(0) = \alpha(1)$ does not lie on an edge of Dome(Ω). Recall that Lemma 6.4 implies that $i(\gamma) \geq 2\pi$.

Let $\{P_t \mid 0 \leq t \leq i(\alpha)\}$ be the full family of support planes to α. If we proceed as in the proof of Lemma 6.1, we must be in case (3). Therefore, we obtain support planes P_0, P_{t_1} and P_{t_2}, such that ∂P_0 and ∂P_{t_1} are tangent on $\hat{\mathbb{C}}$, as are P_{t_1} and P_{t_2}, and a subarc α_1 of α which begins at $\alpha(0) \in P_0$, ends at $\alpha(s_1) \in P_{t_2}$ and intersects P_{t_1}. Lemma 11.3 implies that P_0 and P_{t_2} intersect. Lemma 11.4 implies that if θ is the angle of intersection of P_0 and P_{t_2}, then $\theta \geq R^{-1}(l_h(\gamma))$. As R is decreasing and $R(\pi/2) = 2\sinh^{-1}(1) \geq l_h(\gamma)$ we see that $\theta \geq \pi/2$.

Therefore,

$$\theta \geq R^{-1}(l_h(\gamma)) = 2\tan^{-1}\left(\frac{1}{\sinh\left(\frac{l_h(\gamma)}{2}\right)}\right)$$

Let $\phi = \pi - \theta \leq \pi/2$ be the exterior dihedral angle between P_0 and P_{t_2}. Then,

$$\phi \leq 2\tan^{-1}(\sinh(l_h(\gamma)/2)).$$

Let β_0 be the shortest path on $P_0 \cup P_{t_2}$ joining $\alpha(0)$ to $\alpha(s_1)$ and let z_0 be the point of intersection of β_0 with $P_0 \cap P_{t_2}$. Since

$$d_h(\alpha(0), \alpha(s_1)) \leq l_h(\gamma)/2,$$

Lemma 11.2 implies that

$$l_h(\beta_0) \leq 2\sinh^{-1}\left(\frac{\sinh(d_h(\alpha(0), \alpha(s_1))/2)}{\sin(\theta/2)}\right),$$

so

$$\sinh(l_h(\beta_0)/2) \leq \frac{\sinh(l_h(\gamma)/4)}{\sin(\theta/2)}.$$

Since $\theta \geq \frac{\pi}{2}$ and $l_h(\gamma) < 2\sinh^{-1}(1)$, we see that

$$\sinh(l_h(\beta_0)/2) \leq \frac{\sinh(\frac{\sinh^{-1}(1)}{2})}{\sin(\frac{\pi}{4})} \approx .6436 < 1.$$

It follows that $l_h(\beta_0)/2 < \sinh^{-1}(1)$.

We let $\delta_0 = \alpha_1 \cup \beta_0$. We may argue, exactly as in the proof of Lemma 6.6, that $r(\delta_0)$ is homotopically non-trivial on Dome(Ω). Since $\alpha_1 \subset \gamma$, $l_h(\beta_0)/2 < \sinh^{-1}(1)$ and $N(\gamma)$ has width $w(\gamma) > \sinh^{-1}(1)$, we see that $r(\delta_0)$ is contained in $N(\gamma)$, so is homotopic to a non-trivial power of γ.

We now describe a process to replace β_0 by a path β_n on Dome(Ω) such that $i(\beta_n) \leq \phi$ and β_n is homotopic to $r(\beta_0)$. Once we have done so, we can form $\delta_n = \alpha_1 \cup \beta_n$ and observe that

$$i(\delta_n) \leq i(\alpha_1) + \phi \leq 2\pi + \phi \leq 2\pi + 2\tan^{-1}(\sinh(l_h(\alpha_1)/2)) \leq \frac{5\pi}{2}.$$

Our result follows, since δ_n is homotopic to a non-trivial power of γ and a geodesic minimizes intersection number in its homotopy class.

We label $Q_0 = P_{t_2}$ and let $P_0 \cap Q_0 = h_0$. If h_0 is an edge then P_0 and Q_0 are adjacent and β_0 is on Dome(Ω) and we are done.

Otherwise β_0 intersects an edge on Q_0 labelled e_0, such that h_0 and e_0 are disjoint. We choose e_0 to be the last edge on Q_0 that α intersects. Let y_0 be the first point of intersection of β_0 and

e_0. Let F_1 be the face of Dome(Ω) which abuts e_0 and does not contain a subarc of β_0. Let Q_1 be the support plane containing F_1.

Define $h_1 = P_0 \cap Q_1$. Since h_0 and h_1 are both contained in support planes to e_0, they are disjoint. Since $\alpha(0) \in$ Dome(Ω), h_0 must separate h_1 from $\alpha(0)$ in P_0. Therefore, β_0 intersects h_1 at a point z_1 between z_0 and $\alpha(0)$. We construct β_1 by replacing the subarc of β_0 joining y_0 to z_1 by the geodesic arc in F_1 joining y_0 to z_1. Notice that $r(\beta_1)$ is homotopic to $r(\beta_0)$ in Dome(Ω). If we let θ_1 be the exterior dihedral angle between Q_0 and Q_1 and let ϕ_1 be the exterior dihedral angle between Q_1 and P_0, then

$$\theta_1 + \phi_1 \leq \phi.$$

If h_1 is an edge, then β_1 lies on Dome(Ω) and we are done since

$$i(\beta_1) = \theta_1 + \phi_1 \leq \phi.$$

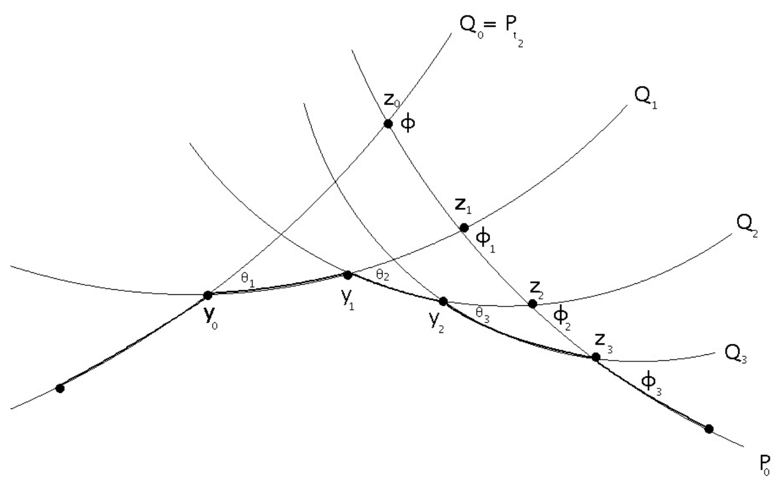

FIGURE 3. $\beta_n = \beta_3$

If h_1 is not an edge, we define β_n by an iterative procedure. Assume Q_k are defined as above, for $k = 0, \ldots, m$, such that $h_m = Q_m \cap P_0$ is not an edge. Define θ_k to be the exterior dihedral angle between Q_{k-1} and Q_k and ϕ_k to be the exterior dihedral angle between Q_k and P_0. We assume, by induction, that

$$\sum_{k=1}^{m} \theta_k + \phi_m \leq \phi.$$

Let h_m be the last edge on Q_m which β_m intersects, and let y_m be the first point of intersection of β_m and h_m. Let F_{m+1} be the face adjacent to e_m which does not agree with F_m and let Q_{m+1} be the support plane it is contained in. As before, if we let $h_{m+1} = Q_{m+1} \cap P_0$, then h_{m+1} separates h_m from $\alpha(0)$. Therefore, β_m intersects h_{m+1} in a point z_{m+1}. We construct β_{m+1} by replacing the piecewise geodesic subarc of β_m connecting y_{m+1} to z_{m+1} by the geodesic arc in F_{m+1} joining y_{m+1} to z_{m+1}. Again, notice that $r(\beta_{m+1})$ is homotopic to $r(\beta_m)$. Furthermore,

$$\theta_{m+1} + \phi_{m+1} \leq \phi_m.$$

Therefore,

$$\sum_{k=1}^{m+1} \theta_k + \phi_{m+1} = \sum_{k=1}^{n} \theta_k + (\theta_{m+1} + \phi_{m+1}) \leq \sum_{k=1}^{m} \theta_m + \phi_m \leq \phi.$$

Notice that the support planes $\{Q_k\}$ in this procedure have disjoint intersection with P_0 and each contain a face F_k. Therefore, as there are only a finite number of faces, the process terminates and some h_n is an edge of Dome(Ω). Then, β_n lies on Dome(Ω) and our argument is complete. □

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