DYNAMICS ON $\mathrm{PSL}(2,\mathbb{C})$-CHARACTER VARIETIES: 3-MANIFOLDS WITH TOROIDAL BOUNDARY COMPONENTS

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ABSTRACT. Let M be a compact, orientable, hyperbolizable 3-manifold with incompressible boundary which is not an interval bundle. We study the dynamics of the action of $\mathrm{Out}(\pi_1(M))$ on the relative $\mathrm{PSL}(2,\mathbb{C})$-character variety $X_T(M)$.

1. Introduction

We continue the investigation of the action of the outer automorphism group $\mathrm{Out}(\pi_1(M))$ of the fundamental group of a compact, orientable, hyperbolizable 3-manifold M with non-abelian fundamental group on its relative $\mathrm{PSL}(2,\mathbb{C})$-character variety

$$X_T(M) = \mathrm{Hom}_T(\pi_1(M), \mathrm{PSL}(2,\mathbb{C})//\mathrm{PSL}(2,\mathrm{C}).$$

Here $\mathrm{Hom}_T(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$ is the space of representations of $\pi_1(M)$ into $\mathrm{PSL}(2,\mathbb{C})$ such that if an element of $\pi_1(M)$ lies in a rank two free abelian subgroup of $\pi_1(M)$ then its image is either parabolic or the identity. The set $AH(M)$ of (conjugacy classes of) discrete, faithful representations is a closed subset of $X_T(M)$ ([16, 22]). One may think of $AH(M)$ as the space of marked, hyperbolic 3-manifolds homotopy equivalent to M. The classical deformation theory of Kleinian groups implies that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on the interior $\mathrm{int}(AH(M))$ of $AH(M)$. Our main theorem describes a domain of discontinuity which is strictly larger than $\mathrm{int}(AH(M))$ in the case that M has non-empty incompressible boundary and is not an interval bundle. Moreover, we characterize when $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on an open neighborhood of $AH(M)$.

Theorem 1.1. *Let M be a compact, orientable, hyperbolizable 3-manifold with nonempty incompressible boundary, which is not an interval bundle. Then there exists an open $\mathrm{Out}(\pi_1(M))$-invariant subset $W(M)$ of $X_T(M)$ such that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $W(M)$, $\mathrm{int}(AH(M))$ is a proper subset of $W(M)$, and $W(M)$ intersects $\partial AH(M)$.*

Our proof yields a domain of discontinuity which contains $AH(M)$ in the case where M contains no primitive essential annuli. We recall that a properly embedded annulus

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$A \subset M$ is *essential* if $\pi_1(A)$ injects into $\pi_1(M)$) and A is not properly homotopic into ∂M. An essential annulus A is *primitive* if $\pi_1(A)$ is a maximal abelian subgroup of $\pi_1(M)$. One may apply results of Canary-Storm [15] to show that if M does contain a primitive essential annulus, then no such domain of discontinuity can exist.

Corollary 1.2. *If M is a compact, orientable, hyperbolizable 3-manifold with incompressible boundary and non-abelian fundamental group, then* $\mathrm{Out}(\pi_1(M))$ *acts properly discontinuously on an open* $\mathrm{Out}(\pi_1(M))$*-invariant neighborhood of $AH(M)$ in $X_T(M)$ if and only if M contains no primitive essential annuli.*

If there are no essential annuli with one boundary component contained in a toroidal boundary component of M, then our main theorem can be extended readily to the full character variety

$$X(M) = \mathrm{Hom}(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))//\mathrm{PSL}(2,\mathrm{C}).$$

Theorem 1.3. *Let M be a compact, orientable, hyperbolizable* 3*-manifold with nonempty incompressible boundary, which is not an interval bundle. If M does not contain an essential annulus with one boundary component contained in a toroidal boundary component of M, then there exists an open* $\mathrm{Out}(\pi_1(M))$*-invariant subset $\hat{W}(M)$ of $X(M)$ such that* $\mathrm{Out}(\pi_1(M))$ *acts properly discontinuously on $\hat{W}(M)$ and*

$$W(M) = \hat{W}(M) \cap X_T(M).$$

In particular, $\hat{W}(M)$ intersects $\partial AH(M)$.

On the other hand, if M does contain an essential annulus with one boundary component in a toroidal boundary component of M, then no point in $AH(M)$ can lie in a domain of discontinuity for the action of $\mathrm{Out}(\pi_1(M))$.

Proposition 1.4. *Let M be a compact, orientable, hyperbolizable* 3*-manifold with nonempty incompressible boundary and non-abelian fundamental group. If M contains an essential annulus with one boundary component contained in a toroidal boundary component of M, then every point in $AH(M)$ is a limit of representations in $X(M)$ which are fixed points of infinite order elements of* $\mathrm{Out}(\pi_1(M))$.

Corollary 1.2 then extends to:

Corollary 1.5. *If M is a compact, orientable, hyperbolizable 3-manifold with incompressible boundary and non-abelian fundamental group, then* $\mathrm{Out}(\pi_1(M))$ *acts properly discontinuously on an open,* $\mathrm{Out}(\pi_1(M))$*-invariant neighborhood of $AH(M)$ in $X(M)$ if and only if M does not contain a primitive essential annulus or an essential annulus with one boundary component contained in a toroidal boundary component of M.*

Historical overview: One may view the study of actions of outer automorphism groups on character varieties as a natural generalization of the study of the action of mapping class groups on Teichmüller spaces. When S is a closed, oriented, hyperbolic surface, then the mapping class group $\mathrm{Mod}(S)$ acts properly discontinuously on Teichmüller space $\mathcal{T}(S)$. Teichmüller space may be identified with a component of the $\mathrm{PSL}(2,\mathbb{R})$-character variety $X_2(S)$ of $\pi_1(S)$ and $\mathrm{Mod}(S)$ is identified with an index two subgroup of $\mathrm{Out}(\pi_1(S))$. Goldman [18] showed that $X_2(S)$ has $4g-3$ components, one of which is identified with $\mathcal{T}(S)$ and another of which is identified with $\mathcal{T}(\overline{S})$, where $\bar{S}$ is S given the opposite orientation. The representations in the other components are not discrete and faithful. The mapping class group preserves each component of $X_2(S)$ and acts properly discontinuously on the components associated to $\mathcal{T}(S)$ and $\mathcal{T}(\overline{S})$. Goldman has conjectured that $\mathrm{Mod}(F)$ acts ergodically on each of the remaining $4g-5$ components. A resolution of Goldman's conjecture would give a very satisfying dynamical decomposition.

In this case when $M = S \times [0,1]$ is an untwisted interval bundle, $\mathrm{Out}(\pi_1(S))$ acts properly discontinuously on the interior $QF(S)$ of $AH(S\times[0,1])$. $QF(S)$ is the space of quasifuchsian representations, i.e. the convex cocompact representations. Goldman conjectured that the maximal domain of discontinuity for the action of $\mathrm{Out}(\pi_1(S))$ on $X(S\times[0,1])$ is exactly $QF(S)$. For evidence in support of Goldman's conjectures see Bowditch [7], Lee [26], Souto-Storm [39], and Tan-Wong-Zhang [40]. In particular, it is known that no point in the boundary of $QF(S)$ can lie in a domain of discontinuity for $\mathrm{Out}(\pi_1(S))$ (see Lee [26]).

Minsky [33] studied the case where M is a handlebody H_g and exhibited a domain of discontinuity PS_g for $\mathrm{Out}(\pi_1(H_g)) = \mathrm{Out}(F_g)$ which is strictly larger than the the set of convex cocompact representations, and in fact contains representations which are neither discrete nor faithful. (Lee [27] has generalized Minsky's results in [33] to the more general case where M is a compression body.) Gelander and Minsky [17] showed that $\mathrm{Out}(F_g)$ acts ergodically on the set R_g of redundant representations. It remains open whether or not $R_g \cup PS_g$ has full measure in $X(H_g)$.

If M is a twisted interval bundle, Lee [26] exhibits an explicit domain of discontinuity $U(M)$ for the action of $\mathrm{Out}(\pi_1(M))$ on $X(M)$ which is strictly larger than $\mathrm{int}(AH(M))$ and contains points in $\partial AH(M)$ and points in the complement of $AH(M)$. Moreover, she shows that if $\rho \in AH(M) - U(M)$, then ρ cannot lie in a domain of discontinuity for the action of $\mathrm{Out}(\pi_1(M))$ on $X(M)$.

Canary and Storm [15] studied the case where M has non-empty incompressible boundary and has no toroidal boundary components. If M is not an interval bundle, they again exhibited a domain of discontinuity for the action of $\mathrm{Out}(\pi_1(M))$ which is strictly larger than the interior of $AH(M)$. Moreover, they showed that there is a domain of discontinuity for the action of $\mathrm{Out}(\pi_1(M))$ which contains $AH(M)$ if and only if M contains no primitive essential annuli. Our results build on their results. The major new difficulties in our case result from the facts that the characteristic submanifold can contain thickened tori and that that the mapping class group of M

need not have finite index in $\mathrm{Out}(\pi_1(M))$ (see Canary-McCullough [14]). Our analysis of $\mathrm{Out}(\pi_1(M))$ is necessarily much more intricate than what was developed in [15].

Outline of argument: Our proof relies on exhibiting a finite index subgroup of $\mathrm{Out}(\pi_1(M))$ which is built from groups of homotopy equivalences associated to components of the characteristic submanifold $\Sigma(M)$ of M. We then consider subgroups of $\pi_1(M)$ which are preserved by these groups of homotopy equivalences and study the action of the outer automorphism group of the subgroup of $\pi_1(M)$ on its associated relative character variety. We can combine these separate analyses to construct our domain of discontinuity for the action of $\mathrm{Out}(\pi_1(M))$.

If M is a compact, orientable, hyperbolizable 3-manifold with incompressible boundary, then its characteristic submanifold $\Sigma(M)$ consists of solid tori, thickened tori and interval bundles. Johannson [21] showed that every homotopy equivalence can be homotoped so that it preserves $\Sigma(M)$ and restricts to a homeomorphism of $M - \Sigma(M)$. He also showed that only finitely many homotopy classes of homeomorphisms of $M - \Sigma(M)$ arise, so one can restrict to a finite index subgroup of $\mathrm{Out}(\pi_1(M))$ such that every automorphism is realized by a homeomorphism which restricts to the identity on $M - \Sigma(M)$. In section 3, we build on techniques developed by McCullough [31] and Canary-McCullough [14] to construct a finite index subgroup $\mathrm{Out}_0(\pi_1(M))$ of $\mathrm{Out}(\pi_1(M))$ and a short exact sequence

$$1 \longrightarrow B \to \mathrm{Out}_0(\pi_1(M)) \xrightarrow{\Phi} A \longrightarrow 1$$

where A is a direct product of mapping class groups of base surfaces of interval bundle components of $\Sigma(M)$ and cyclic subgroups generated by Dehn twists in vertical annuli in thickened torus components of M and B is the direct product of the free abelian groups generated by Dehn twists in frontier annuli of $\Sigma(M)$ and free abelian groups generated by "sweeps" in thickened torus components of $\Sigma(M)$. Guirardel and Levitt [19] have recently developed a related short exact sequence for outer automorphism groups of certain classes of relatively hyperbolic groups.

In section 4 we decompose the frontier of $\Sigma(M)$ into characteristic collections of annuli each of which is either the entire frontier of a solid torus or thickened torus component of $\Sigma(M)$ or is a single component of the frontier of an interval bundle component of $\Sigma(M)$. If a characteristic collection of annuli C is the frontier of a solid torus or a component of the frontier of the interval bundle, we construct a class of free subgroups of $\pi_1(M)$ which register C, in the sense that the group generated by Dehn twists in C injects into the outer automorphism group of the free subgroup. If C is the frontier of a thickened torus component, then our registering subgroups are the free product of the fundamental group of the thickened torus component and a free group. One may use Klein's combination theorem (i.e. the ping-pong lemma), to show that such registering subgroups exist (see section 5). This entire analysis generalizes the analysis of the mapping class group $\mathrm{Mod}(M)$ of M used in Canary-Storm [15] in the case that M has no toroidal boundary components.

In section 6, we show that if H is a registering subgroup, then $\mathrm{Out}(H)$ acts properly discontinuously on the set $GF(H)$ of geometrically finite, minimally parabolic, discrete, faithful representations and that $GF(H)$ is an open subset of $X_T(H)$. Similarly, if Σ is an interval bundle component of $\Sigma(M)$, we find an open subset $GF(\Sigma, \partial_1\Sigma)$ of the appropriate relative character variety on which $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously. Our region $W(M) \subset X_T(M)$ is defined to consist of representations ρ so that for every characteristic collection of annuli there is a registering subgroup H such that $\rho|_H \in GF(H)$ and for every interval bundle component Σ of $\Sigma(M)$, $\rho|_{\pi_1(\Sigma)} \in GF(\Sigma, \partial_1\Sigma)$. Proposition 7.2 establishes that $W(M)$ is an open $\mathrm{Out}(\pi_1(M))$-invariant subset of $X_T(M)$ which contains all discrete, faithful, minimally parabolic representations. In particular, $\mathrm{int}(AH(M))$ is a proper subset of $W(M)$ and $W(M) \cap \partial AH(M) \neq \emptyset$.

Proposition 8.1 shows that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $W(M)$, which completes the proof of our main result. In the proof we consider a sequence $\{\alpha_n\}$ of distinct elements of $\mathrm{Out}_0(\pi_1(M))$. We can restrict to a subsequence so that either (1) there exists a thickened torus or interval bundle component V of $\Sigma(M)$ so that α_n gives rise to a sequence of distinct elements of $\mathrm{Out}(\pi_1(V))$, or (2) there exists a fixed element γ such that $\alpha_n = \beta_n \circ \gamma$ and a characteristic collection of annuli C such that each β_n preserves every registering subgroup H for C and $\{\beta_n\}$ gives rise to a sequence of distinct elements of $\mathrm{Out}(H)$. Let D be a compact subset of $W(M)$. We then use the proper discontinuity of the actions of $\mathrm{Out}(\pi_1(V))$ and $\mathrm{Out}(H)$ to show that $\{\alpha_n(D)\}$ leaves every compact set. This allows us to conclude that $\mathrm{Out}_0(\pi_1(M))$ and hence $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $W(M)$.

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2. Background

2.1. Deformation spaces of hyperbolic 3-manifolds and character varieties.

Let M be a compact, orientable, hyperbolizable 3-manifold with incompressible boundary. Recall that Thurston (see Morgan [34]) proved that a compact, orientable 3-manifold with non-empty boundary is hyperbolizable (i.e., its interior admits a complete hyperbolic metric) if and only if it is atoroidal and irreducible. We will assume throughout the remainder of the paper that M has non-abelian fundamental group. Let $\partial_T M$ denote the non-toroidal boundary components of M.

The action of $\mathrm{Out}(\pi_1(M))$ on the character variety $X(M)$ and the relative character variety $X_T(M)$ is given by

$$\alpha([\rho]) = [\rho \circ \alpha^{-1}]$$

where $\alpha \in \mathrm{Out}(\pi_1(M))$ and $[\rho] \in X(M)$. (See Kapovich [23, Chapter 4.3] for a discussion of the character variety and the relative character variety.)

Sitting within $X_T(M)$ is the space $AH(M)$ of (conjugacy classes of) discrete, faithful representations. If $\rho \in AH(M)$, then $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$ is a hyperbolic 3-manifold and there exists a homotopy equivalence $h_\rho : M \to N_\rho$ in the homotopy class determined by ρ. One may thus think of $AH(M)$ as the space of marked hyperbolic 3-manifolds homotopy equivalent to M.

The interior $\mathrm{int}(AH(M))$ of $AH(M)$, as a subset of $X_T(M)$, consists of discrete, faithful representations which are geometrically finite and minimally parabolic (see Sullivan [41]). A representation $\rho \in AH(M)$ is *geometrically finite* if there is a convex, finite-sided fundamental polyhedron for the action of $\rho(\pi_1(M))$ on $\mathbb{H}^3$. It is *minimally parabolic* if $\rho(g)$ is parabolic if and only if g is a non-trivial element of a rank two free abelian subgroup of $\pi_1(M)$.

The components of $\mathrm{int}(AH(M))$ are in one-to-one correspondence with the set $\mathcal{A}(M)$ of marked, compact, oriented, hyperbolizable 3-manifolds homotopy equivalent to M and each component is parameterized by an appropriate Teichmüller space (see Bers [4] or Canary-McCullough [14, Chapter 7] for complete discussions of this theory).

If $\rho \in AH(M)$, then there is a compact core M_ρ for N_ρ, i.e. a compact submanifold of N_ρ such that the inclusion is a homotopy equivalence (see Scott [38]). We may assume that $h_\rho(M) \subset M_\rho$, so that $h : M \to M_\rho$ is a homotopy equivalence. Formally, $\mathcal{A}(M)$ is the set of pairs (M', h') where M' is a compact, oriented, hyperbolizable 3-manifold and $h' : M \to M'$ is a homotopy equivalence, where (M_1, h_1) is equivalent to (M_2, h_2) if and only if there is an orientation-preserving homeomorphism $j : M_1 \to M_2$ such that $j \circ h_1$ is homotopic to h_2. There is a natural map

$$\Theta : \mathrm{int}(AH(M)) \to \mathcal{A}(M)$$

given by $\Theta(\rho) = [(M_\rho, h_\rho)]$. Thurston's Geometrization Theorem (see [34]) implies that Θ is surjective, while Marden's Isomorphism Theorem [28] implies that the preimage of any element of $\mathcal{A}(M)$ is a component. The work of Bers [3], Kra [24] and Maskit [29], then implies that if $(M', h') \in \mathcal{A}(M)$, then

$$\Theta^{-1}(M', h') \cong \mathcal{T}(\partial_T M')$$

where $\mathcal{T}(\partial_T M')$ is the Teichmüller space of marked conformal structures on the non-toroidal components of $\partial M'$.

One may use this parameterization to show that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $\mathrm{int}(AH(M))$.

Proposition 2.1. *If M is a compact, orientable, hyperbolizable 3-manifold with incompressible boundary, then* $\mathrm{Out}(\pi_1(M))$ *acts properly discontinuously on* $\mathrm{int}(AH(M))$.

Proof. Notice that $\mathrm{Out}(\pi_1(M))$ preserves $\mathrm{int}(AH(M))$. If Q is a component of $\mathrm{int}(AH(M))$ and $\Theta(Q) = (M', h')$, let

$$\mathrm{Mod}_+(M', h') \subset \mathrm{Out}(\pi_1(M))$$

denote the set of outer automorphisms which preserve Q. An outer automorphism α lies in $\mathrm{Mod}_+(M', h')$ if and only if $(h')_* \circ \alpha \circ (h')_*^{-1}$ is realized by an orientation-preserving homeomorphism of M'. Thus, the action of $\mathrm{Mod}_+(M', h')$ on $Q \cong \mathcal{T}(\partial_T M')$ may be identified with the action of a subgroup of $\mathrm{Mod}(\partial_T M')$ on $\mathcal{T}(\partial_T M')$. Therefore, since $\mathrm{Mod}(\partial_T M')$ acts properly discontinuously on $\mathcal{T}(\partial_T M')$, $\mathrm{Mod}_+(M', h')$ acts properly discontinuously on Q. So, $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $\mathrm{int}(AH(M))$. □

Remark: Proposition 2.1 remains true when M has compressible boundary. The proof above must be altered to take into account that components of $\mathrm{int}(AH(M))$ are identified with quotients of the relevant Teichmüller spaces.

2.2. **The characteristic submanifold.** If M is a compact, orientable, hyperbolizable 3-manifold with incompressible boundary, its characteristic submanifold $\Sigma(M)$ contains only interval bundles, solid tori and thickened tori and the frontier $Fr(\Sigma(M))$ consists entirely of essential annuli. The result below recalls the key properties of the characteristic submanifold in our setting. (The general theory of the characteristic submanifold was developed by Jaco-Shalen [20] and Johannson [21]). For a discussion of the characteristic submanifold in our hyperbolic setting see Morgan [34, Sec. 11] or Canary-McCullough [14, Chap. 5]).)

Theorem 2.2. *Let M be a compact, orientable, hyperbolizable 3-manifold with incompressible boundary. There exists a codimension zero submanifold $\Sigma(M) \subseteq M$ with frontier $Fr(\Sigma(M)) = \overline{\partial\Sigma(M) - \partial M}$ satisfying the following properties:*

1. *Each component Σ_i of $\Sigma(M)$ is either*
   (i) an interval bundle over a compact surface with negative Euler characteristic which intersects ∂M in its associated ∂I-bundle,
   (ii) a thickened torus such that $\partial M \cap \Sigma_i$ contains a torus, or
   (iii) a solid torus.
2. *The frontier $Fr(\Sigma(M))$ is a collection of essential annuli.*
3. *Any essential annulus in M is properly isotopic into $\Sigma(M)$.*
4. *If X is a component of $M - \Sigma(M)$, then either $\pi_1(X)$ is non-abelian or $(\overline{X}, Fr(X)) \cong (S^1 \times [0,1] \times [0,1], S^1 \times [0,1] \times \{0,1\})$ and X lies between an interval bundle component of $\Sigma(M)$ and a thickened or solid torus component of $\Sigma(M)$. Moreover, the component of $\Sigma(M) \cup X$ which contains X is not an interval bundle which intersects ∂M in its associated ∂I-bundle.*

A submanifold with these properties is unique up to isotopy, and is called the characteristic submanifold *of M.*

Remark: In Johannson's work, every toroidal boundary component is contained in some component of the characteristic submanifold. We use Jaco and Shalen's definition which requires that no component of the frontier of the characteristic submanifold be properly homotopic into the boundary. In our setting, one obtains Jaco and Shalen's characteristic submanifold from Johannson's characteristic submanifold by simply removing components which are regular neighborhoods of toroidal boundary components.

Johannson [21] proved that every homotopy equivalence between compact, orientable, irreducible 3-manifolds with incompressible boundary may be homotoped so that it preserves the characteristic submanifold and is a homeomorphism on its complement.

Johannson's Classification Theorem: ([21, Theorem 24.2]) *Let M and Q be compact, orientable, irreducible 3-manifolds with incompressible boundary and let $h : M \to Q$ be a homotopy equivalence. Then h is homotopic to a map $g : M \to Q$ such that*

(1) $g^{-1}(\Sigma(Q)) = \Sigma(M)$,
(2) $g|_{\Sigma(M)} : \Sigma(M) \to \Sigma(Q)$ *is a homotopy equivalence, and*
(3) $g|_{\overline{M-\Sigma(M)}} : \overline{M - \Sigma(M)} \to \overline{Q - \Sigma(Q)}$ *is a homeomorphism.*

Moreover, if h is a homeomorphism, then g is a homeomorphism.

2.3. **Ends of hyperbolic 3-manifolds and the Covering Theorem.** In this section, we recall the Covering Theorem which will be used to show that minimally parabolic, discrete faithful representations lie in our domain of discontinuity (see Proposition 7.2).

We first discuss the ends of the non-cuspidal portion N^0 of a hyperbolic 3-manifold with finitely generated fundamental group. Let $N = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold with finitely generated fundamental group. A *precisely invariant system of horoballs* $\mathcal{H}$ for Γ is a Γ-invariant collection of disjoint open horoballs based at parabolic fixed points of Γ, such that there is a horoball based at every parabolic fixed point. It is a consequence of the Margulis Lemma (see [2, Theorems D.2.1, D.2.2] or [30, II.E.3, IV.J.17]) that every Kleinian group has a precisely invariant system of horoballs. Let

$$N^0 = (\mathbb{H}^3 - \mathcal{H})/\Gamma.$$

Each component of ∂N^0 is either an incompressible torus or an incompressible infinite annulus. A *relative compact core* for N^0 is a compact submanifold R of N^0 such that the inclusion of R into N is a homotopy equivalence and R contains every toroidal component of ∂N^0 and intersects every annular component of ∂N^0 in an incompressible annulus. (McCullough [31] and Kulkarni-Shalen [25] established the existence of a relative compact core). The *ends* of N^0 are in one-to-one correspondence with the components of $N^0 - R$ (see Bonahon [5, Proposition 1.3]). An end E of N^0 is *geometrically finite* if there exists a neighborhood of E which does not contain any closed geodesics. Otherwise, the end is called *geometrically infinite*. A hyperbolic 3-manifold N with finitely generated fundamental group is *geometrically finite* if and only if each end of N^0 is geometrically finite (see Bowditch [6] for the equivalence of the many definitions of geometric finiteness for a hyperbolic 3-manifold).

The Covering Theorem asserts that geometrically infinite ends usually cover finite-to-one. (The version of the Covering Theorem we state below incorporates the Tameness Theorem of Agol [1] and Calegari-Gabai [11].)

Covering Theorem: (Thurston [42], Canary [12]) *Let N be a hyperbolic 3-manifold with finitely generated fundamental group which covers another hyperbolic 3-manifold $\hat{N}$ by a local isometry $\pi : N \to \hat{N}$. If E is a geometrically infinite end of N^0, then either*

a) E has a neighborhood U such that π is finite-to-one on U, or

b) $\hat{N}$ has finite volume and has a finite cover N' which fibers over the circle such that, if N_S denotes the cover of N' associated to the fiber subgroup, then N is finitely covered by N_S.

3. The outer automorphism group

In this section, we introduce a finite index subgroup $\mathrm{Out}_0(\pi_1(M))$ of $\mathrm{Out}(\pi_1(M))$ and show that there exists a short exact sequence

$$1 \longrightarrow K(M) \oplus (\oplus_i \mathrm{Sw}(T_i)) \longrightarrow \mathrm{Out}_0(\pi_1(M)) \longrightarrow (\oplus_i D(T_i)) \oplus (\oplus_j E(\Sigma_j, Fr(\Sigma_j)) \longrightarrow 1$$

where $K(M)$ is generated by Dehn twists in annuli of $Fr(\Sigma(M))$, each $\mathrm{Sw}(T_i)$ is a free abelian group generated by sweeps supported on a thickened torus component T_i of $\Sigma(M)$, each $D(T_i)$ is an infinite cyclic group generated by a Dehn twist in a vertical annulus in a thickened torus component T_i of $\Sigma(M)$, and each $E(\Sigma_j, Fr(\Sigma_j))$ is identified with the mapping class group of the base surface of an interval bundle component Σ_j of $\Sigma(M)$. Our proof combines work of Johannson [21] and Canary-McCullough [14] with a new explicit analysis of homotopy equivalences associated to thickened torus components of the characteristic submanifold. A similar short exact sequence for a finite index subgroup of the mapping class group $\mathrm{Mod}(M)$ was developed by McCullough [32] and used in a crucial manner in [15]. Guirardel and Levitt [19] have developed a related short exact sequence for finite index subgroups of the outer automorphism groups of torsion-free, one-ended relatively hyperbolic groups which are hyperbolic relative to families of free abelian subgroups.

3.1. A first short exact sequence and $K(M)$.

Let $\mathrm{Out}_2(\pi_1(M))$ denote the subgroup of $\mathrm{Out}(\pi_1(M))$ consisting of outer automorphisms which are realized by homotopy equivalences which preserves $\Sigma(M)$ and restrict to the identity on $M - \Sigma(M)$. Lemma 10.1.7 (see also Theorem 10.1.9) in [14] implies that $\mathrm{Out}_2(\pi_1(M))$ has finite index in $\mathrm{Out}(\pi_1(M))$.

If V is a component of $\Sigma(M)$, let $E(V, Fr(V))$ be the group of path components of the space of homotopy equivalences of V which restrict to homeomorphisms of $Fr(V)$ which are isotopic to the identity. Note that with this definition, a Dehn twist about a frontier annulus of V is a trivial element of $E(V, Fr(V))$. Proposition 10.1.4 in [14] guarantees that the obvious homomorphism

$$\Psi : \mathrm{Out}_2(\pi_1(M)) \to \oplus E(V_i, Fr(V_i))$$

is well-defined, where the sum is taken over all components of $\Sigma(M)$. Lemma 10.1.8 in [14] implies that Ψ is surjective.

We next show that the kernel $K(M)$ of Ψ is generated by Dehn twists about frontier annuli. This generalizes Lemma 4.2.2 in McCullough [32].

Lemma 3.1. *The kernel $K(M)$ of Ψ is generated by Dehn twists about the frontier annuli of $\Sigma(M)$.*

Proof. If α lies in the kernel of Ψ, then it has a representative which is trivial on $M - \Sigma(M)$, preserves $\Sigma(M)$ and its restriction to $\Sigma(M)$ is homotopic to the identity via a homotopy preserving $Fr(\Sigma(M))$. We may therefore choose the representative $h : M \to M$ to be the identity off of a regular neighborhood $\mathcal{N}$ of $Fr(\Sigma(M))$.

We may choose coordinates so that $\mathcal{N} \cong Fr(\Sigma(M)) \times [-1, 1]$ and $Fr(\Sigma(M) \subset M$ is identified with $Fr(\Sigma(M) \times \{0\}$ in these coordinates. We further choose a Euclidean metric on $Fr(\Sigma(M))$ so that each component is a straight cylinder with geodesic boundary. We can then homotope h on $\mathcal{N}$ so that each arc of the form $\{x\} \times [-1, 1]$ is taken to a geodesic in the product Euclidean metric on $\mathcal{N}$. It is easy to check that the resulting map is a product of Dehn twists in the components of $Fr(\Sigma(M))$. □

We obtain a first approximation to our desired short exact sequence by considering:

$$1 \longrightarrow K(M) \longrightarrow \mathrm{Out}_2(\pi_1(M)) \xrightarrow{\Psi} \oplus_i E(V_i, Fr(V_i)) \longrightarrow 1.$$

3.2. **The analysis of $E(V, Fr(V))$.** Our next goal is to understand $E(V, Fr(V))$ in the various cases. We first recall that $E(V, Fr(V))$ is finite when V is a solid torus component of $\Sigma(M)$.

Lemma 3.2. ([14, Lemma 10.3.2]) *Let M be a compact, orientable, hyperbolizable 3-manifold with incompressible boundary. If V is a solid torus component of $\Sigma(M)$, then $E(V, Fr(V))$ is finite.*

If Σ is an interval bundle component of $\Sigma(M)$ with base surface F, then we say Σ is *tiny* if its base surface is either a thrice-punctured sphere or a twice-punctured projective plane.

The following result combines Propositions 5.2.3 and 10.2.2 in [14], see also the discussion in Section 5 in Canary-Storm [15].

Lemma 3.3. *Let M be a compact, orientable, hyperbolizable 3-manifold with incompressible boundary. Suppose Σ is an interval bundle component of $\Sigma(M)$ whose base surface F has negative Euler characteristic.*

1. *$E(\Sigma, \partial\Sigma)$ is identified with the group $\mathrm{Mod}_0(F, \partial F)$ of (isotopy classes of) homeomorphisms of F whose restriction to the boundary is isotopic to the identity.*
2. *$E(\Sigma, Fr(\Sigma))$ injects into $\mathrm{Out}(\pi_1(\Sigma))$.*
3. *$E(\Sigma, \partial\Sigma)$ is finite if and only if Σ is tiny.*

It remains to analyze the case when T is a thickened torus component of $\Sigma(M)$. We view $(T, Fr(T))$ as a S^1-bundle over (B, b) where B is an annulus and b is a non-empty collection of arcs in one boundary component $\partial_1 B$ of B, so that

$$(T, Fr(T)) = (B \times S^1, b \times S^1).$$

Let $\partial_0 B$ denote the other boundary component. Let $p_1 : T \to B$ and $p_2 : T \to S^1$ be the projections onto the two factors and let $s : B \to T$ be the section of p_1 given by $s(b) = (b, 1)$. Proposition 10.2.2 in [14] guarantees that if $f : (T, Fr(T)) \to (T, Fr(T))$ is a homotopy equivalence, then it is homotopic, as a map of pairs, to a fibre-preserving homotopy equivalence $\bar{f} : (T, Fr(T)) \to (T, Fr(T))$. Moreover, there is a homomorphism

$$P : E(T, Fr(T)) \to E(B, b)$$

given by letting $P([f]) = [p_1 \circ \bar{f} \circ s]$ where $E(B, b)$ is the group of path components of the space of homotopy equivalences of B which restrict to homeomorphisms of b which are isotopic to the identity. We will analyze $E(B, b)$ and the kernel of P in order to understand $E(T, \partial T)$.

If γ is an arc in B with boundary in $\partial_1 B - b$ and β is a loop based at a point x on γ, then one may define a *sweep* $h(\gamma, \beta) : (B, b) \to (B, b)$ by requiring that h fixes the complement of a regular neighborhood N of γ and maps a transversal t of N through x to $t_1 * \beta * t_2$ (where $t = t_1 * t_2$ and t_1 and t_2 intersect at x). See Figure 1. Since $(T, Fr(T)) = (B, b) \times S^1$, we may define a sweep $H(\gamma, \beta) : (T, Fr(T)) \to (T, Fr(T))$ where $H(\gamma, \beta) = h(\gamma, \beta) \times id_{S^1}$. Sweeps are discussed more fully and in greater generality in section 10.2 of [14].

Let $b = \{b_1, \dots, b_n\}$ where each b_j is an arc and b_j is adjacent to b_{j+1} on ∂B. Let β be a core curve of B. Let γ_j be an embedded arc in B joining the components of $\partial_1 B - b$ adjacent to b_j which intersects β at a single point x_j.

Let $E_0(B, b)$ denote the index two subgroup of $E(B, b)$ consisting of elements inducing the identity map on $\pi_1(B)$.

Lemma 3.4. *If B is an annulus and $b = \{b_1, \dots, b_n\}$ is a non-empty collection of arcs in one component of ∂B, then $E_0(B, b)$ is generated by $\{h(\gamma_j, \beta)\}_{j=1}^{n-1}$. Moreover, $E_0(B, b) \cong \mathbb{Z}^{n-1}$.*

We will call $\{h(\gamma_j, \beta)\}_{j=1}^{n-1}$ a *generating system of sweeps* for $E_0(B, b)$.

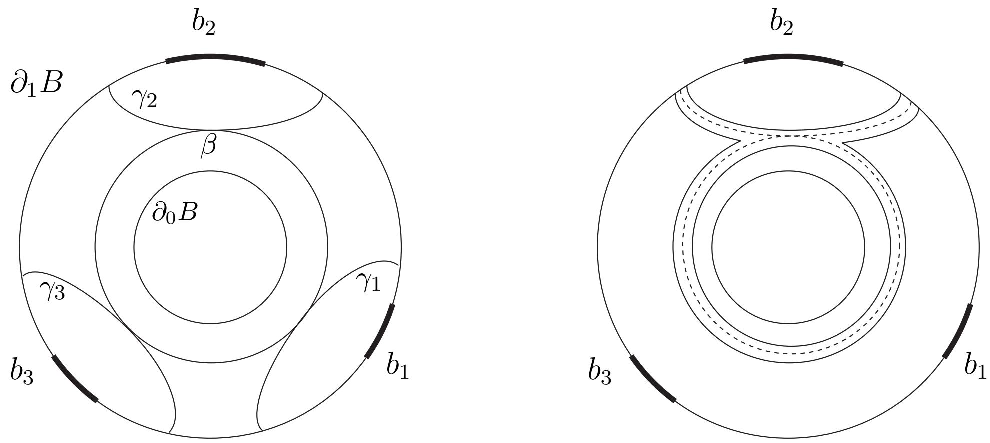

FIGURE 1. The core curve of B is β and $b = \{b_1, b_2, b_3\}$ are three arcs in $\partial_1 B$. On the right is the image of a neighborhood of γ_2 after a sweep $h(\gamma_2, \beta)$.

Proof. We fix an identification of B with $S^1 \times [0,1]$. Choose a collection $\{\lambda_1, \dots, \lambda_n\}$ of disjoint radial arcs in B such that each $\lambda_j = \{a_j\} \times [0,1]$ where $a_j \in b_j$. See Figure 2.

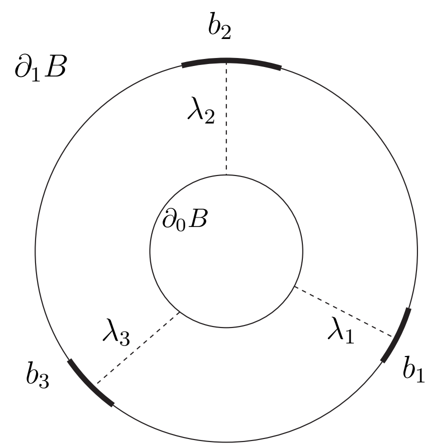

FIGURE 2. The annulus (B, b) when $n = 3$. The arcs λ_j divide B into disks.

Given an element of $E_0(B, b)$ we can choose a representative homotopy equivalence f which is the identity on $b \cup \partial_0 B$. We may further choose f so that f fixes each point on λ_n. Let $p : S^1 \times [0,1] \to S^1$ be projection. Then, for each j, $(p \circ f)_*(\lambda_j)$ will be an element of $\pi_1(S^1, p(a_j))$. For each j, fix an identification of $\pi_1(S^1, p(a_j))$ with $\mathbb{Z}$. We define a map

$$\psi : E_0(B, b) \to \mathbb{Z}^{n-1}$$

by letting

$$\psi([f]) = ((p \circ f)_*(\lambda_1), \ldots, (p \circ f)_*(\lambda_{n-1})).$$

We first show that ψ is well-defined. Suppose that $f_1 : B \to B$ is another choice of representative of $[f] \in E_0(B, b)$ which is the identity on $b \cup \partial_0 B \cup \lambda_n$. Since $[f] = [f_1]$ in $E_0(B, b)$, there is a homotopy $F : B \times [0, 1]$ from $f = f_0$ and f_1, such that $F(\cdot, t) : \partial_0 B \to \partial_0 B$ is a homeomorphism isotopic to the identity for all t, We may deform F to a new homotopy, still called F, so that $F(\{x\} \times [0, 1])$ is a geodesic, in the product Euclidean metric on B, for all $x \in B$. In particular, for all t, $F(\cdot, t) : \partial_0 B \to \partial_0 B$ is a rotation. The fact that $(p \circ f_1)_*(\lambda_n) = (p \circ f_0)_*(\lambda_n)$ implies that $F(\cdot, t) : \partial_0 B \to \partial_0 B$ is the identity for all t, so that the homotopy is constant on $\partial_0 B$. Therefore, $(p \circ f_0)_*(\lambda_j) = (p \circ f_1)_*(\lambda_j)$ for $j = 1, \ldots, n-1$, and so ψ is well-defined.

Notice that if f and g are representatives of $[f]$ and $[g]$ in $E_0(B, b)$ which are the identity on $b \cup \partial_0 B \cup \lambda_n$, then $f \circ g$ is a representative of $[f][g]$ which is the identity on $b \cup \partial_0 B \cup \lambda_n$, so ψ is a homomorphism.

One may easily check that $\psi(h(\gamma_j, \beta)) = (0, \ldots, \pm 1, \ldots 0)$ for all j (where the only non-zero entry is in the j^{th} place. In particular, ψ is surjective.

The proof will be completed by showing that ψ is injective. The collection of arcs $\{\lambda_j\}$ divides B into n disks $\{D_1, \ldots, D_n\}$. Each D_i has the form $[a_j, a_{j+1}] \times [0, 1]$ where indices are taken modulo n. Suppose that $\psi(f) = 0$. We may assume as above, that f is the identity on $b \cup \partial_0 B \cup \lambda_n$. Since $\psi(f) = 0$, we can further homotope f, keeping it the identity on $b \cup \partial_0 B \cup \lambda_n$, so that f fixes λ_j for all $j = 1, \ldots, n-1$. Finally, we homotope f, keeping it the identity on $b \cup \partial_0 B \cup \lambda_1 \cup \cdots \lambda_n$, so that for each j and each $t \in [0, 1]$, $f([a_j, a_{j+1}]) \times \{t\}$ is a geodesic. This final map must be the identity map, so we have shown that $[f] = [id]$ in $E_0(B, b)$ which completes the proof. ☐

If a is an embedded arc in B joining the two boundary components of B such that $a \cap b = \emptyset$, then $A = p_1^{-1}(a)$ is called a *vertical* essential annulus for T. We see that Dehn twists about a vertical annulus generate the kernel of P.

Lemma 3.5. *Let T be a thickened torus component of $\Sigma(M)$ and let A be a vertical essential annulus in T. Then the kernel of P is generated by the Dehn twist D_A about A. In particular, $ker(P) \cong \mathbb{Z}$.*

Proof. Let $f : T \to T$ be a homotopy equivalence which restricts to a homeomorphism of $Fr(T)$ which is isotopic to the identity and such that $P([f]) = [id]$. Proposition 10.2.2 of [14] allows us to assume that f is fiber-preserving and that $p_1 f s = id$. Then the homotopy class of f is determined by the homotopy class of the map $p_2 f s : B \to S^1$. However, every homotopy class of map from B to S^1 occurs when we choose $f = D_A^r$ for some r. Therefore, the kernel is generated by D_A. ☐

Let $E_0(T, Fr(T)) = P^{-1}(E_0(B, b))$. Then $E_0(T, Fr(T))$ is an index two subgroup of $E(T, Fr(T))$ and there is a short exact sequence

$$1 \to ker(P) \to E_0(T, Fr(T)) \to E_0(B, b) \to 1.$$

Lemma 3.5 shows that $ker(P) \cong \langle D_A \rangle \cong \mathbb{Z}$, while Lemma 3.4 shows $E_0(B, b) \cong \mathbb{Z}^{n-1}$ is generated by sweeps $\{h(\gamma_j, \beta)\}_{j=1}^{n-1}$. One may define a section

$$\sigma : E_0(B, b) \to E_0(T, Fr(T))$$

by setting

$$\sigma(h(\gamma_j, \beta)) = H(\gamma_j, \beta)$$

for all j. (Notice that σ is a homomorphism, since any homotopy between maps in $E_0(B, b)$ extends, by simply taking the product with the identity map on S^1, to a homotopy between the corresponding maps in $E_0(T, Fr(T))$.) Let

$$\mathrm{Sw}_0(T) = \sigma(E_0(B, b)) \cong \mathbb{Z}^{n-1}.$$

For any sweep $H = H(\gamma_j, \beta)$ in $E_0(T, Fr(T))$, HD_AH^{-1} lies in the kernel of P. Since H preserves level sets of p_2, we see that $p_2(HD_AH^{-1})s = p_2(D_A)s$. One then sees that $HD_AH^{-1} = D_A \in E_0(T, Fr(T))$, just as in the proof of Lemma 3.5. Therefore, $E_0(T, Fr(T))$ splits as a direct product

$$E_0(T, Fr(T)) = \langle D_A \rangle \oplus \mathrm{Sw}_0(T) \cong \mathbb{Z} \oplus \mathbb{Z}^{n-1} \cong \mathbb{Z}^n.$$

One may combine all of the above analysis to obtain:

Proposition 3.6. *Let M be a compact, orientable, hyperbolizable 3-manifold with incompressible boundary and let T be a thickened torus component of $\Sigma(M)$ with base surface (B, b) and $Fr(T) = \{A_1, \ldots, A_n\}$. Then there is a subgroup $E_0(T, Fr(T))$ of index two in $E(T, Fr(T))$ which is a rank n free abelian group freely generated by sweeps $\{H(\gamma_j, \beta)\}_{j=1}^{n-1}$ (where $\{h(\gamma_j, \beta)\}_{j=1}^{n-1}$ is a generating set of sweeps for $E_0(B, b)$) and a Dehn twist D_A about a vertical essential annulus A.*

3.3. **Assembling the sequence.** We are now ready to define $\mathrm{Out}_0(\pi_1(M))$. If V is a solid torus component of $\Sigma(M)$, we let $E_0(V, Fr(V))$ be the trivial group. If V is an interval bundle component of $\Sigma(M)$, we let $E_0(V, Fr(V)) = E(V, Fr(V))$. We have already defined $E_0(V, Fr(V))$ when V is a thickened torus component of $\Sigma(M)$. Then $\oplus_i E_0(V_i, Fr(V_i))$ is a finite index subgroup of $\oplus_i E(V_i, Fr(V_i))$, so

$$\mathrm{Out}_0(\pi_1(M)) = \Psi^{-1}\left(\oplus_i E_0(V_i, Fr(V_i))\right)$$

is a finite index subgroup of $\mathrm{Out}_2(\pi_1(M))$ and hence of $\mathrm{Out}(\pi_1(M))$.

If T is a thickened torus component of $\Sigma(M)$, then

$$E_0(T, Fr(T)) = D(T) \oplus \mathrm{Sw}_0(T)$$

where $D(T)$ is generated by a Dehn twist about a vertical essential annulus in T and $\mathrm{Sw}_0(T)$ is generated by sweeps $\{H(\gamma_j, \beta)\}_{j=1}^{n-1}$. We may extend each $H(\gamma_j, \beta)$ to a homotopy equivalence $\hat{H}(\gamma_j, \beta)$ of M which is the identity on the complement of T. Up to homotopy, we may assume that for all $i \neq j$, the support of $\hat{H}(\gamma_i, \beta)$ is disjoint from the support of $\hat{H}(\gamma_j, \beta)$. Also, we may assume the support of $\hat{H}(\gamma_i, \beta)$ is disjoint from the image of $\text{supp}(\hat{H}(\gamma_j, \beta))$ under $\hat{H}(\gamma_j, \beta)$. It follows that $\hat{H}(\gamma_i, \beta)$ and $\hat{H}(\gamma_j, \beta)$ commute. (The argument of Lemma 3.4 can also be adapted to show that $\hat{H}(\gamma_i, \beta)$ and $\hat{H}(\gamma_j, \beta)$ commute.) One may thus define a homomorphism

$$s_T : \text{Sw}_0(T) \to \text{Out}_0(\pi_1(M))$$

by letting $s_T(H(\gamma_j, \beta)) = \hat{H}(\gamma_j, \beta)$ for all j. Since $\Psi \circ s_T$ is the identity map, s_T is an isomorphism onto its image and we define

$$\text{Sw}(T) = s_T(\text{Sw}_0(T)) \cong \mathbb{Z}^{n-1}.$$

We may similarly note that elements of $Sw(T)$ and $K(M)$ commute since one can choose representatives so that the supports and the images of the supports are disjoint. (However, we note that s_T cannot be extended to a homomorphism defined on all of $E_0(T, Fr(T))$, since the commutator in $\text{Out}(\pi_1(M))$ of a Dehn twist in a vertical annulus and a sweep is a Dehn twist in a frontier annulus.)

If we let $\{T_i\}$ be the set of thickened torus components of $\Sigma(M)$ and $\{\Sigma_j\}$ denote the collection of interval bundle components of $\Sigma(M)$, then there is an obvious projection map

$$p : \oplus_k E_0(V_k, Fr(V_k)) \to (\oplus_i D(T_i)) \oplus (\oplus_j E(\Sigma_j, Fr(\Sigma_j))\,.$$

Then we consider the map

$$\Phi = p \circ \Psi : \text{Out}_0(\pi_1(M)) \to (\oplus_i D(T_i)) \oplus (\oplus_j E(\Sigma_j, Fr(\Sigma_j))$$

which has kernel $K(M) \oplus (\oplus_i \text{Sw}(T_i))$.

We summarize the above discussion in the following proposition.

Proposition 3.7. *Let M be a compact, orientable, hyperbolizable 3-manifold with incompressible boundary. Then there exists a finite index subgroup $\text{Out}_0(\pi_1(M))$ of $\text{Out}(\pi_1(M))$ and a short exact sequence*

$$1 \longrightarrow K(M) \oplus (\oplus_i \text{Sw}(T_i)) \longrightarrow \text{Out}_0(\pi_1(M)) \xrightarrow{\Phi} (\oplus_i D(T_i)) \oplus (\oplus_j E(\Sigma_j, Fr(\Sigma_j)) \longrightarrow 1$$

4. Characteristic collections of annuli and registering subgroups

We will divide the annuli in $Fr(\Sigma(M))$ into collections, called characteristic collections of annuli. Each isotopy class of annulus in $Fr(\Sigma(M))$ will appear in exactly one collection. A *characteristic collection of annuli* C for $\Sigma(M)$ is either

1. the collection of frontier annuli in a solid torus component of $\Sigma(M)$,
2. a component of the frontier of an interval bundle component of $\Sigma(M)$ which is not isotopic into either a solid torus or thickened torus component of $\Sigma(M)$, or
3. the collection of frontier annuli in a thickened torus component of $\Sigma(M)$.

Let $\{C_1, \ldots, C_m\}$ denote the collections of characteristic annuli for M. Let $K(C_j)$ be the subgroup of $K(M)$ generated by Dehn twists about the annuli in C_j. Notice that

$$K(M) \cong \oplus K(C_j).$$

We extend this decomposition of $K(M)$ into a decomposition of $\ker(\Phi)$. If C is the collection of frontier annuli of a thickened torus T, we define $\hat{K}(C) = K(C) \oplus \mathrm{Sw}(T)$. Otherwise, we define $\hat{K}(C) = K(C)$. With this convention,

$$\ker(\Phi) = K(M) \oplus (\oplus_i \mathrm{Sw}(T_i)) = \oplus_j \hat{K}(C_j).$$

If C is a characteristic collection of annuli, then we may define the projection map

$$q_C : \ker(\Phi) \to \hat{K}(C).$$

We next define subgroups of $\pi_1(M)$ which "register" the action of $\hat{K}(C)$ on $\pi_1(M)$, in the sense that the subgroup is preserved by any element of $\mathrm{Out}(\pi_1(M))$ and $\hat{K}(C)$ injects into the outer automorphism group of the subgroup.

Let $M_C = M - \mathcal{N}(C_1 \cup C_2 \cup \ldots \cup C_m)$ be the complement of a regular neighborhood of the characteristic collections of annuli. If X is a component of M_C, then X is either properly isotopic to a component of $\Sigma(M)$ or to a component of $M - \Sigma(M)$. In particular, $\pi_1(X)$ is non-abelian if it is not properly isotopic to a solid torus or thickened torus component of $\Sigma(M)$. Moreover, no two adjacent components of M_C have abelian fundamental group.

First suppose that $C = Fr(V) = \{A_1, \ldots, A_l\}$ where V is a solid torus component of $\Sigma(M)$. For each $i = 1, \ldots, l$, let X_i be the component of $M_C - V$ abutting A_i. Let a be a core curve for V and let x_0 be a point on a. We say that a subgroup H of $\pi_1(M, x_0)$ is *C-registering* if there exist, for each $i = 1, \ldots, l$, a loop g_i in $T_j \cup X_i$ based at x_0 intersecting A_i exactly twice, such that

$$H = < a > * < g_1 > * \cdots * < g_l > \cong F_{l+1}$$

Now suppose that $C = \{A\}$ is a component of $Fr(\Sigma)$ where Σ is an interval bundle component of $\Sigma(M)$. Let a be a core curve for A and let x_0 be a point on a. We say that a subgroup H of $\pi_1(M, x_0)$ is *C-registering* if there exist two loops g_1 and g_2 based at x_0 each of whose interiors misses A, and which lie in the two distinct components of M_C abutting A, such that

$$H = < a > * < g_1 > * < g_2 > \cong F_3$$

Finally, suppose that $C = Fr(T) = \{A_1, \ldots, A_l\}$ where T is a thickened torus component of $\Sigma(M)$. For each $i = 1, \ldots, l$, let X_i be the component of $M_C - T$ abutting A_i. Pick $x_0 \in T$. We say that a subgroup H of $\pi_1(M, x_0)$ is *C-registering* if there exist, for each $i = 1, \ldots, l$, a homotopically non-trivial loop g_i in $T \cup X_i$ based at x_0 intersecting A_i exactly twice, such that

$$H = \pi_1(T, x_0) * < g_1 > * \cdots * < g_l > \cong (\mathbb{Z} \oplus \mathbb{Z}) * F_l$$

If H is a subgroup of $\pi_1(M)$, then there is an obvious map

$$r_H : X(M) \to X(H) = \mathrm{Hom}(H, \mathrm{PSL}(2,\mathbb{C}))//\mathrm{PSL}(2,\mathbb{C})$$

given by taking ρ to $\rho|_H$.

The following lemma records the key properties of registering subgroups.

Lemma 4.1. *Let M be a compact, orientable, hyperbolizable 3-manifold with incompressible boundary. If C is a characteristic collection of annuli for M and H is a C-registering subgroup of $\pi_1(M)$, then H is preserved by each element of $\hat{K}(C)$ and there is a natural injective homomorphism*

$$s_H : \hat{K}(C) \to \mathrm{Out}(H).$$

Moreover, if $\eta \in \ker(\Phi) = K(M) \oplus (\oplus_i \mathrm{Sw}(T_i))$, then

$$r_H(\rho \circ \eta) = r_H(\rho) \circ s_H(q_C(\eta))$$

for all $\rho \in X(M)$, where $q_C : \ker(\Phi) \to \hat{K}(C)$ is the projection map.

Proof. If C lies in the frontier of a solid torus or interval bundle component of $\Sigma(M)$, this was established as Lemma 6.1 in [15].

Now suppose that $C = Fr(T) = \{A_1, \ldots, A_n\}$ where T is a thickened torus component of $\Sigma(M)$ and that H is a C-registering subgroup of $\pi_1(M, x_0)$ (where $x_0 \in \mathrm{int}(T)$) generated by $\pi_1(T, x_0)$ and $\{g_1, \ldots, g_n\}$. Then $K(C) \cong \mathbb{Z}^{n-1}$ and is generated by the Dehn twists $\{D_{A_1}, \ldots, D_{A_{n-1}}\}$ and $\mathrm{Sw}(T) \cong \mathbb{Z}^{n-1}$ and is generated by sweeps $\{\hat{H}(\gamma_1, \beta), \cdots, \hat{H}(\gamma_{n-1}, \beta)\}$. We choose generators a and b for $\pi_1(T, x_0)$ so that a is homotopic to the core curve of A_1 and b is the core curve β of the annulus B (here we adapt the notation of Proposition 3.6). One may check that each $(D_{A_k})_*$ preserves H, fixes $\pi_1(T, x_0)$ and each g_i where $i \neq k$, and maps g_k to $a g_k a^{-1}$. Similarly, each $(\hat{H}(\gamma_k, \beta))_*$ preserves H and fixes $\pi_1(T, x_0)$ and each g_i where $i \neq k$ and takes g_k to $b g_k b^{-1}$.

This explicit description of each generator allows one to immediately check that $\hat{K}(C)$ preserves H and injects into $\mathrm{Out}(H)$. □

5. The existence of registering subgroups

In this section, we prove that every characteristic collection of annuli admits a registering subgroup. We will make use of the existence of minimally parabolic hyperbolic structures on M to do so.

In the case that the characteristic collection of annuli lies in the frontier of a solid torus or interval bundle component of $\Sigma(M)$, the proof of Lemma 8.3 in [15] immediately yields:

Lemma 5.1. *Suppose that M is a compact, orientable, hyperbolizable 3-manifold with incompressible boundary and C is a characteristic collection of annuli for M such that either (1) $C = Fr(V)$ for a solid torus component V of $\Sigma(M)$ or (2) C is a* *component of the frontier of an interval bundle component of $\Sigma(M)$. If $\rho \in AH(M)$ and $\rho(\pi_1(C))$ is purely hyperbolic, then there exists a C-registering subgroup H of $\pi_1(M)$ such that $\rho|_H$ is discrete, faithful, geometrically finite and purely hyperbolic (and therefore Schottky).*

We may give a variation on the argument in [15] to prove:

Lemma 5.2. *Suppose that M is a compact, orientable, hyperbolizable 3-manifold with incompressible boundary and C is a characteristic collection of annuli for M such that $C = Fr(T)$ for a thickened torus component T of $\Sigma(M)$. If $\rho \in AH(M)$, then there exists a C-registering subgroup H of $\pi_1(M)$ such that $\rho|_H$ is discrete, faithful, geometrically finite and minimally parabolic.*

Proof. Let $\rho \in AH(M)$ and let $C = \{A_1, \ldots, A_l\}$. Let X_i be the component of M_C abutting A_i. Pick $x_0 \in T$ and, in order to be precise, let $\pi_1(T)$ denote $\pi_1(T, x_0) \subset \pi_1(M, x_0)$. Since $\rho \in AH(M)$, $\rho(\pi_1(T))$ consists of parabolic elements fixing a common fixed point $p \in \widehat{\mathbb{C}}$. We may assume that $p = \infty$ and pick a fundamental domain F for the action of $\rho(\pi_1(T))$ on $\mathbb{C}$ which is a quadrilateral. Since $\pi_1(X_i \cup T, x_0)$ is not abelian, we can find $\gamma_i \in \pi_1(X_i \cup T_0, x_0)$ such that $\rho(\gamma_i)$ is a hyperbolic element with both fixed points contained in the interior of F. If $i \neq j$, then $\pi_1(X_i \cup T_0, x_0) \cap \pi_1(X_j \cup T_0, x_0) = \pi_1(T, x_0)$, so γ_i and γ_j have distinct fixed points. One may then find a collection $\{D_1^{\pm}, \ldots, D_l^{\pm}\}$ of $2l$ disjoint disks in the interior of F and integers $\{s_1, \ldots, s_l\}$, so that, for each i, $\gamma_i^{s_i}$ takes the interior of D_i^- homeomorphically onto the exterior of D_i^+. For each i, let g_i be a curve in $X_i \cup T$ which intersects A_i exactly twice and represents $\rho^{-1}(\gamma_i^{s_i})$. Let

$$H = \langle \pi_1(T), g_1, \ldots, g_l \rangle \subset \pi_1(M).$$

Klein's Combination Theorem (see [30, Theorem A.13, Theorem C.2]) then implies that ρ is geometrically finite and minimally parabolic and that

$$\rho(H) \cong \rho(\pi_1(T)) * \langle \gamma_1^{s_1} \rangle * \cdots * \langle \gamma_l^{s_l} \rangle .$$

Therefore, H is a registering subgroup with the desired properties. □

Thurston's Hyperbolization Theorem, see Morgan [34], implies that there exists a geometrically finite, minimally parabolic element $\rho \in AH(M)$. If C is a characteristic collection of annuli contained in the frontier of a solid torus or interval bundle component of $\Sigma(M)$, then no annulus in C is homotopic into a toroidal boundary component of M, so $\rho(\pi_1(C))$ is purely hyperbolic. In these cases, Lemma 5.1 implies the existence of a registering subgroup for C. Otherwise, C is the frontier of a thickened torus component of $\Sigma(M)$ and Lemma 5.2 guarantees the existence of a registering subgroup for C. Therefore, we have established:

Proposition 5.3. *Suppose that M is a compact, orientable, hyperbolizable 3-manifold with incompressible boundary and C is a characteristic collection of annuli for M, then there exists a C-registering subgroup of $\pi_1(M)$.*

6. Registering subgroups and relative deformation spaces

If H is a C-registering subgroup for a characteristic collection of annuli C, then we define $GF(H)$ to be the set of conjugacy classes of discrete, faithful, geometrically finite, minimally parabolic representations. (If H is a free group, $GF(H)$ is the space of Schottky representations.) $GF(H)$ naturally sits inside

$$X_T(H) = \mathrm{Hom}_T(H, \mathrm{PSL}(2,\mathbb{C}))//\mathrm{PSL}(2,\mathbb{C})$$

where $\mathrm{Hom}_T(H, \mathrm{PSL}(2,\mathbb{C}))$ denotes the set of representations of H into $\mathrm{PSL}(2,\mathbb{C})$ such that if an element of H lies in a rank two free abelian subgroup of H then its image is either parabolic or the identity. In turn, $X_T(H)$ is a subvariety of the full character variety $X(H)$.

Lemma 6.1. *Let M be a compact, orientable, hyperbolizable 3-manifold with non-empty incompressible boundary. If H is a registering subgroup for some characteristic collection of annuli, then*

(1) *$GF(H)$ is an open subset of $X_T(H)$.*
(2) *If $\{\alpha_n\}$ is a sequence of distinct elements in $\mathrm{Out}(H)$ and D is a compact subset of $GF(H)$, then $\{\alpha_n(D)\}$ exits every compact subset of $X_T(H)$.*

Proof. Theorem 10.1 in Marden [28] implies that $GF(H)$ is an open subset of $X_T(H)$, which establishes (1).

If (2) fails, there exists a sequence $\{\alpha_n\}$ of distinct elements of $\mathrm{Out}(H)$ and a compact subset D of $GF(H)$ such that $\alpha_n(D)$ intersects a fixed compact subset of $X_T(H)$ for all n.

We will call an element of H toroidal if it lies in a rank two free abelian subgroup. Given $\rho \in GF(H)$ and $g \in G$, let $l_\rho(g)$ denote the translation distance of $\rho(g)$.

Fix, for the moment, an element $\tau \in D$. Then, given any $P > 0$ there exists finitely many conjugacy classes of non-toroidal elements g in H such that $l_\tau(g) < P$. Moreover, there exists a positive lower bound on the translation distance $l_\tau(g)$ whenever g is non-toroidal. Let $\{h_1, \ldots, h_r\}$ be a generating set for H consisting of non-toroidal elements. If $\{\alpha_n\}$ is a sequence of distinct elements of $\mathrm{Out}(H)$, then we may pass to a subsequence $\{\alpha_j\}$ such that either

(1) there exists a generator h_k such that $l_\tau(\alpha_j^{-1}(h_k)) \to \infty$, or
(2) there exist generators h_i and h_k such that the distance between the axes of $\tau(\alpha_j^{-1}(h_i))$ and $\tau(\alpha_j^{-1}(h_k))$ goes to infinity.

In the second case $l_\tau(\alpha_j^{-1}(h_i h_k)) \to \infty$. Therefore, $\{\alpha_j(\tau)\}$ leaves every compact subset of $X_T(H)$.

Without loss of generality, we may assume that D is contained in a single component of $GF(H)$. Since all elements in a component of $GF(H)$ are quasiconformally conjugate (see, e.g. [14, Section 7.3]) and D is compact, there exists L such that all the representations in D are L-quasiconformally conjugate. Therefore, there exists

$K > 0$ such that all the actions on $\mathbb{H}^3$ are K-bilipschitz conjugate ([14, Proposition 7.2.6]). In particular, if $\rho \in D$ and $g \in H$, then

$$l_\rho(g) \geq \frac{1}{K} l_\tau(g).$$

It follows that $\{\alpha_j(D)\}$ exits every compact subset of $X_T(H)$, which contradicts our assumption and completes the proof of (2). □

If Σ is an interval bundle component of $\Sigma(M)$, let $\partial_1\Sigma$ denote the collection of components of $Fr(\Sigma)$ which are homotopic into toroidal boundary components of M. Let $GF(\Sigma, \partial_1\Sigma)$ denote the set of conjugacy classes of discrete, faithful, geometrically finite representations such that the image of a non-trivial element is parabolic if and only if it is conjugate into $\pi_1(\partial_1\Sigma)$. $GF(\Sigma, \partial_1\Sigma)$ naturally sits inside

$$X(\Sigma, \partial_1\Sigma) = \mathrm{Hom}(\Sigma, \partial_1\Sigma, \mathrm{PSL}(2, \mathbb{C}))//\mathrm{PSL}(2, \mathbb{C})$$

where $\mathrm{Hom}(\Sigma, \partial_1\Sigma, \mathrm{PSL}(2, \mathbb{C}))$ denotes the representations such that $\rho(g)$ is parabolic or trivial if g is conjugate into $\pi_1(\partial_1\Sigma)$. $X(\Sigma, \partial_1\Sigma)$ is a subvariety of $X(\Sigma)$.

We may use the same argument as in the proof of Lemma 6.1, replacing non-toroidal elements with elements not conjugate into $\pi_1(\partial_1\Sigma)$, to establish:

Lemma 6.2. *Let M be a compact, orientable, hyperbolizable 3-manifold with non-empty incompressible boundary. Let Σ be an interval bundle component of $\Sigma(M)$, then*

1. *$GF(\Sigma, \partial_1\Sigma)$ is an open subset of $X(\Sigma, \partial_1\Sigma)$.*
2. *If D is a compact subset of $GF(\Sigma, \partial_1\Sigma)$ and $\{\alpha_n\}$ is a sequence of distinct elements of $E(\Sigma, Fr(\Sigma))$, then $\{\alpha_n(D)\}$ exits every compact subset of $X(\Sigma, \partial_1\Sigma)$.*

7. The domains of discontinuity

We are now ready to define the domains of discontinuity which occur in the statements of Theorems 1.1 and 1.3. We first define $W(M) \subset X_T(M)$.

Definition 7.1. A representation $\rho \in X_T(M)$ lies in $W(M)$ if and only if the following hold:

(a) if C is a characteristic collection of annuli for M, then there exists a C-registering subgroup H such that $\rho|_H \in GF(H)$, i.e. $\rho|_H$ is discrete, faithful, geometrically finite and minimally parabolic, and

(b) if Σ is an interval bundle component of $\Sigma(M)$ which is not tiny, then $\rho|_{\pi_1(\Sigma)} \in GF(\Sigma, \partial_1\Sigma)$, i.e. $\rho|_{\pi_1(\Sigma)}$ is discrete, faithful, geometrically finite and $\rho|_{\pi_1(\Sigma)}(g)$ is parabolic if and only if g is conjugate to a non-trivial element of $\pi_1(\partial\Sigma_1)$.

Proposition 7.2. *Let M be a compact, orientable, hyperbolizable 3-manifold with non-empty incompressible boundary which is not an interval bundle. Then*

1. *$W(M)$ is an $\mathrm{Out}(\pi_1(M))$-invariant open subset of $X_T(M)$.*

(2) *The interior of $AH(M)$ is a proper subset of $W(M)$.*
(3) *Every minimally parabolic representation in $AH(M)$ lies in $W(M)$. In particular, $W(M)$ contains a dense subset of $\partial AH(M)$.*
(4) *$AH(M) \subset W(M)$ if and only if M contains no primitive essential annuli.*

Proof. We first show that $W(M)$ is open in $X_T(M)$.

Recall that if H is a registering subgroup for a characteristic collection of annuli, then $r_H : X_T(M) \to X_T(H)$ is continuous. Since $GF(H)$ is an open subset of $X_T(H)$ (see Lemma 6.1), $r_H^{-1}(GF(H))$ is an open subset of $X_T(M)$. Therefore, the set of representations satisfying condition (a) in the definition of $W(M)$ is open.

If Σ is an interval bundle component of $\Sigma(M)$, then the map $r_\Sigma : X_T(M) \to X(\Sigma, \partial_1\Sigma)$ obtained by restriction is continuous. Since $GF(\Sigma, \partial_1\Sigma)$ is an open subset of $X(\Sigma, \partial_1\Sigma)$ (see Lemma 6.2), $r_\Sigma^{-1}(GF(\Sigma, \partial_1\Sigma))$ is an open subset of $X_T(M)$. It follows that the set of representations satisfying condition (b) in the definition of $W(M)$ is open. Therefore, $W(M)$ is open in $X_T(M)$.

Johannson's Classification Theorem implies that every homotopy equivalence h of M is homotopic to one which preserves $\Sigma(M)$ and $M - \Sigma(M)$. In particular, we may assume that h takes every interval bundle component of $\Sigma(M)$ to an interval bundle component of M and takes each characteristic collection of annuli to a characteristic collection of annuli. Moreover, if H is a registering subgroup for C, we see that $h_*(H)$ is a registering subgroup for $h(C)$. Since every outer automorphism of $\pi_1(M)$ is realized by a homotopy equivalence, one easily verifies that $W(M)$ is invariant under $\mathrm{Out}(\pi_1(M))$. This completes the proof of (1).

Since all representations in $\mathrm{int}(AH(M))$ are minimally parabolic, (2) follows from (3). We now turn to the proof of (3).

Suppose that $\rho \in AH(M)$ is minimally parabolic. Lemmas 5.1 and 5.2 imply that if C is a characteristic collection of annuli, then there is a C-registering subgroup H such that $\rho|_H$ is discrete, faithful, geometrically finite and minimally parabolic. Therefore, ρ satisfies condition (a) in the definition of $W(M)$.

Now suppose that Σ is an interval bundle component of $\Sigma(M)$ which is not tiny. Let M_ρ be a relative compact core for N_ρ^0 and let $h : M \to M_\rho$ be a homotopy equivalence in the homotopy class of ρ. Johannson's Classification Theorem implies that we may assume that $h(\Sigma) = \Sigma_\rho$ is an interval bundle component of $\Sigma(M_\rho)$ and that h restricts to a homeomorphism from $Fr(\Sigma)$ to $Fr(\Sigma_\rho)$. Let $\partial_1\Sigma_\rho = h(\partial_1\Sigma)$. Since ρ is minimally parabolic, $r_\Sigma(\rho)(g)$ is parabolic if and only if g is conjugate to a non-trivial element of $\pi_1(\partial_1\Sigma)$.

The interval bundle Σ_ρ lifts to a compact core for the cover N_Σ of N_ρ associated to $\rho(\pi_1(\Sigma)) = \pi_1(\Sigma_\rho)$. However, the lift need not be a relative compact core, since it need not intersect every component of N_Σ^0 in an incompressible annulus. (Here we choose the invariant system of horoballs for $\rho(\pi_1(\Sigma))$ to be a subset of the precisely invariant system of horoballs for $\rho(\pi_1(M))$, so the covering map from N_Σ to N_ρ restricts to a covering map from ∂N_Σ^0 to its image in ∂N_ρ^0.) In order to extend Σ_ρ to a submanifold which does lift to a relative compact core, we construct a submanifold Y_ρ of M_ρ which is homeomorphic to $\partial_1\Sigma_\rho \times [0,1]$ by a homeomorphism identifying $\partial_1\Sigma_\rho$ with $\partial_1\Sigma_\rho \times \{0\}$, so that $Y_\rho \cap \Sigma_\rho = \partial_1\Sigma_\rho$ and $Y_\rho \cap \partial N_\rho^0$ is a collection of incompressible annuli which is identified with $\partial_1\Sigma_\rho \times \{1\}$. If we let

$$\Sigma_\rho^+ = \Sigma_\rho \cup Y_\rho$$

then Σ_ρ^+ does lift to a relative compact core for ∂N_Σ^0. Moreover, the lift of Σ_ρ^+ intersects N_Σ^0 exactly in the lift of $\Sigma_\rho^+ \cap \partial N_\rho^0$. The ends of N_Σ^0 are in one-to-one correspondence with the components of $\partial\Sigma_\rho^+ - (\partial\Sigma_\rho^+ \cap \partial N_\rho^0)$, each of which is homotopic to a component of $\partial\Sigma_\rho - \partial_1\Sigma$. In particular, N_Σ^0 has one or two ends.

If the manifold N_Σ^0 has only one end, then the covering map $N_\Sigma \to N_\rho$ is infinite-to-one on this end. The Covering Theorem implies that the single end of N_Σ^0 is geometrically finite, so N_Σ is geometrically finite. If the manifold N_Σ^0 has two ends, then $Fr(\Sigma_\rho) = \partial_1\Sigma_\rho$, so each component of $\partial\Sigma_\rho - \partial_1\Sigma$ is identified with a proper subsurface of a component of ∂M_ρ. Again $N_\Sigma \to N_\rho$ is infinite-to-one on each end of N_Σ^0 and the Covering Theorem may be used to show that N_Σ is geometrically finite. Thus, in all cases, $\rho|_{\pi_1(\Sigma)} \in GF(\Sigma, \partial_1\Sigma)$, so ρ satisfies condition (b) in the definition of $W(M)$. Therefore, minimally parabolic representations in $AH(M)$ lie in $W(M)$.

Since minimally parabolic representations are dense in the boundary of $AH(M)$, $W(M)$ contains a dense subset of $\partial AH(M)$. (The density of minimally parabolic representations in $\partial AH(M)$ follows from Lemma 4.2 in [13], which shows that minimally parabolic representations are dense in the boundary of any component of int$(AH(M))$ and the Density Theorem, see Brock-Canary-Minsky [8], Bromberg-Souto [10], Namazi-Souto [35] or Ohshika [37], which asserts that $AH(M)$ is the closure of its interior.) This completes the proof of (3).

Suppose that M contains no primitive essential annuli. Then $\Sigma(M)$ contains no interval bundle components which are not tiny, since otherwise a non-peripheral essential annulus in the interval bundle would be a primitive essential annulus (see [15, Lemma 7.3]). Similarly, every component of the frontier of a tiny interval bundle component is isotopic into a solid torus or thickened torus component of $\Sigma(M)$, since otherwise it would be a primitive essential annulus (see Johannson [21, Lemma 32.1]). Therefore, every characteristic collection of annuli is the frontier of either a solid torus or thickened torus component of $\Sigma(M)$. Moreover, the core curve of each solid torus component V of $\Sigma(M)$ is non-peripheral, since otherwise its frontier annuli would be primitive essential annuli (again see Johannson [21, Lemma 32.1]). Therefore, just as in the proof of Lemma 8.1 in [15], $\rho(\pi_1(V))$ is purely hyperbolic if $\rho \in AH(M)$ and V is a solid torus component of $\Sigma(M)$. Therefore, if $\rho \in AH(M)$ and C is any characteristic collection of annuli for M, then Lemma 5.1 or 5.2 guarantees that there exists a C-registering subgroup H such that $\rho|_H$ is discrete, faithful, geometrically finite and minimally parabolic. Since every interval bundle is tiny, it follows that $AH(M) \subset W(M)$.

On the other hand, if M contains a primitive essential annulus A, then there exists $\rho \in AH(M)$ such that $\rho(\pi_1(A))$ is purely parabolic (see Ohshika [36]). Since A is either isotopic to a component of a characteristic collection of annuli or isotopic into an interval bundle component of $\Sigma(M)$, ρ does not lie in $W(M)$. In particular, $AH(M)$ is not a subset of $W(M)$. Therefore, $AH(M) \subset W(M)$ if and only if M contains no primitive essential annuli.

□

If M does not contain an essential annulus which intersects a toroidal boundary component of M, then we define $\hat{W}(M) \subset X(M)$. Notice that M contains an essential annulus with one boundary component contained in a toroidal boundary component of M if and only if $\Sigma(M)$ has a thickened torus component.

Definition 7.3. We say that $\rho \in X(M)$ lies in $\hat{W}(M)$ if and only if the following hold:

(a) if C is a characteristic collection of annuli for M, then there exists a C-registering subgroup H such that $\rho|_H \in GF(H)$, and
(b) if Σ is an interval bundle component of $\Sigma(M)$ with base surface F, which is not tiny, then $\rho|_{\pi_1(\Sigma)} \in GF(\Sigma, \emptyset)$, i.e. $\rho|_{\pi_1(\Sigma)}$ is discrete, faithful, geometrically finite and purely hyperbolic.

We obtain the following analogue of Proposition 7.2 whenever $\hat{W}(M)$ is defined.

Proposition 7.4. *Let M be a compact, orientable, hyperbolizable 3-manifold with non-empty incompressible boundary which is not an interval bundle and so that M contains no essential annuli which intersects a toroidal boundary component. Then*

(1) $\hat{W}(M)$ *is an* $\mathrm{Out}(\pi_1(M))$*-invariant open subset of* $X(M)$.
(2) $\hat{W}(M) \cap X_T(M) = W(M)$.
(3) $AH(M) \subset \hat{W}(M)$ *if and only if M contains no primitive essential annuli.*

Sketch of proof: Since $\Sigma(M)$ does not contain any thickened torus components, every characteristic collection of annuli is either the frontier of a solid torus component of $\Sigma(M)$ or a component of the frontier of an interval bundle component of $\Sigma(M)$. Moreover, every registering subgroup H is a free group and $\partial_1\Sigma$ is empty for every interval bundle component of $\Sigma(M)$. The proof of (1) mimics the proof of Proposition 7.2. If H is a registering subgroup for some characteristic collection of annuli, then we can define $r_H : X(M) \to X(H)$, and $r_H^{-1}(GF(H))$ is an open subset of $X(M)$. In the case that Σ is an interval bundle component of $\Sigma(M)$, we define $r_\Sigma : X(M) \to X(\Sigma)$ and $GF(\Sigma, \emptyset)$ is an open subset of $X(\Sigma)$, so $\rho_\Sigma^{-1}(GF(\Sigma, \emptyset))$ is open in $X(M)$. Therefore, as in the proof of property (1) in Proposition 7.2, $W(M)$ is an open subset of $X(M)$. The $\mathrm{Out}(\pi_1(M))$-invariance of $\hat{W}(M)$ follows from Johannson's Classification Theorem, much as in the proof of Proposition 7.2.

Property (2) follows immediately from the definitions of $W(M)$ and $\hat{W}(M)$ and the restrictions on the characteristic submanifold of M discussed in the previous paragraph. Property (3) follows from property (2) and part (4) of Theorem 7.2. □

Remark: If one, more generally, allowed $\rho|_H$ and $\rho|_{\pi_1(\Sigma)}$ to be primitive-stable (see Minsky [33]) in the definition of $\hat{W}(M)$, then $\hat{W}(M)$ would agree with the domain of discontinuity obtained in [15] in the case that M has no toroidal boundary components.

8. Proof of Main Theorem

We are now prepared to complete the proof of our main theorem, which we recall below:

Theorem 1.1. *Let M be a compact, orientable, hyperbolizable 3-manifold with nonempty incompressible boundary, which is not an interval bundle. Then there exists an open $\mathrm{Out}(\pi_1(M))$-invariant subset $W(M)$ of $X_T(M)$ such that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $W(M)$, $\mathrm{int}(AH(M))$ is a proper subset of $W(M)$, and $W(M)$ intersects $\partial AH(M)$.*

Theorem 1.1 follows immediately from Proposition 7.2, which gives the key properties of $W(M)$, and the following proposition which establishes the proper discontinuity of the action of $\mathrm{Out}(\pi_1(M))$ on $W(M)$.

Proposition 8.1. *If M is a compact, orientable, hyperbolizable 3-manifold with nonempty incompressible boundary which is not an interval bundle, then $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $W(M)$.*

Proof. Since $\mathrm{Out}_0(\pi_1(M))$ has finite index in $\mathrm{Out}(\pi_1(M))$, it suffices to prove that $\mathrm{Out}_0(\pi_1(M))$ acts properly discontinuously on $W(M)$.

Suppose that $\mathrm{Out}_0(\pi_1(M))$ does not act properly discontinuously on $W(M)$. Then there exists a compact subset R of $W(M)$ and a sequence $\{\alpha_n\}$ of distinct elements in $\mathrm{Out}_0(\pi_1(M))$ such that $\alpha_n(R) \cap R$ is non-empty for all n. We may pass to a subsequence so that either

1. $\{\Phi(\alpha_n)\}$ is a sequence of distinct elements of $(\oplus_i D(T_i)) \oplus (\oplus_j E(\Sigma_j, Fr(\Sigma_j))$, or
2. $\{\Phi(\alpha_n)\}$ is a constant sequence.

In case (1) we may pass to a further subsequence, still called $\{\alpha_n\}$, so that either

(a) there exists an interval bundle component Σ of $\Sigma(M)$ so that $\{p_\Sigma(\Phi(\alpha_n)\}$ is a sequence of distinct elements of $E(\Sigma, Fr(\Sigma))$ where

$$p_\Sigma : (\oplus_i D(T_i)) \oplus (\oplus_j E(\Sigma_j, Fr(\Sigma_j)) \to E(\Sigma, Fr(\Sigma)),$$

is the obvious projection map onto $E(\Sigma, Fr(\Sigma)$, or

(b) there exists a thickened torus component T of $\Sigma(M)$ so that $\{p_T(\Phi(\alpha_n)\}$ is a sequence of distinct elements of $D(T)$ where

$$p_T : (\oplus_i D(T_i)) \oplus (\oplus_j E(\Sigma_j, Fr(\Sigma_j)) \to D(T)$$

is the obvious projection map onto $D(T)$.

In case (1a), $r_\Sigma(R)$ is a compact subset of $GF(\Sigma, \partial_1\Sigma)$ and $\{p_\Sigma(\Phi(\alpha_n)\}$ is a sequence of distinct elements of $E(\Sigma, Fr(\Sigma))$. Recall, see Lemma 3.3, that $E(\Sigma, Fr(\Sigma))$ is identified with a subgroup of $\mathrm{Out}(\pi_1(\Sigma))$. Notice that, by construction,

$$r_\Sigma(\alpha(\rho)) = p_\Sigma(\Phi(\alpha))(r_\Sigma(\rho)).$$

or stated differently,

$$r_\Sigma(\rho \circ \alpha^{-1}) = r_\Sigma(\rho) \circ p_\Sigma(\Phi(\alpha))^{-1}.$$

for all $\rho \in W(M)$ and all $\alpha \in \mathrm{Out}_0(\pi_1(M))$. Therefore, $p_\Sigma(\Psi(\alpha_n))(r_\Sigma(R)) \cap r_\Sigma(R)$ is non-empty for all n. Since $r_\Sigma(R)$ is a compact subset of $GF(\Sigma, \partial_1\Sigma)$, this contradicts the proper discontinuity of the action of $E(\Sigma, Fr(\Sigma))$ on $GF(\Sigma, \partial_1\Sigma)$, see Lemma 6.2. This contradiction rules out case (1a).

In case (1b), notice that if $\rho \in W(M)$, then $\rho|_{\pi_1(T)} : \pi_1(T) \to \mathrm{PSL}(2,\mathbb{C})$ is discrete and faithful. Therefore, there exists a continuous restriction map $r_T : W(M) \to AH(\mathbb{Z}^2)$, where $AH(\mathbb{Z}^2)$ is the space of conjugacy classes of discrete faithful representations from $\pi_1(T)$ into $\mathrm{PSL}(2,\mathbb{C})$. There is a natural identification of $D(T)$ with a subgroup of $\mathrm{Out}(\pi_1(T))$. Since all the elements of $\ker(p_T \circ \Phi)$ act trivially on $\pi_1(T)$,

$$r_T(\alpha(\rho)) = p_T(\Phi(\alpha))(r_T(\rho)).$$

for all $\rho \in W(M)$ and $\alpha \in \mathrm{Out}_0(\pi_1(M))$. Therefore, $p_T(\Phi(\alpha_n))(r_T(R)) \cap r_T(R)$ is non-empty for all n. It is easy to check that $\mathrm{Out}(\pi_1(T))$ acts properly discontinuously on $AH(\mathbb{Z}^2)$. (One may identify $AH(\mathbb{Z}^2)$ with $\mathbb{C}\backslash\mathbb{R}$ and the action of $\mathrm{Out}(\pi_1(T))$ is identified with the action of $\mathrm{GL}(2,\mathbb{Z})$ as a group of conformal and anti-conformal automorphisms of $\mathbb{C}\backslash\mathbb{R}$, which is well-known to act properly discontinuously on $\mathbb{C}\backslash\mathbb{R}$.) So, we have again obtained a contradiction and case (1b) cannot occur.

In case (2), there exists $\gamma \in \mathrm{Out}_0(\pi_1(M))$ and a sequence $\beta_n \in \ker(\Phi)$ such that $\alpha_n = \beta_n \circ \gamma$ for all n. Since γ induces a homeomorphism of $X_T(M)$ which preserves $W(M)$, $\gamma(R)$ is a compact subset of $W(M)$ and $\beta_n(\gamma(R)) \cap R$ is non-empty for all n. Recall that $\ker(\Phi) = \oplus \hat{K}(C_j)$. Therefore, after passing to a further subsequence, we may find a characteristic collection of annuli C so that $q_C(\beta_n)$ is a sequence of distinct elements of $\hat{K}(C)$ (where q_C is the projection of $\ker(\Phi)$ onto $\hat{K}(C)$). Since $X_T(M)$ is locally compact, for each $x \in W(M)$, there exists an open neighborhood U_x of x and a C-registering subgroup H_x such that the closure $\bar{U}_x$ is a compact subset of $W(M)$ and $r_{H_x}(\bar{U}_x) \subset GF(H_x)$. Since $\gamma(R)$ is compact, there exists a finite collection of points $\{x_1, \ldots, x_r\}$ such that $\gamma(R) \subset U_{x_1} \cup \cdots \cup U_{x_r}$. Therefore, again passing to subsequence if necessary, there must exist x_i such that $\beta_n(U_{x_i}) \cap R$ is non-empty for all n. Let $U = U_{x_i}$ and $H = H_{x_i}$. Lemma 4.1 implies that $\{s_H(q_C(\beta_n))\}$ is a sequence of distinct elements of $\mathrm{Out}(H)$ and that

$$s_H(q_C(\beta_n))(r_H(\bar{U})) = r_H(\beta_n(\bar{U})).$$

Lemma 6.1 then implies that

$$\{s_H(q_C(\beta_n))(r_H(\bar{U}))\} = \{r_H(\beta_n(\bar{U}))\}$$

exits every compact subset of $X_T(H)$. Therefore, $\{\beta_n(U)\}$ exits every compact subset of $X_T(M)$ which is again a contradiction. Therefore, case (2) cannot occur and we have completed the proof. □

Corollary 1.2, which we restate here, follows readily from Theorem 1.1, Proposition 7.2 and Theorem 1.2 from [15].

Corollary 1.2: *If M is a compact, orientable, hyperbolizable 3-manifold with incompressible boundary and non-abelian fundamental group, then* $\mathrm{Out}(\pi_1(M))$ *acts properly discontinuously on an open* $\mathrm{Out}(\pi_1(M))$*-invariant neighborhood of $AH(M)$ in $X_T(M)$ if and only if M contains no primitive essential annuli.*

Proof of Corollary 1.2: Proposition 8.1 shows that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $W(M)$, and Proposition 7.2 implies that $W(M)$ is an open neighborhood of $AH(M)$ when M contains no primitive essential annuli.

If M contains a primitive essential annulus, then Theorem 1.2 of [15] asserts that $\mathrm{Out}(\pi_1(M))$ does not act properly discontinuously on $AH(M)$, so $\mathrm{Out}(\pi_1(M))$ cannot act properly discontinuously on any $\mathrm{Out}(\pi_1(M))$-invariant neighborhood of $AH(M)$ in $X_T(M)$. □

Remark: One could also prove that $\mathrm{Out}(\pi_1(M))$ cannot act properly discontinuously on an open neighborhood of $AH(M)$ when M contains a primitive essential annulus using the technique of Lemma 15 in Lee [26].

9. Dynamics in the absolute character variety

In this section, we study the action of $\mathrm{Out}(\pi_1(M))$ on the full character variety $X(M)$. We begin by showing that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $\hat{W}(M)$, which is a nearly immediate generalization of Proposition 8.1.

Proposition 9.1. *If M is a compact, orientable, hyperbolizable 3-manifold with nonempty incompressible boundary which is not an interval bundle, and no essential annulus in M has a boundary component contained in a toroidal boundary component of M, then* $\mathrm{Out}(\pi_1(M))$ *acts properly discontinuously on $\hat{W}(M)$.*

Sketch of proof: Again, it suffices to prove that $\mathrm{Out}_0(\pi_1(M))$ acts properly discontinuously on $\hat{W}(M)$. If $\mathrm{Out}_0(\pi_1(M))$ does not act properly discontinuously on $\hat{W}(M)$, then there exists a compact subset R of $\hat{W}(M)$ and a sequence $\{\alpha_n\}$ of distinct elements in $\mathrm{Out}_0(\pi_1(M))$ such that $\alpha_n(R) \cap R$ is non-empty for all n. We may again pass to a subsequence so that either (1) $\{\Phi(\alpha_n)\}$ is a sequence of distinct elements or (2) $\{\Phi(\alpha_n\}$ is a constant sequence. Since $\Sigma(M)$ contains no thickened torus components, in case (1) we can assume that there exists an interval bundle component Σ of $\Sigma(M)$ such that $\{p_\Sigma(\alpha_n)\}$ is a sequence of distinct elements of $E(\Sigma, Fr(\Sigma))$. We then proceed, exactly as in the consideration of cases (1)(a) and (2) in the proof of Proposition 8.1, to obtain a contradiction. □

Propositions 7.4 and 9.1 immediately imply Theorem 1.3.

Theorem 1.3: *Let M be a compact, orientable, hyperbolizable 3-manifold with nonempty incompressible boundary, which is not an interval bundle. If M does not contain an essential annulus with one boundary component contained in a toroidal boundary component of M, then there exists an open $\mathrm{Out}(\pi_1(M))$-invariant subset $\hat{W}(M)$ of $X_T(M)$ such that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $\hat{W}(M)$ and*

$$W(M) = \hat{W}(M) \cap X_T(M).$$

In particular, $\hat{W}(M)$ intersects $\partial AH(M)$.

We next adapt the proof of Lemma 15 in Lee [26] to establish Proposition 1.4:

Proposition 1.4: *Let M be a compact, orientable, hyperbolizable 3-manifold with nonempty incompressible boundary and non-abelian fundamental group. If M contains an essential annulus with one boundary component contained in a toroidal boundary component, then every point in $AH(M)$ is a limit of representations in $X(M)$ which are fixed points of infinite order elements of $\mathrm{Out}(\pi_1(M))$.*

Proof. Let A be an essential annulus in M with one boundary component contained in a toroidal boundary component of M and let a be the core curve of A.

First suppose that $\rho \in \mathrm{int}(AH(M))$. Theorem 5.7 in Bromberg [9] implies that there exists a neighborhood U of $\rho \in X(M)$ and an open holomorphic map $Tr_a : U \to \mathbb{C}$ such that if $\rho' \in U$, then the trace of $\rho'(a)$ is given by $\pm Tr_a(\rho')$. (Recall that the trace of a representation into $\mathrm{PSL}(2,\mathbb{C})$ is only well-defined up to sign.) Therefore, there exists a sequence $\{\rho_n\} \subset X(M)$ such that $\{\rho_n\}$ converges to ρ and $\rho_n(a)^n = Id$ for all large enough n. (Simply choose a sequence of representations $\{\rho_n\}$ converging to ρ such that $Tr_a(\rho_n) = \pm 2\cosh(\frac{\pi}{n})$.) For each n, ρ_n is fixed by the infinite order element $(D_A)_*^n \in \mathrm{Out}(\pi_1(M))$ where D_A is the Dehn twist about A. Therefore, ρ is a limit of fixed points of infinite order elements of $\mathrm{Out}(\pi_1(M))$.

The Density Theorem ([8, 10, 35, 37]) assures us that $AH(M)$ is the closure of its interior, so, by diagonalization, every representation in $AH(M)$ is also a limit of fixed points of infinite order elements of $\mathrm{Out}(\pi_1(M))$. □

One may combine Proposition 1.4 with Proposition 7.4, Theorem 1.3 and Theorem 1.2 from [15] to prove Corollary 1.5.

Corollary 1.5: *If M is a compact, orientable, hyperbolizable 3-manifold with incompressible boundary and non-abelian fundamental group, then $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on an open, $\mathrm{Out}(\pi_1(M))$-invariant neighborhood of $AH(M)$ in $X(M)$ if and only if M does not contain a primitive essential annulus or an essential annulus with one boundary component contained in a toroidal boundary component of M.*

Proof. Suppose that M does not contain a primitive essential annulus or an essential annulus with one boundary component contained in a toroidal boundary component of M. Theorem 1.3 implies that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $W(M)$, while Proposition 7.4 implies that $AH(M) \subset W(M)$ and that $W(M)$ is open in $X(M)$. Therefore, $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on an open neighborhood of $AH(M)$ in $X(M)$.

On the other hand, if M contains a primitive essential annulus, then $\mathrm{Out}(\pi_1(M))$ does not act properly discontinuously on $AH(M)$, by Theorem 1.2 of [15], so it cannot act properly discontinuously on an open $\mathrm{Out}(\pi_1(M))$-invariant neighborhood of $AH(M)$. If M contains an essential annulus with a boundary component contained in a toroidal boundary component of M, Proposition 1.4 shows that no point in $AH(M)$ can be contained in a domain of discontinuity for the action of $\mathrm{Out}(\pi_1(M))$ on $W(M)$. The consideration of these two cases completes the proof of Corollary 1.5. □

References

[1] I. Agol, "Tameness of hyperbolic 3-manifolds," preprint, available at: http://front.math.ucdavis.edu/math.GT/0405568

[2] R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry*, Springer-Verlag Universitext, 1992.

[3] L. Bers, "Spaces of Kleinian groups," in *Maryland Conference in Several Complex Variables I.* Springer-Verlag Lecture Notes in Math, No. 155 (1970), 9–34.

[4] L. Bers, "On moduli of Kleinian groups," *Russian Math. Surveys* **29** (1974), 88–102.

[5] F. Bonahon, "Bouts des variétés hyperboliques de dimension 3," *Annals of Math.* **124** (1986), 71–158.

[6] B. Bowditch, "Geometrical finiteness for hyperbolic groups," *J. Funct. Anal.* **113** (1993), 245–317.

[7] B. Bowditch, "Markoff triples and quasi-Fuchsian groups," *Proc. London Math. Soc.* **77** (1998), 697–736.

[8] J. Brock, R. Canary and Y. Minsky, "The classification of Kleinian surface groups, II: The Ending Lamination Conjecture," *Annals of Math.*, **176**(2012), 1–149.

[9] K. Bromberg, "Rigidity of geometrically finite hyperbolic cone-manifolds," *Geom. Ded.* **105**(2004), 143–170.

[10] K. Bromberg and J. Souto, "The density conjecture: a prehistoric approach," in preparation.

[11] D. Calegari and D. Gabai, "Shrinkwrapping and the taming of hyperbolic 3-manifolds," *J. Amer. Math. Soc.* **19**(2006), 385–446.

[12] R.D. Canary, "A covering theorem for hyperbolic 3-manifolds and its applications," *Topology* **35**(1996), 751–778.

[13] R.D. Canary and S. Hersonsky, "Ubiquity of geometric finiteness in boundaries of deformation spaces of hyperbolic 3-manifolds," *Amer. J. Math.* **126**(2004), 1193–1220.

[14] R.D. Canary and D. McCullough, *Homotopy equivalences of 3-manifolds and deformation theory of Kleinian groups,* Mem. Amer. Math. Soc. **172**(2004), no. 812.

[15] R.D. Canary and P. Storm, "Moduli spaces of hyperbolic 3-manifolds and dynamics on character varieties," *Comm. Math. Helv.*, **88**(2013), 221–251.

[16] V. Chuckrow, "On Schottky groups with applications to Kleinian groups," *Annals of Mathematics*, **88** (1968), 47-61.

[17] T. Gelander and Y. Minsky, "The dynamics of $\mathrm{Aut}(F_n)$ on redundant representations," *Groups, Geom., and Dyn.*, to appear. preprint available at: `http://front.math.ucdavis.edu/1104.4774`.

[18] W. Goldman, "Topological components of spaces of representations," *Invent. Math.* **93**(1988), 557-607.

[19] V. Guirardel and G. Levitt, "Splittings and automorphisms of relatively hyperbolic groups," preprint, available at: http://front.math.ucdavis.edu/1212.1434

[20] W. Jaco and P. Shalen, *Seifert fibered spaces in* 3*-manifolds*, Mem. Amer. Math. Soc. **21** (1979), no. 220.

[21] K. Johannson, *Homotopy Equivalences of* 3*-manifolds with Boundary*, Lecture Notes in Mathematics, vol. 761, Springer-Verlag, 1979.

[22] T. Jørgensen, "On discrete groups of Möbius transformations," *American Journal of Mathematics*, **98** (1976), 739-749.

[23] M. Kapovich, *Hyperbolic manifolds and discrete groups*, Progr. Math. **183**, Birkhäuser, 2001.

[24] I. Kra, "On spaces of Kleinian groups," *Comm. Math. Helv.* **47** (1972), 53–69.

[25] R. Kulkarni and P. Shalen, "On Ahlfors' finiteness theorem," *Adv. in Math.* **76** (1989), 155–169.

[26] M. Lee, "Dynamics on the PSL$(2,\mathbb{C})$-character variety of a twisted I-bundle," *Groups, Geom., and Dyn.*, to appear, preprint, available at: `http://front.math.ucdavis.edu/1103.3479`.

[27] M. Lee, "Dynamics on the PSL$(2,\mathbb{C})$-character variety of a compression body," in preparation.

[28] A. Marden, "The geometry of finitely generated Kleinian groups," *Ann. of Math.* **99**(1974), 383–462.

[29] B. Maskit,"Self-maps of Kleinian groups," *Amer. J. Math.* **93** (1971), 840–856.

[30] B. Maskit, *Kleinian groups*, Springer-Verlag, 1988.

[31] D. McCullough, "Compact submanifolds of 3-manifolds with boundary," *Quart. J. Math. Oxford* **37** (1986), 299–307.

[32] D. McCullough, "Virtually geometrically finite mapping class groups of 3-manifolds," *J. Diff. Geom.* **33**(1991), 1–65.

[33] Y. Minsky, "Note on dynamics of $\mathrm{Out}(F_n)$ on $\mathrm{PSL}_2(\mathbb{C})$-characters," *Israel J. Math.*, **193**(2013), 47–70.

[34] J. Morgan, "On Thurston's uniformization theorem for three-dimensional manifolds" in *The Smith Conjecture*, Pure Appl. Math. **112**, Academic Press, 1984, 37–125.

[35] H. Namazi and J. Souto, "Non-realizability and ending laminations - Proof of the Density Conjecture," *Acta Math.* **209**(2012), 323–395.

[36] K. Ohshika, "Geometrically finite Kleinian groups and parabolic elements," *Proc. Edin. Math. Soc.* **41**(1998), 141–159.

[37] K. Ohshika, "Realising end invariants by limits of minimally parabolic, geometrically finite groups," *Geom. and Top.* **15**(2011), 827–890.
[38] P. Scott, "Compact submanifolds of 3-manifolds," *J. London Math. Soc.* **7** (1974), 246–250.
[39] J. Souto and P. Storm, "Dynamics of the mapping class group action on the variety of $\mathrm{PSL}_2(\mathbf{C})$-characters," *Geom. Topol.* **10** (2006), 715–736.
[40] S.P. Tan, Y. L. Wong, and Y. Zhang, "Generalized Markoff maps and McShane's identity," *Adv. Math.* **217**(2008), 761–813.
[41] D. Sullivan, "Quasiconformal homeomorphisms and dynamics. II. Structural stability implies hyperbolicity for Kleinian groups," *Acta Math.* **155** (1985), 243–260.
[42] W.P. Thurston, *The geometry and topology of 3-manifolds*, lecture notes, Princeton University.