Metric spaces: definitions and examples

If $X$ is a set, then a function $d : X \times X \to [0, \infty)$ is a metric if
(M1) $d(x, y) = 0$ if and only if $x = y$,
(M2) $d(x, y) = d(y, x)$ for all $x, y \in X$, and
(M3) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

$\mathbb{R}^n$ admits the metrics
$$d_2(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$
and
$$d_\infty(\vec{x}, \vec{y}) = \max\{|x_i - y_i| \mid i = 1, \ldots, n\}.$$

$d_2$ is called the Euclidean metric.

If $X$ is a set, then the discrete metric $d : X \times X \to [0, \infty)$ is given by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$.

The space $C([a, b], \mathbb{R})$ of all continuous functions $f : [a, b] \to \mathbb{R}$ admits the metrics $d_1$ and $d_\infty$ where
$$d_1(f, g) = \int_{a}^{b} |f(x) - g(x)| \, dx$$
and
$$d_\infty(f, g) = \max\{|f(x) - g(x)| \mid x \in [a, b]\}.$$

Open sets and continuity

If $(X, d)$ is a metric space, $x \in X$ and $\epsilon > 0$, then
$$B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}.$$  

A subset $U$ of $X$ is open if for all $x \in U$, there exists $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subset U$.

Lemma: If $\epsilon > 0$ and $x \in X$, then $B(x, \epsilon)$ is an open subset of $X$.

Proposition: If $(X, d)$ is a metric space, then
(i) The emptyset $\emptyset$ and $X$ are open.
(ii) If $\{U_\alpha\}_{\alpha \in \Lambda}$ is a collection of open sets, then $\bigcup_{\alpha \in \Lambda} U_\alpha$ is an open set.
(iii) If $\{U_1, \ldots, U_n\}$ is a finite collection of open sets, then $\bigcap_{i=1}^{n} U_i$ is open.

Proposition: If $(X, d)$ is a metric space and $x$ and $y$ are distinct points in $X$, then there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $y \in V$, i.e. metric spaces are Hausdorff.

A function between metric spaces $f : (X_1, d_1) \to (X_2, d_2)$ is continuous at $x \in X_1$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $y \in X$ and $d_1(x, y) < \delta$, then $d_2(f(x), f(y)) < \epsilon$. $f$ is continuous if it is continuous at every point in $X_1$.

Theorem: A function $f : (X_1, d_1) \to (X_2, d_2)$ is continuous if and only if whenever $U$ is an open subset of $X_2$, then $f^{-1}(U)$ is open in $X_1$.  

Closed sets and continuity

A subset $C$ of a metric space $(X, d)$ is said to be closed in $X$ if its complement $X \setminus C$ is an open set in $X$.

**Proposition:** If $(X, d)$ is a metric space, then

(i) The emptyset $\emptyset$ and $X$ are closed.

(ii) If $\{C_\alpha\}_{\alpha \in \Lambda}$ is a collection of closed sets, then $\bigcap_{\alpha \in \Lambda} C_\alpha$ is a closed set.

(iii) If $\{C_1, \ldots, C_n\}$ is a finite collection of closed sets, then $\bigcup_{i=1}^n C_i$ is closed.

**Lemma:** If $(X, d)$ is a metric space, $x \in X$ and $\epsilon \geq 0$, then the closed disk

$$D(x, \epsilon) = \{y \in X \mid d(y, x) \leq \epsilon\}$$

of radius $\epsilon$ about $x$ is a closed subset of $X$.

**Theorem:** A function $f : (X_1, d_1) \to (X_2, d_2)$ is continuous if and only if whenever $C$ is a closed subset of $X_2$, then $f^{-1}(C)$ is closed in $X_1$.

**Sequences in metric spaces:**

A sequence $\{x_n\}$ in a metric space $(X, d)$ is said to be convergent with limit $x \in X$ if for all $\epsilon > 0$, there exists $N$ such that if $n \geq N$, then $d(x_n, x) < \epsilon$. We write $\lim x_n = x$.

**Theorem:** Suppose that $f : X \to Y$ is a function between metric spaces. Then $f$ is continuous at $x$ if and only if whenever $\{x_n\}$ is a sequence in $X$ such that $\lim x_n = x$, then $\lim f(x_n) = f(\lim x_n)$.

**Lemma:** A convergent sequence $\{x_n\}$ in a metric space $X$ is bounded (i.e. there exists $x_0 \in X$ and $R > 0$ so that $x_n \in D(x_0, R)$ for all $n$).

A sequence $\{x_n\}$ in a metric space $X$ is said to be a Cauchy sequence if for all $\epsilon > 0$ there exists $N$ such that if $n, m \geq N$, then $d(x_n, x_m) < \epsilon$.

**Lemma:** Any convergent sequence in a metric space is a Cauchy sequence.

**Closure and interior:**

If $x \in X$, then an open set $U$ in $X$ is an open neighborhood of $x$ if it contains $x$. If $A$ is a subset of a metric space $X$, then $x \in \bar{A}$ if and only if whenever $U$ is an open neighborhood of $x$, $U \cap A$ is non-empty. Moreover, $a \in A^0$ if and only if there exists an open neighborhood $U$ of $a$ which is contained in $A$. $\bar{A}$ is called the closure of $A$ and $A^0$ is called the interior of $A$.

**Lemma:** If $A$ is a subset of a metric space $X$, then

1. $A^0 \subseteq A \subseteq \bar{A}$.
2. $A^0$ is open and $\bar{A}$ is closed in $X$.
3. If $C$ is a closed subset of $X$ and $A \subseteq C$, then $\bar{A} \subseteq C$.
4. $x \in A$ if and only if there exists a convergent sequence $\{x_n\} \subseteq A$ such that $x = \lim x_n$.
5. $A$ is closed if and only if $A = \bar{A}$.
6. $A$ is open if and only if $A = A^0$.

**Useful Lemma:** A subset $V$ of a metric space $(X, d)$ is open in $X$ if and only if for all $x \in V$, there exists an open neighborhood $U_x$ of $x$ which is contained in $V$, i.e. $x \in U_x \subseteq V$.

**Proposition:** A subset $C$ of a metric space $(X, d)$ is closed if and only if whenever $\{x_n\}$ is a convergent sequence such that $\{x_n\} \subseteq C$, then $\lim x_n \in C$. 

---

2
**Sequential compactness:**

A subset $C$ of a metric space $X$ is **sequentially compact** if any sequence $\{x_n\}$ in $C$ has a convergent subsequence $\{x_{n_j}\}$ with limit in $C$, i.e. $\lim x_{n_j} \in C$.

A subset $A$ of a metric space $X$ is **bounded** if there exists $x_0 \in X$ and $R > 0$ such that $A \subset D(x_0, R)$.

**Proposition:** A sequentially compact subset of a metric space is closed and bounded.

**Proposition:** If $C$ is a closed subset of a sequentially compact metric space $X$, then $C$ is sequentially compact.

**Proposition:** If $f : X \rightarrow Y$ is a continuous map between metric spaces and $C \subset X$ is sequentially compact, then $f(C)$ is sequentially compact.

**Theorem:** If $f : X \rightarrow \mathbb{R}$ is continuous and $C \subset X$ is sequentially compact, then there exists $c \in C$ such that $f(c) = \sup f(C)$, i.e. $f$ achieves its supremum on $C$.

**Theorem:** A subset of $\mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded.

**Lemma:** A Cauchy sequence $\{x_n\}$ in a sequentially compact metric space is convergent.

**Topological spaces**

If $X$ is a set, then a collection $\mathcal{T}$ of subsets of $X$ is a **topology** if

1. $\emptyset, X \in \mathcal{T}$,
2. If $\{U_\alpha\}_{\alpha \in \Lambda}$ is a collection of elements of $\mathcal{T}$, then their union $\bigcup_{\alpha \in \Lambda} U_\alpha$ is also an element of $\mathcal{T}$, and
3. If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$

The pair $(X, \mathcal{T})$ is called a topological space and elements of $\mathcal{T}$ are called **open** sets in $X$. A subset $C$ of $X$ is **closed** if and only if $X - C$ is open.

A function $f : (X_1, \mathcal{T}_1) \rightarrow (X_2, \mathcal{T}_2)$ between topological spaces is **continuous** if whenever $V \in \mathcal{T}_2$, then $f^{-1}(V) \in \mathcal{T}_1$.

A map $f : (X_1, \mathcal{T}_1) \rightarrow (X_2, \mathcal{T}_2)$ is a **homeomorphism** if it is one-to-one, onto and continuous and its inverse map $f^{-1} : (X_2, \mathcal{T}_2) \rightarrow (X_1, \mathcal{T}_1)$ is also continuous.

If $\mathcal{T}$ is a topology for $A$, then a subset $\mathcal{B}$ of $\mathcal{T}$ is a **basis** if every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$.

**Facts:**
1. A function $f : (X_1, \mathcal{T}_1) \rightarrow (X_2, \mathcal{T}_2)$ between topological spaces is continuous if and only if whenever $C$ is closed in $X_2$, $f^{-1}(C)$ is closed in $X_1$.
2. Suppose that $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are topological spaces and $\mathcal{B}_2$ is a basis for $\mathcal{T}_2$. A function $f : (X_1, \mathcal{T}_1) \rightarrow (X_2, \mathcal{T}_2)$ is continuous if and only if whenever $B$ is an element of $\mathcal{B}_2$, $f^{-1}(B)$ is open in $X_1$.

If $X$ is any set, then the **indiscrete** topology on $X$ is given by $\mathcal{T}_{\text{indiscrete}} = \{\emptyset, X\}$, while the **discrete** topology $\mathcal{T}_{\text{discrete}}$ is given by the collection of all subsets of $X$.

**Fact:** If $(X, \mathcal{D})$ is a metric space, then the collection $\mathcal{T}_d$ of all open subsets in the metric space forms a topology on $X$.

If $(X, \mathcal{T})$ is a topological space and $A \subset X$, then the **subspace topology** on $A$ is given by $\mathcal{T}_A = \{A \cap U \mid U \in \mathcal{T}\}$.
A topological space \((X, \mathcal{T})\) is **metrizable** if there exists a metric \(d\) on \(X\) such that \(\mathcal{T} = \mathcal{T}_d\).

A topological space \(X\) is **Hausdorff** if whenever \(x\) and \(y\) are distinct points in \(X\), then there exist disjoint open sets \(U\) and \(V\) in \(X\) so that \(x \in U\) and \(y \in V\).

**Fact:** Any metrizable topological space is Hausdorff.

If \(A\) is a subset of a topological space \(X\), then \(x \in \overline{A}\) if every open neighborhood of \(x\) intersects \(A\). Moreover, \(x \in A^0\) if there exists an open neighborhood of \(x\) which is contained in \(A\).

**Facts:**

1. \(A^0 \subset A \subset \overline{A}\).

2. \(A^0\) is open and \(\overline{A}\) is closed in \(X\).

3. If \(C\) is a closed subset of \(X\) and \(A \subset C\), then \(\overline{A} \subset C\).

4. \(A\) is closed if and only if \(A = \overline{A}\).

5. \(A\) is open if and only if \(A = A^0\).

**The product topology**

If \((X, \mathcal{T}_X)\) and \((Y, \mathcal{T}_Y)\) are topological spaces then we say that \(W \in \mathcal{T}_{X \times Y}\) if for each \((x, y) \in W\), there exists \(U_{(x,y)} \in \mathcal{T}_X\) and \(V_{(x,y)} \in \mathcal{T}_Y\) such that 
\[(x, y) \in U_{(x,y)} \times V_{(x,y)} \subset W.\]

\(\mathcal{T}_{X \times Y}\) is called the **product topology** on \(X \times Y\).

**Fact:** If \((X_1, \mathcal{T}_1)\) and \((X_2, \mathcal{T}_2)\) are topological spaces, then 
\[\mathcal{B} = \{V_1 \times V_2 \mid V_1 \in \mathcal{T}_1, V_2 \in \mathcal{T}_2\}\]

is a basis for the **product topology** on \(X_1 \times X_2\).

If \((X, d_X)\) and \((Y, d_Y)\) are metric spaces, then
\[d_\infty((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}\]

is a metric on \(X \times Y\) and \(d_\infty\) induces the product topology associated to \((X, \mathcal{T}_{d_X})\) and \((Y, \mathcal{T}_{d_Y})\) on \(X \times Y\).

**Facts:** (Suppose that \((X, \mathcal{T}_X)\) and \((Y, \mathcal{T}_Y)\) are topological spaces and \(X \times Y\) is given the product topology.

1. \(X \times Y\) is Hausdorff if and only \(X\) and \(Y\) are Hausdorff.

2. \(\pi_X : X \times Y \to X\) (given by \(\pi_X(x, y) = x\)) and \(\pi_Y : X \times Y \to Y\) (given by \(\pi_Y(x, y) = y\)) are continuous, open maps.

3. If \((Z, \mathcal{T}_Z)\) is a topological space and \(f : Z \to X\) and \(g : Z \to Y\) are continuous and \(h : Z \to X \times Y\) is given by \(h(z) = (f(z), g(z))\), then \(h\) is continuous.

**Connectedness**

A topological space \((X, \mathcal{T})\) is **disconnected** if there exist disjoint non-empty open subsets \(U\) and \(V\) such that \(X = U \cup V\). A topological space is called **connected** if it is not disconnected.

**Fact:** A topological space \(X\) is disconnected if and only if there exists a continuous, onto function \(f : X \to \{0, 1\}\) (where \(\{0, 1\}\) is given the discrete topology).
A subset $A$ of a topological space $X$ is said to be disconnected if it is disconnected in the subspace topology, i.e. $(A, T_A)$ is disconnected. Equivalently, a subset $A$ of $X$ is disconnected if and only if there exist open sets $U$ and $V$ in $X$ such that $A \subset U \cup V$, $(U \cap V) \cap A = \emptyset$, and $A \cap U$ and $A \cap V$ are both non-empty.

A non-empty subset $S$ of $\mathbb{R}$ is an interval if and only whenever $x, y \in S$ and $x < z < y$, then $z \in S$. An interval $J \subset \mathbb{R}$ has the form either $[a, b]$, $]a, b[$, $(a, b]$, $(-\infty, b]$, $(-\infty, b)$, $(a, \infty)$, $[a, \infty)$, $\mathbb{R}$ or $\{a\}$ where $a, b \in \mathbb{R}$ and $b > a$.

**Facts:**

1. A non-empty subset of $\mathbb{R}$ is connected if and only if it is an interval.
2. If $f : X \to Y$ is continuous and $X$ is a connected topological space, then $f(X)$ is connected.
3. If $f : X \to \mathbb{R}$ is continuous, $X$ is a connected topological space, $a, b \in f(X)$ and $c$ lies between $a$ and $b$, then there exists $x \in X$ such that $f(x) = c$.
4. If $A \subset X$ and $A$ is connected, then $\overline{A}$ is also connected.
5. If $A$ and $B$ are connected subsets of a topological space $X$ and $A \cap B \neq \emptyset$, then $A \cup B$ is a connected subset of $X$.
6. If $(X, T_X)$ and $(Y, T_Y)$ are topological spaces, then $X \times Y$ is connected in the product topology if and only if $X$ and $Y$ are connected.

If $a$ and $b$ are points in a topological space $X$, then a path joining $a$ to $b$, is a continuous function $\gamma : [0, 1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = b$. A topological space is path connected if any two points in the space may be joined by a path.

**Facts:**

1. A path connected topological space is connected.
2. If $f : X \to Y$ is continuous and $X$ is a path connected topological space, then $f(X)$ is path connected.
3. A product of two path connected topological spaces is path connected.

**Compactness**

A collection $\mathcal{U} = \{U_a\}_{a \in \Lambda}$ of open subsets of a topological space $X$ is an open cover of $X$ if $\bigcup_{a \in \Lambda} U_a = X$. A subcover of $\mathcal{U}$ is a subcollection $\mathcal{V} \subset \mathcal{U}$ which is also an open cover of $X$. A topological space is compact if every open cover has a finite subcover.

A subset $A$ of $X$ is said to be compact if it is compact in the subspace topology. Equivalently, $A$ is a compact subset of $X$ if and only if whenever $\mathcal{U} = \{U_a\}_{a \in \Lambda}$ is a collection of open subsets of $X$ such that

$$A \subset \bigcup_{a \in \Lambda} U_a,$$

then there exists a finite subcollection

$$\{U_{a_1}, \ldots, U_{a_n}\} \subset \{U_a\}_{a \in \Lambda}$$

such that

$$A \subset U_{a_1} \cup \cdots \cup U_{a_n}.$$

**Facts:**

1. If $f : X \to Y$ is continuous and $X$ is a compact topological space, then $f(X)$ is compact.
2. Any compact subset of a Hausdorff topological space is closed.
3. Any closed subset of a compact topological space is compact.
4. If $f : X \to \mathbb{R}$ is a continuous function and $X$ is a compact topological space, then there exists
z \in X \text{ such that } f(z) = \sup f(X), \text{ i.e. } f \text{ achieves its supremum.}

(5) If \((X, d)\) is a metric space, then \(X\) is compact if and only if it is sequentially compact.

(6) A product of two topological spaces is compact if and only if each space is compact.

(7) If \(X\) is a sequentially compact metric space, and \(U\) is an open cover of \(X\), then there exists \(\delta > 0\) such that if \(x \in X\), then there exists \(U \in U\) such that \(B(x, \delta) \subset U\).

(8) If \(X\) is sequentially compact and \(\epsilon > 0\), then there exists a finite collection of points \(\{x_1, \ldots, x_n\}\) in \(X\) so that the corresponding collection of \(\epsilon\)-balls \(\{B(x_1, \epsilon), \ldots, B(x_n, \epsilon)\}\) covers \(X\).

(9) If \(X\) is a compact and Hausdorff topological space and \(C\) is a closed subset of \(X\) and \(x \in X \setminus C\), then there exist disjoint open sets \(U\) and \(V\) in \(X\) so that \(C \subset U\) and \(x \in V\).

**DeMorgan’s Laws:** Let \(X\) be a set and let \(\{A_\alpha\}_{\alpha \in \Lambda}\) be a collection of subsets of \(X\).

1. \(X \setminus \bigcup_{\alpha \in \Lambda} A_\alpha = \bigcap_{\alpha \in \Lambda} (X \setminus A_\alpha)\)
2. \(X \setminus \bigcap_{\alpha \in \Lambda} A_\alpha = \bigcup_{\alpha \in \Lambda} (X \setminus A_\alpha)\)