Bounding the bending of a hyperbolic 3-manifold

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Abstract

In this paper we obtain bounds on the total bending of the boundary of the convex core of a hyperbolic 3-manifold. These bounds will depend on the geometry of the boundary of the convex hull of the limit set.

1 Introduction

The boundary of the convex core of a hyperbolic 3-manifold is a hyperbolic surface in its intrinsic metric. This surface is totally geodesic except along a lamination, called the bending lamination. The bending lamination inherits a transverse measure which keeps track of how much the surface is bent along the lamination. The length (or mass) of the bending lamination, regarded as a measured lamination, records the total bending of the boundary of the convex core. For example, if the boundary of the convex core is bent by an angle of θ along a single simple closed geodesic of length L, then the length of the bending lamination is $L\theta$.

Our main result is an upper bound on the mass of the bending lamination which depends on a lower bound for the injectivity radius of the boundary of the convex hull of the limit set. An upper bound on the mass of the bending lamination is also implicit in the techniques developed by Bonahon and Otal, see Lemma 12 in [5].

If $N = \mathbf{H}^3/\Gamma$ is an orientable hyperbolic 3-manifold and Γ is a non-abelian group of orientation-preserving isometries of $\mathbf{H}^3$, then the limit set L_Γ of Γ is the smallest closed non-empty Γ-invariant subset of $\partial_\infty \mathbf{H}^3 = \widehat{\mathbf{C}}$. The convex core $C(N)$ of N is simply $CH(L_\Gamma)/\Gamma$ where $CH(L_\Gamma)$ is the convex hull in $\mathbf{H}^3$ of L_Γ. Notice that ρ_0 is a lower bound for the injectivity radius of the boundary $\partial CH(L_\Gamma)$ of the convex hull of the limit set if and only if $2\rho_0$ is a lower bound for the length of a compressible curve on the boundary of the convex core (i.e. a closed curve in $\partial C(N)$ which is null-homotopic in $C(N)$ but not in $\partial C(N)$.)

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Theorem 1: *There exist constants S and T such that if N is an orientable hyperbolic 3-manifold with finitely generated, non-abelian fundamental group, β_N is its bending lamination and $\rho_0 \in (0,1]$ is a lower bound for the injectivity radius of the boundary $\partial CH(L_\Gamma)$ of the convex hull of the limit set, then*

$$l_{\partial C(N)}(\beta_N) \leq |\chi(\partial C(N))|(S\log(\frac{1}{\rho_0})+T)$$

where $l_{\partial C(N)}(\beta_N)$ denotes the length of β_N and $\chi(\partial C(N))$ denotes the Euler characteristic of the boundary of the convex core.

We also obtain a lower bound for the mass of the bending lamination, in the case that $\partial C(N)$ has a short compressible curve. This lower bound makes clear that the dependence on the geometry of the convex hull of the limit set in our first result cannot be removed and that the form of the estimate cannot be substantially improved. Also notice that if one passes to a degree d cover of N both the length of the bending lamination and the Euler characteristic of the boundary of the convex core multiply by d, while the convex hull of the limit set is the same, so any upper bound must depend linearly on $|\chi(\partial C(N))|$.

Theorem 2: *Let $N = \mathbf{H}^3/\Gamma$ be an orientable hyperbolic 3-manifold with finitely generated, non-abelian fundamental group. If $\partial CH(L_\Gamma)$ contains a closed geodesic of length $\rho \leq 2\sinh^{-1}(1)$, then*

$$l_{\partial C(N)}(\beta_N) \geq 4\pi \log\left(\frac{4\sinh^{-1}(1)}{\rho}\right).$$

In the case when the boundary of the convex core is incompressible, Proposition 4.2 gives the following stronger result:

Theorem 3: *If N is an orientable hyperbolic 3-manifold with finitely generated, non-abelian fundamental group and $\partial C(N)$ is incompressible in N, then*

$$l_{\partial C(N)}(\beta_N) \leq \frac{\pi^3}{\sinh^{-1}(1)}|\chi(\partial C(N))|.$$

In related work, Epstein, Marden and Markovic (see, for example, Theorem 4.2 in [11]) have studied the possible bending laminations of embedded convex hyperbolic planes in $\mathbf{H}^3$.

Thurston [15] (see also Kourouniotis [13], Johnson-Millson [12] or Epstein-Marden [10]) studied the operation of obtaining a quasifuchsian group by bending a Fuchsian group along a simple closed geodesic, or more generally along a measured lamination. Theorem 3 may be used to quantify the observation that if this geodesic is "long," then one may only bend by a "small" angle.

This paper is based on earlier work of the authors ([6, 7, 9]) which explored the relationship between the boundary of the convex core and the conformal boundary. In particular, we make central use of the fact (Lemma 4.3 in [7]) that there is a lower bound, which depends only on the injectivity radius of its basepoint, on the length of a geodesic arc in $\partial CH(L_\Gamma)$ whose intersection with the bending lamination is at least 2π. We will combine this estimate with a Crofton-like formula (Lemma 4.1) for the length of the bending lamination to prove Theorem 1.

In section 7, we will apply the results of [7] and [9] to obtain analogues of Theorems 1 and 2 which depend on the geometry of the domain of discontinuity $\Omega(\Gamma)$ for Γ's action on $\widehat{\mathbf{C}}$.

2 Background

Let $N = \mathbf{H}^3/\Gamma$ be an orientable hyperbolic 3-manifold with non-abelian fundamental group. Then, Γ acts properly discontinuously on the domain of discontinuity $\Omega(\Gamma) = \widehat{\mathbf{C}} - L_\Gamma$. The domain of discontinuity admits a canonical conformally invariant hyperbolic metric $p(z)|dz|$ called the *Poincaré metric*. The quotient surface $\partial_c N = \Omega(\Gamma)/\Gamma$, called the conformal boundary of N, is then naturally a hyperbolic surface. The hyperbolic 3-manifold N is said to be *analytically finite* if $\partial_c N$ has finite area in this metric. Ahlfors' Finiteness Theorem [1] asserts that N is analytically finite if Γ is finitely generated. All of our results hold for analytically finite hyperbolic 3-manifolds.

If N is analytically finite then there is always a positive lower bound for the injectivity radius on $\Omega(\Gamma)$. By Lemma 8.1 of [7], a lower bound on the injectivity radius of $\Omega(\Gamma)$ implies a lower bound on the injectivity radius of $\partial CH(L_\Gamma)$. In particular, if N is analytically finite then there is a positive lower bound on the injectivity radius of $\partial CH(L_\Gamma)$. The boundary of the convex hull of the limit set is a hyperbolic surface in its intrinsic metric and is totally geodesic in the complement of a closed union β_Γ of disjoint geodesics, called the bending lamination of $CH(L_\Gamma)$. The bending lamination β_N of the convex core $C(N)$ is simply the projection of β_Γ to $\partial C(N)$.

A *measured lamination* on a hyperbolic surface S consists of a closed subset λ of S which is the disjoint union of simple geodesics, together with countably additive invariant (with respect to projection along λ) measures on arcs transverse to λ. The bending laminations β_Γ and β_N come equipped with bending measures on arcs transverse to the lamination which record the total bending along the arc. These bending measures give β_Γ and β_N the structure of measured laminations. Real multiples of simple closed geodesics are dense in the space $ML(S)$ of all measured laminations on a finite area hyperbolic surface S. Moreover, the length of a simple closed geodesic and the intersection number of two simple closed geodesics extend naturally to continuous functions on $ML(S)$ and $ML(S) \times ML(S)$ respectively. See Thurston [15] or Bonahon [4] for fuller discussions of measured lamination spaces and Thurston [15] or Epstein-Marden [10] for a fuller discussion of convex cores and bending laminations.

3 Local intersection number estimates

In [7] we obtained bounds on the intersection of a transverse geodesic arc with the bending lamination. In that paper we define a function

$$F(x) = \frac{x}{2} + \sinh^{-1}\left(\frac{\sinh\left(\frac{x}{2}\right)}{\sqrt{1-\sinh^2\left(\frac{x}{2}\right)}}\right)$$

and its inverse $G(x) = F^{-1}(x)$. The function F is monotonically increasing and has domain $(0, 2\sinh^{-1}(1))$. The function $G(x)$ has domain $(0,\infty)$, has asymptotic behavior $G(x) \asymp x$ as x tends to 0, and $G(x)$ approaches $2\sinh^{-1}(1)$ as x tends to ∞. Moreover, we define $G_\infty = 2\sinh^{-1}(1) \approx 1.76275$.

Lemma 3.1 (Lemma 4.3 in [7]) *Let $N = \mathbf{H}^3/\Gamma$ be an analytically finite hyperbolic 3-manifold such that L_Γ is not contained in a round circle. Let $\alpha : [0,1) \to \partial CH(L_\Gamma)$ be a geodesic path (in the intrinsic metric on $\partial CH(L_\Gamma)$) with length $l(\alpha)$. If either*

1. *$l(\alpha) \leq G(\mathrm{inj}_{\partial CH(L_\Gamma)}(\alpha(0)))$, or*
2. *$\alpha([0,1))$ is contained in a simply connected component of $\partial CH(L_\Gamma)$ and $l(\alpha) \leq G_\infty$,*

then

$$i(\alpha, \beta_\Gamma) \leq 2\pi.$$

Notice that a geodesic arc α is either transverse to β_Γ or contained within β_Γ, in which case we define $i(\alpha, \beta_\Gamma) = 0$.

If $\alpha : [0,1) \to \partial C(N)$ is a geodesic in the boundary of the convex core, then we consider its lift $\tilde{\alpha} : [0,1) \to \partial CH(L_\Gamma)$. If we subdivide this lift into pieces to which may apply Lemma 3.1, as in the proof of Proposition 5.1 in [7], we obtain the following corollary.

Corollary 3.2 *Let N be an analytically finite hyperbolic 3-manifold. Let $\alpha : [0,1) \to \partial C(N)$ be a geodesic path with length $l(\alpha)$. If α is contained in an incompressible component of $\partial C(N)$, let $G = G_\infty$. Otherwise, let ρ_α be a lower bound on the injectivity radius of $\partial CH(L_\Gamma)$ at every point in $\tilde{\alpha}([0,1))$ and let $G = G(\rho_\alpha)$. Then*

$$i(\alpha, \beta_N) \leq 2\pi \left[\frac{l(\alpha)}{G}\right]^+$$

where $[x]^+$ is the least integer greater than or equal to x.

We have so far avoided, for simplicity of exposition, discussing the case that the limit set is contained in a round circle. In this case, the convex core is a totally geodesic surface with geodesic boundary. It is natural to consider the boundary of the convex core to be the double of the convex core (where one considers the two sheets of the convex core to have opposite normal vectors.) With this convention, the boundary of the convex core is still a finite area hyperbolic surface with boundary if our manifold is analytically finite. One may easily see, just as in the proof of Proposition 5.1 in [7], that Corollary 3.2 remains valid in this situation.

4 A length formula

In order to prove Theorem 1 we first represent the length of the bending lamination as the integral of the intersection number over all geodesics of a fixed length. Our formula is similar to the Crofton formula for the area of a region in the plane. See also Proposition 14 in Bonahon [3].

Let S be a hyperbolic surface. If $v \in T^1(S)$ is a unit tangent vector then let $\bar{\alpha}(v) : (0, \infty) \to S$ be the unit speed geodesic ray originating at the basepoint of v and in the direction of v. Let $\alpha^L(v) = \bar{\alpha}|_{(0,L)}$ be the open geodesic segment of length L emanating from the basepoint of v in the direction v.

Lemma 4.1 *Let β be a measured lamination on a finite area hyperbolic surface S. Then*

$$l_S(\beta) = \frac{1}{4L} \int_{T^1(S)} i(\alpha^L(v), \beta) d\Omega(v)$$

where $d\Omega$ is the volume form on $T_1(S)$.

Proof of 4.1: We define a function F_L on the space $ML(S)$ of measured laminations by setting

$$F_L(\beta) = \frac{1}{4L} \int_{T_1(S)} i(\alpha^L(v), \beta) d\Omega(v).$$

As F_L and l_S are both continuous on $ML(S)$ and real multiples of closed geodesics are dense in $ML(S)$, it suffices to prove that $F_L(\beta) = l_S(\beta)$ for real multiples of closed geodesics. Since $F_L(k\beta) = kF_L(\beta)$ and $l_S(k\beta) = kl_S(\beta)$ for all $\beta \in ML(S)$ and all $k > 0$, we may assume that β is a single closed geodesic with unit transverse measure.

Let C be the hyperbolic cylinder covering S corresponding to β and let $\tilde{\beta}$ be the lift of β to C. If $v \in T^1(S)$, then $i(\alpha^L(v), \beta)$ is precisely the number of lifts of $\alpha^L(v)$ to C which intersect $\tilde{\beta}$. Let

$$U = \left\{ v \in T^1(C) | \ \ \alpha^L(v) \ \text{ intersects } \ \tilde{\beta} \ \right\}.$$

Lifting the integral to C we see that

$$\int_{T^1(S)} i(\alpha^L(v), \beta)\ d\Omega(v) = \int_U d\Omega(v).$$

The metric on C is given by

$$ds^2 = dx^2 + \cosh^2 x\ dl^2,$$

where x is the perpendicular distance to the core geodesic and l is a length coordinate along the core geodesic, see Example 1.3.2 in Buser [8]. Moreover, the hyperbolic area element is given by $dA = \cosh x\ dx\ dl$.

Let

$$N = \{c \in C | 0 < d(\tilde{\beta}, c) < L\}.$$

If $v \in U$ then the basepoint p of v is in N. If $p \in N$, let U_p denote the cone of tangent vectors in $U \cap T^1_p(C)$. Let w_p denote the unit vector tangent to the geodesic ray through p which is perpendicular to $\tilde{\beta}$. Then U_p consists of all vectors in $T^1_p(C)$ which make an angle of at most $\theta(p)$ with w_p, where

$$\theta(p) = \cos^{-1}\left(\frac{\tanh x}{\tanh L}\right).$$

Therefore,

$$\int_U d\Omega(v) = \int_N 2\cos^{-1}\left(\frac{\tanh x}{\tanh L}\right)\ dA.$$

Integrating over the core of the annulus we obtain

$$\begin{aligned}\int_N 2\cos^{-1}\left(\frac{\tanh x}{\tanh L}\right)\ dA &= 2l_S(\beta)\int_{-L}^{L}\cos^{-1}\left(\frac{\tanh x}{\tanh L}\right)\cosh x\ dx \\ &= 4l_S(\beta)\int_0^L \cos^{-1}\left(\frac{\tanh x}{\tanh L}\right)\cosh x\ dx.\end{aligned}$$

Therefore,

$$F_L(\beta) = \frac{l_S(\beta)}{L}\int_0^L \cos^{-1}\left(\frac{\tanh x}{\tanh L}\right)\cosh x \, dx.$$

Substituting $u = \frac{\tanh x}{\tanh L}$ we obtain

$$F_L(\beta) = \frac{l_S(\beta)\tanh L}{L}\int_0^1 \frac{\cos^{-1}(u)}{(1-u^2\tanh^2 L)^{3/2}}\, du.$$

We may then integrate by parts and evaluate the result to check that $F_L(\beta)$ has the claimed form.

4.1

We now prove a version of Theorem 1, which is a direct application of Corollary 3.2 and Lemma 4.1. Recall that $G(x) \asymp x$ as x tends to 0. If $\rho_0 \geq 1$, then this estimate is better than the one provided by Theorem 1, but it is much weaker as ρ_0 approaches 0, since the upper bound provided by Proposition 4.2 is $O(\frac{|\chi(\partial C(N))|}{\rho_0})$ while the estimate provided by Theorem 1 is $O(|\chi(\partial C(N))| \log\left(\frac{1}{\rho_0}\right))$. Notice that Theorem 3 is case (2) of Proposition 4.2.

Proposition 4.2 *Let $N = \mathbf{H}^3/\Gamma$ be an analytically finite hyperbolic 3-manifold with bending lamination β_N.*

1. *If $\rho_0 > 0$ is a lower bound for the injectivity radius of $\partial CH(L_\Gamma)$, then*

$$l_{\partial C(N)}(\beta_N) \leq \frac{2\pi^3}{G(\rho_0)}|\chi(\partial C(N))|$$

2. *If $\partial C(N)$ is incompressible in N, then*

$$l_{\partial C(N)}(\beta_N) \leq \frac{\pi^3}{\sinh^{-1}(1)}|\chi(\partial C(N))|$$

Proof of 4.2: If $\partial C(N)$ is incompressible, let $G = G_\infty = 2\sinh^{-1}(1)$. If not, we let $G = G(\rho_0)$. Corollary 3.2 implies that, for all $v \in T^1(\partial C(N))$,

$$i(\alpha^L(v), \beta_N) \leq 2\pi\left[\frac{L}{G}\right]^+ \leq 2\pi\left(\frac{L}{G}+1\right)$$

Therefore, by Lemma 4.1,

$$\begin{aligned} l_{\partial C(N)}(\beta_N) &\leq \frac{\pi}{2L}\int_{T^1(\partial C(N))}\left(\frac{L}{G}+1\right)d\Omega \\ &\leq \mathrm{vol}(T^1(\partial C(N)))\left(\frac{\pi}{2G}+\frac{\pi}{2L}\right). \end{aligned}$$

The volume of the unit tangent bundle $T^1(\partial C(N))$ is $4\pi^2|\chi(\partial C(N))|$. Thus, by letting L tend to infinity, we see that

$$l_{\partial C(N)}(\beta_N) \leq 4\pi^2|\chi(\partial C(N))|\left(\frac{\pi}{2G}\right) = \frac{2\pi^3}{G}|\chi(\partial C(N))|.$$

4.2

5 Proof of Theorem 1

To obtain the sharper bound on the length of the bending lamination given by Theorem 1, we must decompose $\partial C(N)$ using the Collar Lemma. We will use the following explicit version of the Collar Lemma, which combines Theorem 4.4.6 in Buser [8], and Lemma 7 of Yamada [16], which guarantees that curves of length at most $2\sinh^{-1}(1)$ are simple.

Theorem 5.1 (Collar Lemma) *Let S be a finite area hyperbolic surface of genus g with n punctures. Let $\{\nu_1,\ldots,\nu_k\}$ be the collection of all primitive closed geodesics on S of length at most $2\sinh^{-1}(1)$. Then*

1. $k \leq 3g-3+n$,
2. $\{\nu_1,\ldots,\nu_k\}$ *is a disjoint collection of simple closed geodesics,*
3. *there exists a disjoint collection $\{B_1,\ldots,B_k\}$ of metric collar neighborhoods of $\{\nu_1,\ldots,\nu_k\}$ such that if ν_i has length $l_S(\nu_i)$, then B_i is isometric to the quotient of $[-w(\nu_i),w(\nu_i)]\times[0,l_S(\nu_i)]$ where one identifies $(t,0)$ with $(t,l_S(\nu_i))$ for all t with the metric*
$$ds^2 = dx^2+\cosh^2 x\; dl^2$$
where $w(\nu_i)=\sinh^{-1}(1/\sinh(\frac{1}{2}l_S(\nu_i)))$.
4. *if* $x\in B_i$, *then* $\sinh(\mathrm{inj}_S(x)) = \sinh\left(\frac{1}{2}l_S(\nu_i)\right)\cosh(d(x,\nu_i))$, *and*

5. *if there is a curve through $x \in S$ homotopic to ν_i of length at most $2\sinh^{-1}(1)$, then $x \in B_i$.*

We now restate Theorem 1 for analytically finite hyperbolic 3-manifolds.

Theorem 1: *There exist constants S and T such that if $N = \mathbf{H}^3/\Gamma$ is an analytically finite hyperbolic 3-manifold, β_N is its bending lamination and $\rho_0 \in (0,1]$ is a lower bound for the injectivity radius of the boundary $\partial CH(L_\Gamma)$ of the convex hull of the limit set, then*

$$l_{\partial C(N)}(\beta_N) \leq |\chi(\partial C(N))|(S\log(\frac{1}{\rho_0}) + T)$$

where $l_{\partial C(N)}(\beta_N)$ denotes the length of β_N and $\chi(\partial C(N))$ denotes the Euler characteristic of the boundary of the convex core.

Proof of Theorem 1: As the proof is rather technical, we will begin with a brief outline. We first decompose $\partial C(N)$ into the set X of collars of short compressible geodesics and its complement Y. We choose $\epsilon = \sinh^{-1}(1)$ and $L = G(\epsilon)$. By Lemma 4.1

$$\begin{aligned} l_{\partial C(N)}(\beta_N) &= \frac{1}{4L}\int_{T^1(S)} i(\alpha^L(v), \beta_N)d\Omega(v) \\ &= \frac{1}{4L}\left(\int_{T^1(X)} i(\alpha^L(v), \beta_N)d\Omega(v) + \int_{T^1(Y)} i(\alpha^L(v), \beta_N)d\Omega(v)\right). \end{aligned}$$

Corollary 3.1 implies that if $v \in T^1(Y)$, then $i(\alpha^L(v), \beta_N) \leq 2\pi$, so, just as in the proof of Proposition 4.2,

$$\int_{T^1(Y)} i(\alpha^{G(\epsilon)}(v), \beta_N)d\Omega \leq 2\pi \mathrm{vol}(T^1(Y)).$$

To handle the integral over $T^1(X)$, we use Corollary 3.2 which implies that

$$i(\alpha^L(v), \beta_N) \leq 2\pi\left[\frac{L}{G(r(v))}\right]^+.$$

where $r(v)$ is a lower bound on the injectivity radius of $\partial C(N)$ at any point on $\alpha^L(v)$. If B is a component of X with core geodesic ν and $v \in T^1(B)$, we observe that

$$r(v) \geq \sinh^{-1}\left(\frac{1}{e^{G(\epsilon)}}\sinh\left(\frac{l_S(\nu)}{2}\right)\cosh d(v)\right)$$

where $d(v)$ is the distance from the basepoint of v to ν. Combining the resulting bounds and integrating, we obtain an upper bound on the integral of $i(\alpha^L(v), \beta_N)$

over $T^1(B)$ in terms of the length of ν. Summing the resulting bounds over $T^1(Y)$ and all components of $T^1(X)$ gives our result.

Let $\{\nu_1,\ldots,\nu_k\}$ be the primitive closed geodesics of length at most $2\sinh^{-1}(1)$ on $\partial C(N)$. Let $\{B_1,\ldots,B_k\}$ be the collar neighborhoods of $\{\nu_1,\ldots,\nu_k\}$ provided by the Collar Lemma.

Let $\pi : \partial CH(L_\Gamma) \to \partial C(N)$ be the covering map from the boundary of the convex hull to the boundary of the convex core. Let $\epsilon = \sinh^{-1}(1)$ and let

$$\widetilde{V} = \left\{ x \in \partial CH(L_\Gamma) |\ \mathrm{inj}_{\partial CH(L_\Gamma)}(x) \leq \epsilon \right\}.$$

If $x \in \widetilde{V}$, then x lies on a homotopically non-trivial curve n_x of length at most 2ϵ. Since there is a lower bound on the injectivity radius of $\partial CH(L_\Gamma)$, n_x is homotopic to a closed geodesic $\widetilde{\nu}_x$ of length at most 2ϵ. Then $\widetilde{\nu}_x$ projects to (a multiple of) one of the curves $\{\nu_1,\ldots,\nu_k\}$, so $\pi(x)$ lies in some collar neighborhood B_i and x lies in a lift of B_i to $\partial CH(L_\Gamma)$. Let X denote the union of all collar neighborhoods B_i which contain some component of $\pi(\widetilde{V})$. Let $Y = \partial C(N) - X$. We may renumber $\{B_1,\ldots,B_k\}$, so that $X = \cup_{i=1}^m B_i$ for some $m \leq k$. Notice that if $y \in \pi^{-1}(Y)$, then $\mathrm{inj}_{\partial CH(L_\Gamma)}(y) > \epsilon$.

We choose $L = G(\epsilon)$ in the formula for $l_{\partial C(N)}(\beta_N)$ in Lemma 4.1. We split the integral into two integrals using the decomposition, so that

$$l_{\partial C(N)}(\beta_N) = \frac{1}{4G(\epsilon)} \left(\int_{T^1(X)} i(\alpha^{G(\epsilon)}(v), \beta_N) d\Omega + \int_{T^1(Y)} i(\alpha^{G(\epsilon)}(v), \beta_N) d\Omega \right)$$

We first estimate the portion of the integral with domain $T^1(Y)$. If v has basepoint in Y and $\widetilde{\alpha}^{G(\epsilon)}(v)$ is a lift of $\alpha^{G(\epsilon)}(v)$ to $\partial CH(L_\Gamma)$, then $\widetilde{\alpha}^{G(\epsilon)}(v)$ originates at a point $\widetilde{y}$ such that $\mathrm{inj}_{\partial CH(L_\Gamma)}(\widetilde{y}) > \epsilon$ and has length $G(\epsilon) < G(\mathrm{inj}_{\partial CH(L_\Gamma)}(\widetilde{y}))$. Therefore, Lemma 3.1 implies that $i(\widetilde{\alpha}^{G(\epsilon)}(v), \beta_\Gamma) \leq 2\pi$ and hence that $i(\alpha^{G(\epsilon)}(v), \beta_N) \leq 2\pi$. Therefore

$$\int_{T^1(Y)} i(\alpha^{G(\epsilon)}(v), \beta_N) d\Omega \leq \int_{T^1(Y)} 2\pi d\Omega \leq 2\pi \mathrm{vol}(T^1(Y)) \tag{1}$$

We now estimate the portion of the integral with domain $T^1(X)$. If X is empty, then we are already done. Otherwise, let B_i be a component of X. Let $v \in T_1(B_i)$ and $d_i(v)$ be the distance of the basepoint b_v of v from ν_i.

We now derive a lower bound for the injectivity radius along the geodesic $\alpha^{G(\epsilon)}(v)$ as a function of $d_i(v)$. One may readily check that if S is a hyperbolic surface, $w, z \in S$ and $\delta = d_S(z,w)$, then $\sinh(\mathrm{inj}_S(w)) \geq \frac{1}{e^\delta} \sinh(\mathrm{inj}_S(z))$. (One may derive this, for example, from Theorem 7.35.1 in Beardon [2].) Since, by the Collar Lemma,

$\sinh(\mathrm{inj}_S(b_v)) = \sinh\left(\frac{1}{2}l_S(\nu_i)\right)\cosh\left(d_i(v)\right)$, we see that if x is any point on $\alpha^{G(\epsilon)}(v)$, then

$$\sinh\left(\mathrm{inj}_{\partial C(N)}(x)\right) \geq \frac{1}{e^{G(\epsilon)}}\sinh\left(\frac{l_S(\nu_i)}{2}\right)\cosh\left(d_i(v)\right).$$

We define $R_i : [0, w(\nu_i)] \to \mathbf{R}$ by

$$R_i(t) = \sinh^{-1}\left(\frac{1}{e^{G(\epsilon)}}\sinh\left(\frac{l_S(\nu_i)}{2}\right)\cosh t\right).$$

The injectivity radius at every point of $\alpha^{G(\epsilon)}(v)$ is bounded from below by $R_i(d_i(v))$, so if $\tilde{\alpha}^{G(\epsilon)}(v)$ is a lift of $\alpha^{G(\epsilon)}(v)$ to $\partial CH(L_\Gamma)$, the injectivity radius of $\partial CH(L_\Gamma)$ at every point of $\tilde{\alpha}^{G(\epsilon)}(v)$ is also bounded from below by $R_i(d_i(v))$. Thus, by Corollary 3.2,

$$i(\alpha^{G(\epsilon)}(v), \beta_N) \leq 2\pi\left[\frac{G(\epsilon)}{G(R_i(d_i(v)))}\right]^+.$$

So,

$$\begin{aligned}\int_{T^1(B_i)} i(\alpha^{G(\epsilon)}(v), \beta_N)d\Omega &\leq \int_{T^1(B_i)} 2\pi\left[\frac{G(\epsilon)}{G(R_i(d_i(v)))}\right]^+ d\Omega \\ &\leq 2\pi G(\epsilon)\int_{T^1(B_i)}\frac{1}{G(R_i(d_i(v)))}d\Omega + 2\pi\mathrm{vol}(T^1(B_i)) \quad (2)\end{aligned}$$

Since the integral depends only on $d_i(v)$,

$$\int_{T^1(B_i)}\frac{1}{G(R_i(d_i(v)))}d\Omega \leq 2\pi\int_0^{l_S(\nu_i)}\int_{-\omega(\nu_i)}^{\omega(\nu_i)}\frac{1}{G(R_i(|x|))}\cosh x \; dxdl$$

where x and l are the coordinates on B_i provided by the Collar Lemma.

As $R_i(|x|) < \epsilon$ on B_i, we need only consider G on the domain $[0, \epsilon]$. Since $t/G(t)$ tends to 1 as t tends to 0 and is continuous on $(0, \epsilon]$, there exists a constant $K_1 > 0$ such that $t/G(t) \leq K_1$ for all $t \in (0, \epsilon]$. Therefore we have

$$\int_{T^1(B_i)}\frac{1}{G(R_i(d_i(v)))}d\Omega \leq 2\pi\int_0^{l_S(\nu_i)}\int_{-\omega(\nu_i)}^{\omega(\nu_i)}\frac{K_1\cosh x}{R_i(|x|)}dxdl.$$

Integrating over the core curve and making use of the symmetry about the core geodesic, we see that

$$\int_{T^1(B_i)}\frac{1}{G(R_i(d_i(v)))}d\Omega \leq 4\pi K_1 l_S(\nu_i)\int_0^{w(\nu_i)}\frac{\cosh x}{R_i(x)}dx. \quad (3)$$

Since $\sinh x/x$ is increasing on $(0,\infty)$, $\sinh x/x \le K_2 = \sinh\epsilon/\epsilon$ for all $x \in (0,\epsilon]$. Thus, for all $x \in (0, w(\nu_i))$,

$$\frac{1}{R_i(x)} \le \frac{K_2}{\sinh R_i(x)}.$$

Therefore,

$$\int_0^{w(\nu_i)} \frac{\cosh x}{R_i(x)}\,dx \le \int_0^{w(\nu_i)} \frac{K_2 e^{G(\epsilon)}}{\sinh(l(\nu_i)/2)} dx \le \frac{w(\nu_i) K_2 e^{G(\epsilon)}}{\sinh(l(\nu_i)/2)}. \tag{4}$$

Combining inequalities (3) and (4) we see that

$$\int_{T^1(B_i)} \frac{1}{G(R_i(d_i(v)))} d\Omega \le \frac{4\pi K_1 l(\nu_i) w(\nu_i) K_2 e^{G(\epsilon)}}{\sinh(l(\nu_i)/2)}.$$

Since $\sinh(x) \ge x$, we see that

$$\int_{T^1(B_i)} \frac{1}{G(R_i(d_i(v)))} d\Omega \le 8\pi K_1 K_2 e^{G(\epsilon)} w(\nu_i).$$

Applying the fact that $\sinh^{-1}(x) = \log(x + \sqrt{x^2+1})$ we see that

$$w(\nu_i) = \sinh^{-1}\left(\frac{1}{\sinh(\frac{l(\nu_i)}{2})}\right) = \log\left(\frac{1+\cosh(l(\nu_i)/2)}{\sinh(l(\nu_i)/2)}\right).$$

So,

$$w(\nu_i) \le \log\left(1+\cosh\left(\frac{l(\nu_i)}{2}\right)\right) + \log\left(\frac{1}{\sinh(l(\nu_i)/2)}\right) \le \log(1+\cosh\epsilon) + \log\left(\frac{2}{l(\nu_i)}\right).$$

Thus,

$$\int_{T^1(B_i)} \frac{1}{G(R_i(d_i(v)))} d\Omega \le S_0 \log\left(\frac{2}{l(\nu_i)}\right) + T_0 \tag{5}$$

where $S_0 = 8\pi K_1 K_2 e^{G(\epsilon)}$ and $T_0 = S_0 \log(1+\cosh\epsilon)$.

As $X = \cup_{i=1}^m B_i$, we may combine inequalities (2) and (5) to obtain

$$\begin{aligned}
\int_{T^1(X)} i(\alpha^{G(\epsilon)}(v), \beta_N) d\Omega &= \sum_{i=1}^m \int_{T^1(B_i)} i(\alpha^{G(\epsilon)}(v), \beta_N) d\Omega \\
&\le \sum_{i=1}^m 2\pi G(\epsilon)\left(S_0 \log\left(\frac{2}{l(\nu_i)}\right) + T_0\right) + 2\pi \mathrm{vol}(T^1(B_i))
\end{aligned}$$

Since m is bounded above by the number of disjoint geodesics in $\partial C(N)$,

$$m \leq \frac{3}{2}|\chi(\partial C(N))|.$$

Moreover, as ρ_0 is a lower bound for the injectivity radius of $\partial CH(L_\Gamma)$, $\rho_0 \leq l(\nu_i)/2$ for all i. Therefore,

$$\int_{T^1(X)} i(\alpha^{G(\epsilon)}(v), \beta_N) d\Omega \quad \leq \quad 3\pi G(\epsilon)|\chi(\partial C(N))| \left(S_0 \log\left(\frac{1}{\rho_0}\right) + T_0 \right) + 2\pi \mathrm{vol}(T^1(X)). \quad (6)$$

Combining the estimates (1) and (6) for the integral over $T^1(X)$ and $T^1(Y)$, we see that

$$\begin{aligned} \int_{T^1(\partial C(N))} i(\alpha^{G(\epsilon)}(v), \beta_N) d\Omega &= \int_{T^1(X)} i(\alpha^{G(\epsilon)}(v), \beta_N) d\Omega + \int_{T^1(Y)} i(\alpha^{G(\epsilon)}(v), \beta_N) d\Omega \\ &\leq 3\pi G(\epsilon)|\chi(\partial C(N))| \left(S_0 \log\left(\frac{1}{\rho_0}\right) + T_0 \right) \\ &\qquad + 2\pi \mathrm{vol}(T^1(X)) + 2\pi \mathrm{vol}(T^1(Y)) \\ &\leq 3\pi G(\epsilon)|\chi(\partial C(N))| \left(S_0 \log\left(\frac{1}{\rho_0}\right) + T_0 \right) + 2\pi \mathrm{vol}(T^1(\partial C(N))) \end{aligned}$$

Recalling that

$$l_{\partial C(N)}(\beta_N) = \frac{1}{4G(\epsilon)} \int_{T^1(\partial C(N))} i(\alpha^{G(\epsilon)}(v), \beta_N) d\Omega$$

and that $\mathrm{vol}(T^1(\partial C(N))) = 4\pi^2 |\chi(\partial C(N))|$, we see that this implies that

$$l_{\partial C(N)}(\beta_N) \leq |\chi(\partial C(N))| (S \log(\frac{1}{\rho_0}) + T)$$

where $S = \frac{3\pi S_0}{4}$ and $T = \frac{3\pi T_0}{4} + \frac{2\pi^3}{G(\epsilon)}$.

Theorem 1

Remark: One may evaluate the constants used in the proof to check that $\epsilon = \sinh^{-1}(1) \approx .8814$, $G(\epsilon) = F^{-1}(\epsilon) \approx .8387$, $K_1 = \frac{\epsilon}{G(\epsilon)} \approx 1.0509$ (since $t/G(t)$ is increasing), and $K_2 = \frac{\sinh \epsilon}{\epsilon} \approx 1.1346$. Therefore, $S \leq 164$ and $T \leq 218$.

6 A lower bound on the length of the bending lamination

If the boundary of the convex core contains a short compressible curve we obtain a lower bound on the length of the bending lamination which has the same asymptotic form as the upper bound obtained in Theorem 1. Notice that if N is Fuchsian, then the bending lamination has length zero, so no general lower bound is possible.

Theorem 2: *Let $N = \mathbf{H}^3/\Gamma$ be an analytically finite hyperbolic 3-manifold. If $\partial CH(L_\Gamma)$ contains a closed geodesic of length $\rho \leq 2\sinh^{-1}(1)$, then*

$$l_{\partial C(N)}(\beta_N) \geq 4\pi \log\left(\frac{4\sinh^{-1}(1)}{\rho}\right).$$

Proof of Theorem 2: Let $\tilde{\alpha}$ be the closed geodesic of length ρ on $\partial CH(L_\Gamma)$ and let $\epsilon = \sinh^{-1}(1)$. Let α be the projection of $\tilde{\alpha}$ to $\partial C(N)$. It follows from the Collar Lemma that α is a multiple of a simple closed geodesic ν. Let B be the collar of ν provided by the Collar Lemma. The collar B has width $w \geq \sinh^{-1}(1/\sinh(\rho/2))$. Since $\sinh^{-1}(x) = \log(x + \sqrt{x^2+1})$,

$$w \geq \log\left(\frac{1+\cosh(\rho/2)}{\sinh(\rho/2)}\right) \geq \log\left(\frac{2}{\sinh(\rho/2)}\right).$$

Since $\sinh x/x$ is an increasing function on $(0,\infty)$

$$\sinh\left(\frac{\rho}{2}\right) \leq \left(\frac{\sinh\epsilon}{\epsilon}\right)\frac{\rho}{2} = \frac{\rho}{2\epsilon},$$

so

$$w \geq \log\left(\frac{4\epsilon}{\rho}\right).$$

Any leaf of $\beta_N \cap B$ which intersects α, intersects it exactly once and runs from one boundary component of B to the other and has length at least $2w$. Proposition 4 in Lecuire [14], see also Proposition 7 in Bonahon-Otal [5] for the case when β_N is finite-leaved, implies that $i(\alpha, \beta_N) > 2\pi$. Thus, the total (measured) length of $\beta_N \cap B$ is at least $2\pi(2w) = 4\pi w$. Therefore,

$$l_{\partial C(N)}(\beta_N) \geq 4\pi \log\left(\frac{4\epsilon}{\rho}\right)$$

as claimed.

Theorem 2

7 Bounds which depend on the geometry of $\Omega(\Gamma)$

We observed in [7] that a lower bound on the injectivity radius of the boundary of the convex hull implies a lower bound on the injectivity radius of the domain of discontinuity, while in [9] we saw that a short geodesic in the domain of discontinuity implies the existence of an even shorter geodesic in the boundary of the convex hull. Therefore, we may give versions of Theorems 1 and 2 where the constants depend on the geometry of the domain of discontinuity.

If $N = \mathbf{H}^3/\Gamma$ is an analytically finite hyperbolic 3-manifold, then Lemma 8.1 of [7] implies that if r_0 is a lower bound for the injectivity radius of the domain of discontinuity $\Omega(\Gamma)$ of Γ, then

$$\frac{e^{-m}e^{\frac{-\pi^2}{2r_0}}}{2}$$

is a lower bound for the injectivity radius of $\partial CH(L_\Gamma)$ where $m = \cosh^{-1}(e^2)$. Therefore, we obtain the following version of Theorem 1, where $S' = \frac{\pi^2 S}{2}$ and $T' = S\log 2 + Sm + T$.

Theorem $1'$**:** *There exist constants S and T such that if N is an analytically finite hyperbolic 3-manifold, β_N is it bending lamination, and r_0 is a lower bound for the injectivity radius of the domain of discontinuity $\Omega(\Gamma)$, then*

$$l_{\partial C(N)}(\beta_N) \leq |\chi(\partial C(N))| \left(\frac{S'}{r_0} + T'\right)$$

where $l_{\partial C(N)}(\beta_N)$ denotes the length of β_N and $\chi(\partial C(N))$ denotes the Euler characteristic of the boundary of the convex core.

Theorem 5.1 of [9] implies that if $\Omega(\Gamma)$ contains a closed geodesic of length $r \leq 1$, then $\partial CH(L_\Gamma)$ contains a closed geodesic of length at most

$$\frac{4\pi e^{(.502)\pi}}{e^{\frac{\pi^2}{\sqrt{e}r}}} \leq .153\ r.$$

Thus, we obtain the following version of Theorem 2 where $P = \frac{4\pi^3}{\sqrt{e}}$ and $Q = 4\pi \log\left(\frac{4\pi e^{(.502)\pi}}{\sinh^{-1}(1)}\right)$.

Theorem $2'$**:** *There exist postive constants P and Q such that if $N = \mathbf{H}^3/\Gamma$ is an analytically finite hyperbolic 3-manifold, $\Omega(\Gamma)$ contains a closed geodesic of length $r \leq 1$, then*

$$l_{\partial C(N)}(\beta_N) \geq \frac{P}{r} - Q.$$

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