QUASICONFORMAL HOMOGENEITY OF HYPERBOLIC SURFACES WITH FIXED-POINT FULL AUTOMORPHISMS

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Abstract. We show that any closed hyperbolic surface admitting a conformal automorphism with "many" fixed points is uniformly quasiconformally homogeneous, with constant uniformly bounded away from 1. In particular, there is a uniform lower bound on the quasiconformal homogeneity constant for all hyperelliptic surfaces. In addition, we introduce more restrictive notions of quasiconformal homogeneity and bound the associated quasiconformal homogeneity constants uniformly away from 1 for all hyperbolic surfaces.

1. Introduction

An (orientable) hyperbolic manifold M is *K-quasiconformally homogeneous* if, given any pair of points $x, y \in M$ there is a K-quasiconformal homeomorphism $f : M \to M$ such that $f(x) = y$. If there exists a K so that M is K-quasiconformally homogeneous then we say that M is *uniformly quasiconformally homogeneous*.

In [2], we established that for all dimensions $n \geq 3$ there exists a uniform constant $K_n > 1$ so that if $M \neq \mathbb{H}^n$ is K-quasiconformally homogeneous, then $K \geq K_n$. The proof of this fact depends crucially on rigidity phenomena that occur in dimensions $n \geq 3$, but that do not occur in dimension two. It is natural to ask whether there is a similar constant in dimension 2. In this note we demonstrate the existence of such a uniform constant for classes of closed hyperbolic surfaces which admit conformal automorphism with "many" fixed points.

Main Theorem: *For each $c \in (0, 2]$, there exists $K_c > 1$, such that if S is a K-quasiconformally homogeneous closed hyperbolic surface of genus g that admits a non-trivial conformal automorphism with at least $c(g+1)$ fixed points, then $K \geq K_c$.*

A classical family of examples satisfying the hypotheses of our main theorem are the hyperelliptic surfaces, which admit conformal involutions with $2g + 2$ fixed points. Note that the set of hyperelliptic surfaces of genus g forms a $(2g - 1)$-complex dimensional subvariety of the Moduli space $\mathcal{M}_g$ of all (isometry classes of) closed hyperbolic surfaces of genus g, e.g. see [5].

Corollary: *There exists a constant $K_{hyp} > 1$, such that if S is a closed hyperelliptice surface, then $K \geq K_{hyp}$.*

The first author was supported in part by NSF grant 0305704.
The second author was supported in part by NSF grant 0305634.
The third author was supported in part by NSF grants 0203698 and 0504791.
The fourth author was supported in part by NSF grant 0305704.

We also investigate more restrictive definitions of quasiconformal homogeneity in which it is possible to bound the associated homogeneity constants uniformly away from 1 for all hyperbolic surfaces (other than $\mathbb{H}^2$).

2. Basic Facts

This paper continues a study of uniformly quasiconformally homogeneous hyperbolic manifolds that was initiated in [2]. The study of the quasiconformal homogeneity properties of planar sets was begun by Gehring and Palka in [6] (see also [8] and [9].)

Let M be a uniformly quasiconformally homogenous hyperbolic manifold. We define the *quasiconformal homogeneity constant* $K(M)$ to be

$$K(M) = \inf\{K : M \text{ is } K\text{-quasiconformally homogeneous}\}.$$

One observes, see Lemma 2.1 in [2], that M is in fact $K(M)$-quasicomformally homogeneous and that, see Proposition 2.2 in [2], $K(M) > 1$ unless M is the hyperbolic space $\mathbb{H}^n$.

We recall that if M is an orientable hyperbolic n-manifold then there exists a discrete subgroup Γ of $Isom^+(\mathbb{H}^n)$, called a *Kleinian group*, so that M is isometric to $\mathbb{H}^n/\Gamma$. The group Γ also acts as a group of conformal automorphisms of $\partial_\infty \mathbb{H}^n = \mathbb{S}^{n-1}$. The *domain of discontinuity* $\Omega(\Gamma)$ is the largest open subset of $\mathbb{S}^{n-1}$ on which Γ acts properly discontinuously and the *limit set* $\Lambda(\Gamma) = \mathbb{S}^{n-1} - \Omega(\Gamma)$ is its complement.

We recall that the assumption that M is uniformly quasiconformally homogeneous places strong geometric restrictions on M. We define $l(M)$ to be the infimum of the lengths of homotopically non-trivial curves in M, and we define $d(M)$ to be the supremum of the diameters of embedded hyperbolic balls in M.

Theorem 2.1. *(Theorem 1.1 in* [2]*) For each dimension $n \geq 2$ and each $K \geq 1$, there is a positive constant $m(n, K)$ with the following property. Let $M = \mathbb{H}^n/\Gamma$ be a K-quasiconformally homogeneous hyperbolic n-manifold, which is not $\mathbb{H}^n$. Then*

1. $d(M) \leq Kl(M) + 2Klog4$,
2. $l(M) \geq m(n, K)$, *i.e. there exists a lower bound on the injectivity radius of M that depends only on n and K, and*
3. *every non-trivial element of Γ is hyperbolic, and* $\Lambda(\Gamma) = \partial(\mathbb{H}^n)$.

Using quasiconformal rigidity results we showed in [2] that in dimension at least 3, a uniformly quasiconformally homogeneous hyperbolic manifold, other than hyperbolic space itself, has quasiconformal homogeneity constant uniformly bounded away from 1.

Theorem 2.2. *(Theorem 1.4 in* [2]*) For each $n \geq 3$, there exists a constant $K_n > 1$, such that if M is any uniformly quasiconformally homogeneous hyperbolic n-manifold, other than $\mathbb{H}^n$, then $K(M) \geq K_n > 1$.*

This note is motivated by the following natural question:

Question 2.3. *Does there exist a uniform lower bound $K_2 > 1$ such that if S is an uniformly K-quasiconformally homogenous surface and $S \neq \mathbb{H}^2$, then $K \geq K_2$?*

Theorem 2.1 implies that a hyperbolic surface with finitely generated fundamental group is uniformly quasiconformally homogeneous if and only if it is closed, see Corollary 1.2 in [2]. However it is easy to construct examples of non-finite type hyperbolic surfaces that are also uniformly quasiconformally homogeneous, e.g. non-compact regular covers of closed surfaces.

3. Geometric convergence and quasiconformal homogeneity

We first recall the definition of geometric convergence. We say that a sequence of Kleinian groups $\{\Gamma_i\}$ *converges geometrically* to a Kleinian group Γ_∞ if every accumulation point of $\{\Gamma_i\}$ is in Γ_∞, and if $\gamma \in \Gamma_\infty$ then there exists a sequence $\{\gamma_i \in \Gamma_i\}$ which converges to γ.

We say that a sequence $\{M_i\}$ of hyperbolic manifolds converges *geometrically* to a hyperbolic manifold M_∞ if $M_\infty = \mathbb{H}^n/\Gamma_\infty$ and there exists a sequence of Kleinian groups $\{\Gamma_i\}$ such that $M_i = \mathbb{H}^n/\Gamma_i$ (for all i) and $\{\Gamma_i\}$ converges geometrically to Γ_∞. It is well-known that any sequence of hyperbolic n-manifolds has a geometrically convergent subsequence (see, for example, Proposition 3.5 in [7] or Corollary 3.1.7 in [4]). Moreover, a single sequence of hyperbolic manifolds can have many different geometric limits, exhibiting quite different behaviors. For example, one limit could have trivial fundamental group while another limit could have infinitely generated fundamental group.

We begin with an elementary observation about the behavior of the quasiconformal homogeneity constant under geometric convergence.

Lemma 3.1. *Let $\{M_i\}$ be a sequence of uniformly quasiconformally homogeneous hyperbolic manifolds which converges geometrically to a hyperbolic manifold M_∞. Then*

$$\liminf K(M_i) \geq K(M_\infty).$$

In particular, if $\liminf K(M_i) < \infty$ then the limit manifold M_∞ is uniformly quasiconformally homogeneous.

Proof: Let $\{\Gamma_i\}$ be a sequence of Kleinian groups such that $M_i \cong \mathbb{H}^n/\Gamma_i$ (for all i) and $\{\Gamma_i\}$ converges geometrically to Γ_∞ where $M_\infty = \mathbb{H}^n/\Gamma_\infty$. We first pass to a subsequence, still called $\{M_i\}$, so that $\lim K(M_i)$ exists and is equal to the limit inferior of the original sequence.

Let $x, y \in M_\infty$. Let $\tilde{x}$ and $\tilde{y}$ be pre-images of x and y in $\mathbb{H}^n$. Moreover, let x_i and y_i be the images of $\tilde{x}$ and $\tilde{y}$ in M_i. For all i, there exists a $K(M_i)$-quasiconformal homeomorphism $f_i : M_i \to M_i$ such that $f_i(x_i) = y_i$. There exists a lift $\tilde{f}_i : \mathbb{H}^n \to \mathbb{H}^n$ of f_i such that $\tilde{f}_i(\tilde{x}) = \tilde{y}$. The collection $\{\tilde{f}_i\}$ is a normal family and (possibly passing to a subsequence) it converges to a K-quasiconformal map $\tilde{f}_\infty : \mathbb{H}^n \to \mathbb{H}^n$ which descends to a K-quasiconformal map $f_\infty : M_\infty \to M_\infty$ such that $f_\infty(x) = y$, and $K = \lim K(M_i)$ (see e.g. Väsäilä [12] Theorem 19.2 and Theorem 37.2). □

The following consequence of Lemma 3.1 is a crucial tool in the proof of the Main Theorem.

Proposition 3.2. *Let $\{M_i\}$ be a sequence of uniformly quasiconformally homogeneous hyperbolic manifolds so that $\lim K(M_i) = 1$. Then $\lim l(M_i) = \infty$.*

Proof: Suppose that $\{l(M_i)\}$ does not converge to infinity. Then there exists a geometrically convergent subsequence of $\{M_i\}$, still denoted $\{M_i\}$, so that $\{l(M_i)\}$ is bounded. Since $\{l(M_i)\}$ is bounded and $\lim K(M_i) = 1$, Theorem 2.1 implies that $\{d(M_i)\}$ is bounded. Let R be chosen so that $d(M_i) \leq R$ for all i.

Let M_∞ be a geometric limit of $\{M_i\}$. Lemma 3.1 implies that $K(M_\infty) = 1$, so that $M_\infty = \mathbb{H}^n$. Let $\{\Gamma_i\}$ be an associated sequence of Kleinian groups so that $M_i = \mathbb{H}^n/\Gamma_i$ and so that $\{\Gamma_i\}$ converges geometrically to $\Gamma_\infty = \{1\}$. On the other hand, since $d(M_i) \leq R$, there exists $\gamma_i \in \Gamma_i - \{id\}$ so that $d(0, \gamma_i(0)) \leq 2R$. We may further assume that $d(0, \gamma_i(0)) \geq R$ (by replacing γ_i by a power of γ_i if necessary.) It follows that $\{\gamma_i\}$ has an accumulation point γ_∞ in $Isom_+(\mathbb{H}^n)$ which is non-trivial. This contradicts the fact that $\{\Gamma_i\}$ converges to the trivial group. □

Remark 3.3. Under the assumptions of Lemma 3.1, it is possible that $K(M_\infty)$ is strictly smaller than $\liminf K(M_i)$. One may readily construct a sequence of closed hyperbolic surfaces so that $\{l(M_i)\}$ stays bounded but so that $\lim d(M_i) = \infty$. Theorem 2.1 then implies that $\lim K(M_i) = \infty$, but one may also see that $\{M_i\}$ converges to $M_\infty = \mathbb{H}^2$, so $K(M_\infty) = 1$.

4. Bounds on the geometry of surfaces with many fixed points

In this section, we obtain bounds on the length $l(S)$ of the shortest homotopically non-trivial closed curve when S admits a conformal automorphism with many fixed points. This result will be a key tool in the proof of the Main Theorem. As a corollary, we obtain bounds on $d(S)$ in terms of the quasiconformal homogeneity constant $K(S)$ and the number of fixed points.

Proposition 4.1. *Let S be a closed hyperbolic surface of genus g and let ϕ be a non-trivial conformal automorphism of S with $q \geq 2$ fixed points. Then*

$$\cosh\left(\frac{l(S)}{4}\right) \leq \frac{2g-2}{q} + 1.$$

The key observation in the proof of Proposition 4.1 is that any two fixed points of ϕ are separated by at least $l(S)/2$:

Lemma 4.2. *Let S be a closed hyperbolic surface and let ϕ be a non-trivial conformal automorphism group of S. If x_1 and x_2 are distinct fixed points of ϕ and $[x_1, x_2]$ is a geodesic segment connecting x_1 to x_2, then $[x_1, x_2] \cup \phi([x_1, x_2])$ is a homotopically non-trivial closed curve in S. In particular,*

$$d(x_1, x_2) \geq \frac{l(S)}{2}.$$

Proof: It is clear, since both x_1 and x_2 are fixed, that $[x_1, x_2] \cup \phi([x_1, x_2])$ is a closed curve. If $\phi([x_1, x_2]) = [x_1, x_2]$, then ϕ must fix every point on $[x_1, x_2]$ which would contradict the fact that any non-trivial conformal automorphism of a hyperbolic surface has a finite set of fixed points. Therefore, $\phi([x_1, x_2]) \neq [x_1, x_2]$.

If $[x_1, x_2] \cup \phi([x_1, x_2])$ were homotopically trivial then $[x_1, x_2]$ and $\phi([x_1, x_2])$ would be distinct homotopic geodesics between the points x_1 and x_2, which is impossible. Therefore, $[x_1, x_2] \cup \phi([x_1, x_2])$ must be homotopically non-trivial. □

Remark 4.3. If ϕ is an involution, e.g. a hyperelliptic involution, then one may further conclude in Lemma 4.2 that $[x_1, x_2] \cup \phi([x_1, x_2])$ is a closed geodesic.

Proof of Proposition 4.1: Lemma 4.2 implies that the hyperbolic disks of radius $l(S)/4$ about the fixed points of ϕ are disjoint. Since a hyperbolic disk of radius r

has area $2\pi(\cosh r - 1)$ and S has area $2\pi(2g-2)$, the main estimate of Proposition 4.1 follows easily. □

We now combine Proposition 4.1 and Theorem 2.1 to give bounds on the diameter $d(S)$ of a maximally embedded ball.

Corollary 4.4. *If S is a closed hyperbolic surface of genus g which admits a conformal automorphism with $q \geq 2$ fixed points, then*

$$d(S) \leq 4K(S)\cosh^{-1}\left(\frac{2g-2}{q}+1\right) + 2K(S)\log 4.$$

5. Proof of the Main Theorem

We are now ready to establish our Main Theorem:

Main Theorem: *For each $c \in (0,2]$, there exists $K_c > 1$, such that if S is a K-quasiconformally homogeneous closed hyperbolic surface of genus g that admits a non-trivial conformal automorphism with at least $c(g+1)$ fixed points, then $K \geq K_c$.*

Proof: We will argue by contradiction. Fix $c \in (0,2]$. If the result is false, there exists a sequence $\{S_i\}$ of closed hyperbolic surfaces so that S_i has genus g_i , admits a conformal automorphism with at least $c(g_i+1)$ fixed points, and $\lim K(S_i) = 1$. Proposition 3.2 implies that $\lim l(S_i) = \infty$ (and hence that $g_i \to \infty$.) On the other hand, Proposition 4.1 implies that, for all large enough i,

$$l(S_i) \leq 4\cosh^{-1}\left(\frac{2g_i-2}{c(g_i+1)}+1\right) \leq 4\cosh^{-1}\left(\frac{2}{c}+1\right)$$

which establishes our desired contradiction. □

Remark 5.1. Recall that any non-trivial conformal automorphism of a closed hyperbolic surface of genus g has at most $2g+2$ fixed points, so we limit c to the interval $(0,2]$.

Remark 5.2. It is easy to construct hyperelliptic surfaces (of any genus) with arbitrarily large quasiconformal homogeneity constant, so there is no possible upper bound in the setting of our Main Theorem. One may do so, for example, by constructing a sequence $\{S_n\}$ of hyperelliptic surfaces such that $\{l(S_n)\}$ converges to 0, and then applying part (2) of Theorem 2.1.

6. More restrictive forms of quasiconformal homogeneity

In this section we will consider more restrictive notions of quasiconformal homogeneity. In particular, we will look at situations where one requires that the quasiconformal homeomorphisms are homotopic to either the identity or to a conformal automorphism. In these cases, one can bound the associated quasiconformal homogeneity constants uniformly away from 1.

We will say that S is *strongly K-quasiconformally homogeneous* if for any two points $x, y \in S$, there is a K-quasiconformal homeomorphism of S taking x to y which is homotopic to a conformal automorphism of S. If S is strongly K-quasiconformally homogeneous for some K then we simply say that S is *strongly quasiconformally homogeneous.*

Similarly, we will say that S is *extremely K-quasiconformally homogeneous* if for any two points $x, y \in S$, there is a K-quasiconformal homeomorphism of S taking

x to y which is homotopic to the identity. If S is extremely K-quasiconformally homogeneous for some K then we simply say that S is *extremely quasiconformally homogeneous.* Note that a surface that is extremely quasiconformally homogeneous is strongly quasiconformally homogeneous.

We now introduce some convenient notation. If S is a hyperbolic surface and $f : S \to S$ is a quasiconformal automorphism of S, we let $k(f)$ be the quasiconformal dilatation of f. If $x, y \in S$, we define

$$k(x,y) = \min_f \{k(f) \mid f : S \to S \text{ is quasiconformal and } f(x) = y\},$$

$$k_{aut}(x,y) = \min_f \{k(f) \mid f(x) = y, f \simeq f',\ f' : S \to S \text{ is conformal}\},$$

and

$$k_0(x,y) = \min_f \{k(f) \mid f : S \to S \text{ is quasiconformal and } f(x) = y, f \simeq \mathrm{id}\}$$

where we use the symbol "$\simeq$" to denote the homotopy relation.

Notice that it is clear that each of these quantities is defined since one may easily construct a diffeomorphism homotopic to the identity such that $f(x) = y$ and f is equal to the identity off of a compact set. It is also easy to see that k, k_{aut}, and k_0 are all continuous on $S \times S$ for any Riemann surface S.

If S is uniformly quasiconformally homogeneous, then

$$K(S) = \sup_{(x,y)\in S\times S} k(x,y).$$

If S is strongly quasiconformally homogeneous, we may similarly define

$$K_{aut}(S) = \sup_{(x,y)\in S\times S} k_{aut}(x,y)$$

and, if S is extremely quasiconformally homogeneous, we define

$$K_0(S) = \sup_{(x,y)\in S\times S} k_0(x,y).$$

Since, by definition, $k_o(x,y) \geq k_{aut}(x,y) \geq k(x,y)$, we immediately see that:

Lemma 6.1. (1) *If S is extremely quasiconformally homogeneous, then*

$$K_0(S) \geq K_{aut}(S) \geq K(S).$$

(2) *If S is strongly quasiconformally homogeneous, then*

$$K_{aut}(S) \geq K(S).$$

We next completely characterize extremely and strongly quasiconformally homogeneous hyperbolic surfaces and show that K_0 and K_{aut} can both be bounded uniformly away from 1. We will make central use of the following estimate:

Proposition 6.2. *Let $f : \mathbb{H}^2 \to \mathbb{H}^2$ be a quasiconformal map which extends to the identity on $\partial_\infty \mathbb{H}^2$ and let $x \in \mathbb{H}^2$. Then $k(f) \geq \psi(d(x, f(x)))$, where $\psi : [0,\infty) \to [1,\infty)$ is the increasing homeomorphism given by the function*

$$\psi(d) = \coth^2\left(\frac{\pi^2}{4\mu(e^{-d})}\right) = \coth^2 \mu\left(\sqrt{1 - e^{-2d}}\right),$$

where $\mu(r)$ is the modulus of the Grötsch ring whose complementary components are $\overline{\mathbb{B}^2}$ and $[1/r, \infty]$ for $0 < r < 1$. In particular,

$$\psi(d) \sim \frac{16d^2}{\pi^4} \text{ as } d \to \infty$$
$$\text{and } \psi(d) \sim 1 + \frac{d}{2} \text{ as } d \to 0.$$

Proof: We may assume that $\mathbb{H}^2$ is modelled by the Poincaré disc $\mathbb{B}^2$ and that $x = 0$. In dimension 2 the extremal problem of finding the smallest possible dilatation K of a quasiconformal mapping $f : \mathbb{B}^2 \to \mathbb{B}^2$ that extends to the identity on $\partial\mathbb{B}^2$ and maps the origin 0 to the point $-\sigma \in \mathbb{B}^2$ where $0 < \sigma < 1$ was considered by Teichmüller in [11]. In particular, Teichmüller shows that the dilatation K of this extremal mapping satisfies

$$K = \left(\frac{R + \frac{1}{R}}{R - \frac{1}{R}}\right)^2,$$

where $\log R$ is the conformal modulus of the ring domain given by the unit disk $\mathbb{B}^2$ minus the slit on the imaginary axis from $-i\sqrt{\sigma}$ to $i\sqrt{\sigma}$.

Observe that the ring domain $\mathbb{B}^2 \setminus [-i\sqrt{\sigma}, i\sqrt{\sigma}]$ can be mapped conformally onto the Grötsch ring $R_{G,2}(s)$, where

$$\sqrt{\sigma} = s - \sqrt{s^2 - 1}. \tag{6.1}$$

Here, we denote by $R_{G,2}(s)$ the Grötsch ring whose complementary components are $\overline{\mathbb{B}^2}$ and $[s, \infty]$, where $s > 1$. Following Anderson, Vamanamurthy and Vuorinen [1, 8.35], we define

$$\mu(r) := \quad \text{mod } R_{G,2}\left(\frac{1}{r}\right), \quad 0 < r < 1.$$

Since $\mathbb{B}^2 \setminus [-i\sqrt{\sigma}, i\sqrt{\sigma}]$ and $R_{G,2}(s)$ are conformally equivalent, their conformal moduli agree. Thus we see that

$$\log R = \quad \text{mod } R_{G,2}(s) = \mu\left(\frac{1}{s}\right).$$

But (6.1) implies that $s = \frac{1+\sigma}{2\sqrt{\sigma}}$. Furthermore, if d denotes the hyperbolic distance between 0 and $-\sigma$ then $\sigma = (e^d - 1)/(e^d + 1)$. Thus

$$\frac{1}{s} = \frac{2\sqrt{\sigma}}{1+\sigma} = \frac{2\sqrt{\frac{e^d-1}{e^d+1}}}{1 + \frac{e^d-1}{e^d+1}} = \frac{2\sqrt{e^{2d}-1}}{2e^d} = \sqrt{1 - e^{-2d}}.$$

Hence the conformal radius $\log R$ is given by

$$\log R = \mu\left(\sqrt{1 - e^{-2d}}\right),$$

and thus the dilatation K is

$$\begin{aligned} K &= \left(\frac{R + \frac{1}{R}}{R - \frac{1}{R}}\right)^2 = \left(\frac{e^{\mu(\sqrt{1-e^{-2d}})} + e^{-\mu(\sqrt{1-e^{-2d}})}}{e^{\mu(\sqrt{1-e^{-2d}})} - e^{-\mu(\sqrt{1-e^{-2d}})}}\right)^2 \\ &= \coth^2 \mu\left(\sqrt{1 - e^{-2d}}\right) \\ &= \coth^2\left(\frac{\pi^2}{4\mu(e^{-d})}\right), \end{aligned}$$

where the last equality follows from the first equality in [1, (5.2)].

The asymptotic estimates for ψ can be found using the fact ([1, 5.13(2)]) that

$$\lim_{r\to 0+}(\mu(r)+\log r)=\log 4.$$

□

Remark 6.3. This proposition, without the explicit estimate, follows easily from the compactness of the family of all K–quasiconformal mappings $f:\mathbb{B}^n\to\mathbb{B}^n$ with $f|\partial\mathbb{B}^n=\mathrm{id}$ (see Lemma 3.1 in [3] or Lemma 4.1 in [2]). A proof of a more general version of this lemma in the case of quasiisometries between manifolds of strictly negative curvature can be found for example in [10, 16.11].

Proposition 6.2 allows us to bound K_0 uniformly away from 1 and to obtain a complete characterization of extremely quasiconformally homogeneous hyperbolic surfaces.

Theorem 6.4. *A hyperbolic surface, other than* $\mathbb{H}^2$*, is extremely quasiconformally homogeneous if and only if it is closed. If S is a closed hyperbolic surface, then*

$$K_0(S)\geq \psi(\mathrm{diam}(S))\geq \psi\left(\sinh^{-1}\left(\frac{2}{\sqrt{3}}\right)\right)=1.626\ldots>1.$$

Moreover,

$$K_0(S)\leq\left(e^{\frac{l(S)}{4}}+1\right)^{2\left(\frac{4\mathrm{diam}(S)}{l(S)}+1\right)}.$$

Proof: Let S be an extremely quasiconformally homogeneous hyperbolic surface. Let $x,y\in S$ and let f be a quasiconformal automorphism of S so that $f(x)=y$ and f is homotopic to the identity. Let $\tilde{f}:\mathbb{H}^2\to\mathbb{H}^2$ be a lift of f which extends to the identity map on $\partial_\infty\mathbb{H}^2$. Proposition 6.2 then immediately implies that $k(\tilde{f})=k(f)\geq\psi(d(x,y))$, so $k_0(x,y)\geq\psi(d(x,y))$. Since ψ is proper and increasing, we immediately conclude that S must have finite diameter and that $K_0(S)\geq\psi(\mathrm{diam}(S))$. The main result of Yamada [14] implies that any closed hyperbolic surface has diameter at least $\sinh^{-1}\left(\frac{2}{\sqrt{3}}\right)$, so

$$K_0(S)\geq \psi(\mathrm{diam}(S))\geq \psi\left(\sinh^{-1}\left(\frac{2}{\sqrt{3}}\right)\right)=1.626\ldots>1.$$

On the other hand, if S is a closed surface, then Lemma 2.6 in [2] may be used to show, exactly as in the proof of Proposition 2.4 in [2], that S is K-extremely quasiconformally homogeneous where

$$K\leq\left(e^{\frac{l(S)}{4}}+1\right)^{2\left(\frac{4\mathrm{diam}(S)}{l(S)}+1\right)}.$$

□

With a little more effort, we can characterize strongly quasiconformally homogeneous hyperbolic surfaces and obtain uniform lower bounds on K_{aut}. Theorem 6.5 is a direct 2-dimensional analogue of the main results, Theorems 1.3 and 1.4, of [2]. The key difference in dimensions 3 and above is that every quasiconformal automorphism of a uniformly quasiconformally homogeneous hyperbolic manifold is homotopic to a conformal automorphism, see Proposition 4.2 in [2]. In particular, in dimensions 3 and above, uniformly quasiconformally homogeneous hyperbolic manifolds are strongly quasiconformally homogeneous and $K=K_{aut}$.

Theorem 6.5. *A hyperbolic surface is strongly quasiconformally homogeneous if and only if it is a regular cover of a closed hyperbolic orbifold. If S is a strongly quasiconformally homogeneous surface, other than $\mathbb{H}^2$, then*

$$K_{aut}(S) \geq \psi(\tau) > 1.05951... > 1.$$

where

$$\tau = \sinh^{-1}\left(\frac{4\cos^2(\pi/7) - 3}{8\cos(\pi/7) + 7}\right) \approx 0.131467.$$

Moreover, if S is a regular cover of a closed hyperbolic 2-orbifold $Q = \mathbb{H}^2/G$, then

$$K_{aut}(S) \leq \left(e^{\frac{l'(Q)}{4}} + 1\right)^{2\left(\frac{4\text{diam}(Q)}{l'(Q)}+1\right)}$$

where $l'(Q)$ denotes the minimal translation length of a hyperbolic element of G.

Proof: Let S be a strongly quasiconformally homogeneous hyperbolic surface and let $Aut(S)$ denote its group of conformal automorphisms. Let $Q = S/Aut(S)$ be the (orientable) hyperbolic orbifold obtained by quotienting by the conformal automorphisms of S and let $p : S \to Q$ be the associated covering map. Let x and y be two points in S and let $f : S \to S$ be a quasiconformal automorphism such that $f(x) = y$ and f is homotopic to a conformal automorphism g of S. Then $h = g^{-1} \circ f$ is homotopic to the identity and $k(h) = k(f)$. Moreover, $d(x, h(x)) \geq d(p(x), p(y))$. Let $\tilde{h} : \mathbb{H}^2 \to \mathbb{H}^2$ be a lift of h so that $\tilde{h}$ extends to the identity map on $\partial_\infty \mathbb{H}^2$. Proposition 6.2 then implies that $k(h) = k(\tilde{h}) \geq \psi(d(x, h(x)) \geq \psi(d(p(x), p(y)))$. It follows that $k_{aut}(x, y) \geq \psi(d(p(x), p(y)))$. Since ψ is proper and increasing we can conclude that Q has finite diameter, so is a closed hyperbolic orbifold, and that $K_{aut}(S) \geq \psi(\text{diam}(Q))$. A result of Yamada [13] implies that the diameter of any (orientable) hyperbolic 2-orbifold is at least τ. Therefore, $K_{aut}(S) \geq \psi(\tau)$. By Proposition 6.2

$$\psi(d) = \coth^2 \mu\left(\sqrt{1 - e^{-2d}}\right).$$

Therefore using the fact (see [1, (5.3)]) that

$$\log \frac{1}{r} < \mu(r) < \log \frac{4}{r},$$

we have that

$$\psi(\tau) = \coth^2 \mu\left(\sqrt{1 - e^{-2\tau}}\right) > \coth^2 \log\left(\frac{1}{\sqrt{1 - e^{-2\tau}}}\right) = 1.05951....$$

On the other hand if a hyperbolic surface S is a regular cover of a closed hyperbolic 2-orbifold $Q = \mathbb{H}^2/G$, then one may argue just as in the proof of Proposition 2.7 in [2] to show that S is K-strongly quasiconformally homogeneous where

$$K \leq \left(e^{\frac{l'(Q)}{4}} + 1\right)^{2\left(\frac{4\text{diam}(Q)}{l'(Q)}+1\right)}.$$

□

Remark 6.6. If S is a closed hyperbolic surface and X is an infinite degree regular cover of S, then X is strongly quasiconformally homogeneous, but not extremely quasiconformally homogeneous.

One may construct a hyperbolic surface X' and a quasiconformal automorphism $f : X \to X'$ such that X' is not a regular cover of any closed hyperbolic surface (see Lemma 5.1 in [2]). Then X' will be uniformly quasiconformally homogeneous, but not strongly quasiconformally homogeneous.

7. Quasiconformal homogeneity constants on Moduli space

It is natural to consider how our quasiconformal homogeneity constants vary over the Moduli space $\mathcal{M}_g$ of all (isometry classes of) closed hyperbolic surfaces of a fixed genus g. The constants K, K_0 and K_{aut} all give rise to functions defined on $\mathcal{M}_g$. We finish the section with a few basic observations about these functions. We first observe that K and K_0 are continuous on $\mathcal{M}_g$, while K_{aut} is lower semi-continuous.

Lemma 7.1. *The functions $K : \mathcal{M}_g \to (1, \infty)$ and $K_0 : \mathcal{M}_g \to (1, \infty)$ are continuous, while $K_{aut} : \mathcal{M}_g \to (1, \infty)$ is lower semi-continuous.*

Proof: We first prove that K_{aut} is lower semi-continuous. The proofs that K and K_0 are lower semicontinuous are direct generalizations. Let $\{S_n\}$ be a sequence of (equivalence classes of) hyperbolic surfaces in $\mathcal{M}_g$ converging to S. Then there exists a sequence of quasiconformal homeomorphisms $f_n : S_n \to S$ such that $\lim k(f_n) = 1$. If $x, y \in S$, let $x_n = f_n^{-1}(x)$ and $y_n = f_n^{-1}(y)$ for all n. For all n there exists a $K_{aut}(S_n)$-quasiconformal automorphism g_n of S_n such that $g_n(x_n) = y_n$ and g_n is homotopic to a conformal automorphism h_n. We may pass to a subsequence $\{S_{n_j}\}$ such that $\lim K_{aut}(S_{n_j}) = \liminf K_{aut}(S_n)$. Since $\lim k(f_{n_j} h_{n_j} f_{n_j}^{-1}) = 1$, we may pass to a subsequence, again called $f_{n_j} h_{n_j} f_{n_j}^{-1}$, which converges to a conformal automorphism h of S. Moreover, $f_{n_j} h_{n_j} f_{n_j}^{-1}$ is homotopic to h for all large enough j. Thus, for all large enough j, $f_{n_j} g_{n_j} f_{n_j}^{-1}$ is a quasiconformal automorphism of S which is homotopic to h. Since $k(f_{n_j} g_{n_j} f_{n_j}^{-1}) \leq k(f_{n_j})^2 k(g_{n_j})$, $\lim k(f_{n_j}) = 1$, and $f_{n_j}(g_{n_j}(f_{n_j}^{-1}(x))) = y$ we may conclude that

$$k_{aut}(x, y) \leq \liminf k(g_{n_j}) \leq \lim K_{aut}(S_{n_j}) = \liminf K_{aut}(S_n).$$

Since x and y may be chosen arbitrarily, it follows that $K_{aut}(S) \leq \liminf K_{aut}(S_n)$, so K_{aut} is lower semicontinuous.

We next show that K_0 is upper semi-continuous. The proof that K is upper semi-continuous is much the same, so our result follows. Again, let $\{S_n\}$ be a sequence of (equivalence classes of) hyperbolic surfaces in $\mathcal{M}_g$ converging to S. There exists a sequence of quasiconformal homeomorphisms $f_n : S_n \to S$ such that $\lim k(f_n) = 1$. Fixing n for a moment, if $x_n, y_n \in S_n$, let $x = f_n(x_n)$ and $y = f_n(y_n)$ and let g be a quasiconformal automorphism such that $g(x) = y$, $k(g) \leq K_0(S)$ and g is homotopic to the identity map. Then $h_n = f_n^{-1} g f_n$ is a quasiconformal automorphism of S_n so that $h_n(x_n) = y_n$ and h_n is homotopic to the identity. Therefore, $k_0(x_n, y_n) \leq k(h_n) \leq k(f_n)^2 K_0(S)$. Since x_n and y_n can be chosen arbitrarily, $K_0(S_n) \leq k(f_n)^2 K_0(S)$. Since $\lim k(f_n) = 1$, $\limsup K_0(S_n) \leq K_0(S)$ and we have shown that K_0 is upper semi-continuous. □

Notice that there is no reason to assume that, in the last paragraph, if g is homotopic to a conformal automorphism then h_n will be homotopic to a conformal automorphism, so one does not expect K_{aut} to be upper semicontinuous. In fact, one expects that, if $g \neq 2$, then K_{aut} is discontinuous at any hyperbolic surface with a non-trivial automorphism group and that if $g = 2$, then K_{aut} is discontinuous at any hyperbolic surface whose conformal automorphism group consists of more than the canonical hyperelliptic involution. It is easy to show the discontinuity in high enough genus.

Lemma 7.2. *If g is sufficiently large, then K_{aut} is not continuous on $\mathcal{M}_g$.*

Proof: Let S be a closed hyperbolic surface of genus 2. Let α be a non-separating simple closed geodesic on S and let S' be the surface with geodesic boundary obtained from S by cutting along α. Label the two boundary components of S' with a + and -. We may construct a n-fold regular cover S_n of S from n copies of S' by attaching the + boundary component of the i^{th} copy of S' to the - boundary component of the $(i+1)^{st}$ copy of S' (and the + boundary component of the n^{th} copy of S' to the - boundary component of the first copy of S'.) There is a conformal action of $\mathbb{Z}_n$ on S_n with quotient S and S_n has genus $n+1$.

We next observe that $K_{aut}(S_n) \leq K_0(S)$. If $x, y \in S_n$, then there exists a $K_0(S)$-quasiconformal homeomorphism $f : S \to S$ such that $f(p(x)) = p(y)$ (where $p : S_n \to S$ is the obvious covering map) and f is homotopic to the identity. There is then a lift $\tilde{f} : S_n \to S_n$ of f which is $K_0(S)$-quasiconformal and is homotopic to the identity. It need not be the case that $\tilde{f}(x) = y$, but there always exists a conformal automorphism g of S_n such that $g(\tilde{f}(x)) = y$. Then $h = g \circ \tilde{f}$ is $K_0(S)$-quasiconformal, is homotopic to the conformal automorphism g, and $h(x) = y$. It follows that $k_{aut}(x, y) \leq k(h) \leq K_0(S)$. Since x and y were arbitrary, $K_{aut}(S_n) \leq K_0(S)$.

Since $\{\text{diam}(S_n)\}$ diverges to ∞ and ψ is proper, there exists N such that if $n \geq N$, then $\psi(\text{diam}(S_n)) > K_0(S)$. As the action of the mapping class group on Teichmüller space is properly discontinuous and the fixed point set of each finite order element has topological codimension at least 2, one may find a sequence $\{R_j\}$ of surfaces in $\mathcal{M}_{n+1}$, each of which has trivial conformal automorphism group, which converge to S_n, so $K_{aut}(R_j) = K_0(R_j) \geq \psi(\text{diam}(R_j))$. Since $\lim \text{diam}(R_j) = \text{diam}(S_n)$, we see that

$$\liminf K_{aut}(R_j) \geq \psi(\text{diam}(S_n)) > K_0(S) \geq K_{aut}(S_n)$$

if $n \geq N$. It follows that K_{aut} is discontinuous on $\mathcal{M}_{n+1}$ if $n \geq N$. □

The lower semicontinuity of K, K_{aut}, and K_0, along with their asymptotic properties allow us to see that each achieves its minimum on $\mathcal{M}_g$.

Lemma 7.3. *If $\{S_n\}$ is a sequence of hyperbolic surfaces leaving every compact subset of $\mathcal{M}_g$, then*

$$\lim K(S_n) = \lim K_0(S_n) = \lim K_{aut}(S_n) = \infty.$$

Moreover, K, K_{aut} and K_0 all attain their minima on $\mathcal{M}_g$.

Proof: It is well-known that $\lim l(S_n) = 0$ if $\{S_n\}$ leaves every compact subset of $\mathcal{M}_g$. Part (2) of Theorem 2.1 then implies that $\lim K(S_n) = \infty$. It follows that $\lim K_{aut}(S_n) = \infty$ and $\lim K_0(S_n) = \infty$ as well.

Since, K, K_0 and K_{aut} are all lower semicontinuous, the first claim of our lemma allows us to conclude that they all attain their minima on $\mathcal{M}_g$. □

It is thus natural to define

$$\begin{aligned} K^g &= \min\{K(S) \,|\, S \in \mathcal{M}_g\} \\ K^g_{aut} &= \min\{K_{aut}(S) \,|\, S \in \mathcal{M}_g\} \\ K^g_0 &= \min\{K_0(S) \,|\, S \in \mathcal{M}_g\} \end{aligned}$$

From Lemma 6.1 and the fact that K attains its minimum on $\mathcal{M}_g$, it follows that for each genus g

$$1 < K^g \leq K^g_{aut} \leq K^g_0.$$

We first observe that K^g_0 diverges to ∞ as g goes to ∞.

Lemma 7.4. *For any $g \geq 2$,*

$$K^g_0 \geq \psi(\cosh^{-1}(2g-1)).$$

Proof: It follows from the Gauss-Bonnet theorem, that if S is a closed hyperbolic surface of genus g, then $\operatorname{diam}(S) \geq \cosh^{-1}(2g-1)$. Theorem 6.4 then implies that if $S \in \mathcal{M}_g$, then $K_0(S) \geq \psi(\cosh^{-1}(2g-1))$ which establishes our result. □

The unboundedness of the sequence of minima $\{K^g_0\}$ contrasts with the boundedness of the sequence $\{K^g_{aut}\}$.

Lemma 7.5. *The sequence $\{K^g_{aut}\}_{g=2}^{\infty}$ is universally bounded above. In particular $K^g_{aut} \leq K^2_0$.*

Proof. Let S be a genus two hyperbolic surface such that $K_0(S) = K^2_0$. Let S_n be the genus $n+1$ regular cover of S constructed as in Lemma 7.2. In the proof of Lemma 7.2, we showed that $K_{aut}(S_n) \leq K_0(S)$ for all n, so

$$K^{n+1}_{aut} \leq K_{aut}(S_n) \leq K_0(S) = K^2_0.$$

□

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