Principle of Mathematical Induction: Let \((P_n)_{n \in \mathbb{N}}\) be a sequence of mathematical statements indexed by the natural numbers \(\mathbb{N}\). If \(P_1\) is true and \(P_{n+1}\) is true whenever \(P_n\) is true, then \(P_n\) is true for all \(n \in \mathbb{N}\).

Rational Zeros Theorem: Suppose that \(c_0, c_1, \dotsc, c_n\) are integers such that \(c_0 \neq 0\) and \(c_n \neq 0\) and \(r\) is a rational solution of 
\[
c_0 x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0.
\]
If \(r = \frac{c_n}{d_n}\) where \(c\) and \(d\) are integers with no common factors, then \(c\) divides \(c_0\) and \(d\) divides \(c_n\).

Definitions: Let \(S\) be a non-empty subset of \(\mathbb{R}\). If there exists \(s_0 \in S\), so that \(s \leq s_0\) whenever \(s \in S\), then \(s_0\) is the maximum of \(S\), and we write \(s_0 = \max S\). If there exists \(s_1 \in S\) so that \(s \geq s_1\) for all \(s \in S\), then \(s_1\) is the minimum of \(S\) and we write \(s_1 = \min S\).

If there exists \(M \in \mathbb{R}\) so that \(s \leq M\) for all \(s \in S\), then \(M\) is an upper bound for \(S\). If there exists \(m \in \mathbb{R}\), so that \(s \geq m\) for all \(s \in S\), then \(m\) is a lower bound for \(S\). \(S\) is bounded if it has both an upper bound and a lower bound. Equivalently, \(S\) is bounded if there exist \(R \in \mathbb{R}\) so that if \(s \in S\), then \(|s| \leq R\).

Completeness Axiom: If \(S\) is a non-empty subset of \(\mathbb{R}\) which has an upper bound, then \(S\) has a least upper bound, called the supremum and written \(\sup S\).

Corollary: If \(S\) is a non-empty subset of \(\mathbb{R}\) which has a lower bound, then \(S\) has a greatest lower bound, called the infimum and written \(\inf S\).

Conventions: If \(S\) has no upper bound, we write \(\sup S = +\infty\) and if \(S\) has no lower bound, we write \(\inf S = -\infty\).

Archimidean Property: If \(a > 0\) and \(b > 0\), then there exists a natural number \(n \in \mathbb{N}\) so that \(na > b\).

Density of Rationals: If \(a, b \in \mathbb{R}\) and \(a < b\), then there exists a rational number \(r \in \mathbb{Q}\) so that \(a < r < b\).

Corollary: If \(a, b \in \mathbb{R}\) and \(a < b\), then there exists an irrational number \(x\) so that \(a < x < b\).

Definition: A sequence \((s_n)\) converges to a real number \(s\) if for any \(\epsilon > 0\), there exists \(N \in \mathbb{N}\) so that if \(n > N\), then \(|s_n - s| < \epsilon\). In this case, we write \(\lim s_n = s\).

(Our sequences will typically be indexed by the natural numbers \(\mathbb{N}\), but may also be indexed by any subset of the form \(\{n \in \mathbb{Z} \mid n \geq m\}\) for some \(m \in \mathbb{Z}\), in which case we write \((s_n)_{n=m}^{\infty}\).)

Basic examples: a) If \(p > 0\), then \(\lim \frac{1}{n^p} = 0\). b) If \(|a| < 1\), then \(\lim a^n = 0\). c) \(\lim n^{\frac{1}{n}} = 1\).

d) If \(a > 0\), then \(\lim a^n = 1\).

Fact: If \((s_n)\) is a convergent sequence it is bounded, i.e the set \(\{s_n \mid n \in \mathbb{N}\}\) is a bounded subset of \(\mathbb{R}\).

Limit laws: a) If \(k \in \mathbb{R}\) and \((s_n)\) converges to \(s\), then \((ks_n)\) converges to \(ks\).

b) If \((s_n)\) and \((t_n)\) are convergent sequences, then \((s_n + t_n)\) is convergent and \(\lim s_n + t_n = \lim s_n + \lim t_n\).

c) If \((s_n)\) and \((t_n)\) are convergent sequences, then \((s_n t_n)\) is convergent and \(\lim s_n t_n = (\lim s_n)(\lim t_n)\).

d) If \((s_n)\) and \((t_n)\) are convergent sequences, \(\lim t_n \neq 0\) and \(t_n\) is non-zero for all \(n \in \mathbb{N}\), then then \((\frac{s_n}{t_n})\) is convergent and \(\lim \frac{s_n}{t_n} = \frac{\lim s_n}{\lim t_n}\).

Comparison laws: Let \((a_n)\), \((b_n)\) and \((s_n)\) be sequences.

a) \((s_n)\) converges to 0 if and only if \(|(s_n)|\) converges to 0.

b) If \((s_n)\) is convergent and \(s_n \geq a\) for all but finitely many values of \(n\), then \(\lim s_n \geq a\).

c) If \((s_n)\) is convergent and \(s_n \leq b\) for all but finitely many values of \(n\), then \(\lim s_n \leq b\).

d) (NEW) If \((a_n)\) and \((b_n)\) are convergent and \(a_n \leq b_n\) for all but finitely many values of \(n\), then \(\lim a_n \leq \lim b_n\).

e) (Squeeze principle) If \(a_n \leq s_n \leq b_n\) for all \(n \in \mathbb{N}\) and \(\lim a_n = \lim b_n\), then \((s_n)\) converges and \(\lim s_n = \lim a_n\).

Definitions: A sequence \((s_n)\) is decreasing if, for all \(n \in \mathbb{N}\), \(s_n \geq s_{n+1}\). A sequence \((s_n)\) is increasing if \(s_n \leq s_{n+1}\) for all \(n \in \mathbb{N}\). A sequence is monotone if it is either increasing or decreasing.
Theorem: Every bounded monotone sequence is convergent.

Definition: A sequence \((s_{n_k})_{k \in \mathbb{N}}\) is a subsequence of \((s_n)\) if each \(n_k \in \mathbb{N}\) and \(n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots\).

Fact: If \((s_n)\) converges to \(s\) and \((s_{n_k})\) is a subsequence of \((s_n)\), then \((s_{n_k})\) also converges to \(s\).

Theorem: Every sequence has a monotone subsequence.

Bolzano-Weierstrass Theorem: Every bounded sequence has a convergent subsequence.

Definition: A sequence \((s_n)\) is Cauchy if for any \(\epsilon > 0\), there exists \(N\) so that if \(n, m > N\), then \(|s_n - s_m| < \epsilon\).

Theorem: A sequence \((s_n)\) is a Cauchy sequence if and only if it is convergent.

Definitions: A sequence \((s_n)\) diverges to \(+ \infty\), written \(\lim s_n = + \infty\) if for all \(M > 0\) there exists \(N\) such that if \(n > N\), then \(s_n > M\). A sequence \((s_n)\) diverges to \(- \infty\), written \(\lim s_n = - \infty\), if for all \(M < 0\) there exists \(N\) such that if \(n > N\), then \(s_n < M\).

Facts: a) (Sample Limit Law) If \((s_n)\) and \((t_n)\) are sequences, \(\lim s_n = + \infty\) and \((t_n)\) converges to a positive real number or diverges to \(+ \infty\), then \(\lim s_nt_n = + \infty\).

b) (Sample comparison law) If \((s_n)\) and \((t_n)\) are sequences, \(\lim s_n = + \infty\) and \(t_n \geq s_n\) for all \(n \in \mathbb{N}\), then \(\lim t_n = + \infty\).

c) If \((s_n)\) is unbounded and non-decreasing, then \(\lim s_n = + \infty\).

d) If \((s_n)\) is unbounded and non-increasing, then \(\lim s_n = - \infty\).

Definitions: If \((s_n)\) is a sequence, we define

\[
\limsup s_n = \lim_{N \to \infty} \sup\{s_n \mid n > N\}, \quad \text{and} \quad \liminf s_n = \lim_{N \to \infty} \inf\{s_n \mid n > N\}.
\]

If \((s_n)\) is not bounded above, then we say \(\limsup s_n = + \infty\) and if \((s_n)\) is not bounded below then we define \(\liminf s_n = - \infty\). With these conventions \(\limsup\) and \(\liminf\) are always defined.

Theorem: If \((s_n)\) is a sequence, then \(\lim s_n\) exists if and only if \(\liminf s_n = \limsup s_n\). Moreover, if \(\lim s_n\) exists, then \(\lim s_n = \liminf s_n = \limsup s_n\).

Definitions: If \(\sum_{n=1}^{\infty} a_n\) is an infinite series, we consider the partial sum \(s_n = a_1 + \cdots + a_n\). We say that the series converges to \(s\) if \(s_n = s\). We then write \(\sum_{n=1}^{\infty} a_n = s\). If \(\lim s_n = + \infty\), then we say that \(\sum_{n=1}^{\infty} a_n\) diverges to \(+ \infty\) and we write \(\sum_{n=1}^{\infty} a_n = + \infty\). Similarly, if \(\lim s_n = - \infty\), then we say that \(\sum_{n=1}^{\infty} a_n\) diverges to \(- \infty\) and we write \(\sum_{n=1}^{\infty} a_n = - \infty\).

We say that \(\sum_{n=1}^{\infty} a_n\) satisfies the Cauchy criterion if there exists \(N\) so that if \(n, m > N\), then \(|s_n - s_m| < \epsilon\).

We say that \(\sum_{n=1}^{\infty} a_n\) converges absolutely if \(\sum_{n=1}^{\infty} |a_n|\) converges.

Facts: a) An infinite series is convergent if and only if it satisfies the Cauchy criterion.

b) If \(\sum_{n=1}^{\infty} a_n\) is convergent, then \(\lim a_n = 0\).

Comparison Tests: Let \(\sum_{n=1}^{\infty} a_n\) and \(\sum_{n=1}^{\infty} b_n\) be infinite series.

a) If \(\sum_{n=1}^{\infty} a_n\) converges and \(|b_n| \leq a_n\) for all \(n\), then \(\sum_{n=1}^{\infty} b_n\) converges.

b) If \(\sum_{n=1}^{\infty} a_n = + \infty\) and \(b_n \geq a_n\) for all \(n\), then \(\sum_{n=1}^{\infty} b_n = + \infty\).

c) If \(\sum_{n=1}^{\infty} a_n\) converges absolutely, then it converges.

Root and Ratio Tests: Let \(\sum_{n=1}^{\infty} a_n\) be an infinite series.

a) If \(\limsup_{n \to \infty} |\frac{a_{n+1}}{a_n}| < 1\), then \(\sum_{n=1}^{\infty} a_n\) converges absolutely.

b) If \(\liminf_{n \to \infty} |\frac{a_{n+1}}{a_n}| > 1\), then \(\sum_{n=1}^{\infty} a_n\) does not converge.

c) If \(\limsup_{n \to \infty} |a_n|^{\frac{1}{n}} < 1\), then \(\sum_{n=1}^{\infty} a_n\) converges absolutely.

d) If \(\limsup_{n \to \infty} |a_n|^{\frac{1}{n}} > 1\), then \(\sum_{n=1}^{\infty} a_n\) does not converge.

Examples: a) \(\sum_{n=1}^{\infty} a_n\) is convergent if and only if \(|a| < 1\).

b) \(\sum_{n=1}^{\infty} \frac{1}{n^p}\) is convergent if and only if \(p > 1\).

Alternating Series Test: (NEW) If \((a_n)\) is a non-increasing sequence, \(\lim a_n = 0\) and \(a_n \geq 0\) for all \(n \in \mathbb{N}\), then \(\sum_{n=1}^{\infty} (-1)^n a_n\) converges.
New Material:

**Definition:** A function $f$ is **continuous at** $x_0 \in \text{dom}(f)$ if for every sequence $(x_n)$ in $\text{dom}(f)$ which converges to $x_0$, we have $\lim f(x_n) = f(x_0)$. The function $f$ is said to be **continuous** if it is continuous at every point in $\text{dom}(f)$.

**Theorem:** A function $f$ is continuous at $x_0 \in \text{dom}(f)$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ so that if $x \in \text{dom}(f)$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

**Basic facts:** Suppose that $f$ and $g$ are functions which are continuous at $x_0 \in \mathbb{R}$, then

a) If $k \in \mathbb{R}$, then $kf$ and $g$ are continuous at $x_0$.

b) $f + g$, $f - g$ and $fg$ are continuous at $x_0$.

c) If $g(x_0) \neq 0$, then $\frac{f}{g}$ is continuous at $x_0$.

d) If $h$ is continuous at $f(x_0)$, then $h \circ f$ is continuous at $x_0$.

e) Polynomials and rational functions are continuous. Moreover, $r(x) = \sqrt{x}$ is continuous.

**Theorem:** If $f$ is continuous on the closed interval $[a, b]$, then $f$ is bounded and it achieves its maximum and its minimum on $[a, b]$, i.e. there exists $x_1, x_2 \in [a, b]$ so that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$.

**Intermediate Value Theorem:** If $f$ is continuous on a closed interval $[a, b]$ and $y$ lies between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ so that $f(c) = y$.

**Corollary:** If $f$ is continuous on an interval $I$, then $f(I)$ is an interval or a single point.

**Definition:** A function $f$ is **uniformly continuous** on $S \subset \text{dom}(f)$ if for all $\epsilon > 0$ there exists $\delta > 0$ so that if $x, y \in S$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

**Theorem:** If $f$ is continuous on a closed interval $[a, b]$, then it is uniformly continuous on $[a, b]$.

**Fact:** If $f$ is uniformly continuous on $S$ and $(x_n)$ is a Cauchy sequence in $S$, then $f(x_n)$ is a Cauchy sequence.

**Definition:** If $S \subset \mathbb{R}$ and $a \in \mathbb{R}$ (or $a = \pm \infty$), there exists a sequence $(x_n)$ in $S$ which converges to $a$ and $f$ is a function defined on $S$, then $\lim_{x \to a^+} f(x) = L$ (where $L \in R$ or $L = \pm \infty$) if for any sequence $(x_n)$ in $S$ such that $lim x_n = a$, we have $\lim f(x_n) = L$.

In the case, that $I$ is an open interval about $a$ and $S = I - \{a\}$, then we write $\lim_{x \to a^+} f(x)$ as simply $\lim_{x \to a^+} f(x)$.

In the case, that $S = (a, b)$, we write $\lim_{x \to a^+} f(x)$ as simply $\lim_{x \to a^+} f(x)$.

**Fact:** If $f$ is defined on an open interval about $a \in \mathbb{R}$, then $f$ is continuous at $a$ if and only if $\lim_{x \to a} f(x) = f(a)$.

**Basic facts:** Suppose that $f$ and $g$ are functions defined on $S$ and $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2$ both exist and are finite, then

a) If $k \in \mathbb{R}$, then $\lim_{x \to a} kf(x)$ exists and $\lim_{x \to a} kf(x) = kL_1$.

b) $\lim_{x \to a}(f + g)(x)$, $\lim_{x \to a}(f - g)(x)$, and $\lim_{x \to a}(fg)(x)$ all exist, and $\lim_{x \to a}(f + g)(x) = L_1 + L_2$, $\lim_{x \to a}(f - g)(x) = L_1 - L_2$, and $\lim_{x \to a}(fg)(x) = L_1 L_2$.

c) If $L_2 \neq 0$, then $\lim_{x \to a} \frac{f(x)}{g}(x)$ exists and $\lim_{x \to a} \frac{f(x)}{g}(x) = \frac{L_1}{L_2}$.

d) If $h$ is a function defined on $f(S) \cup \{L_1\}$ and continuous at $L_1$, then $\lim_{x \to a} (h \circ f)(x)$ exists and equals $h(L_1)$.

**Theorem:** If $f$ is a function defined on $S$, $a$ is the limit of some sequence in $S$ and $L \in \mathbb{R}$, then $\lim_{x \to a} f(x) = L$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - a| < \delta$ and $x \in S$, then $|f(x) - L| < \epsilon$.

**Definition:** Suppose that $f$ is defined on an open interval containing $a$. We say that $f$ is **differentiable** at $a$ if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists and is a real number. In this, case we write $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

**Facts:** a) If $f$ is differentiable at $a$, then $f$ is continuous at $a$.

b) If $f(x) = c$ is a constant function and $x_0 \in \mathbb{R}$, then $f$ is differentiable at $x_0$ and $f'(x_0) = 0$.

c) If $n \in \mathbb{N}$, $x_0 \in \mathbb{R}$ and $f(x) = x^n$, then $f$ is differentiable at $x_0$ and $f'(x_0) = nx_0^{n-1}$.
Intermediate Value Theorem for Derivatives:

a) If $k \in \mathbb{R}$, then $kf$ is differentiable at $x_0$ and $(kf)'(x_0) = kf'(x_0)$.
b) $f + g, f - g, \text{ and } fg$ are differentiable at $x_0$ and $(f + g)'(x_0) = f'(x_0) + g'(x_0)$, $(f - g)'(x_0) = f'(x_0) - g'(x_0)$, and $(fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0)$.
c) If $f(x_0) \neq 0$, then $\frac{g}{f}$ is differentiable at $x_0$ and \( \left( \frac{g}{f} \right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(f(x_0))^2} \)  
d) $h$ is differentiable at $f(x_0)$, then $h \circ f$ is differentiable at $x_0$ and $(h \circ f)'(x_0) = h'(f(x_0))f'(x_0)$.
e) Polynomials and rational functions are differentiable on their domains.

Theorem: Suppose that $f$ is defined on an open interval $I$ containing $x_0$ and assumes its maximum or minimum at $x_0$. If $f$ is differentiable at $x_0$, then $f'(x_0) = 0$.

Rolle’s Theorem: If $f$ is continuous on $[a, b]$, differentiable on $(a, b)$ and $f(a) = f(b)$, then there exists $x_0 \in (a, b)$ so that $f'(x_0) = 0$.

Mean Value Theorem: If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there exists $x_0 \in (a, b)$ so that $f'(x_0) = \frac{f(b) - f(a)}{b - a}$.

Corollaries: Suppose that $f$ and $g$ are differentiable on $(a, b)$.

a) If $f'(x) = 0$ for all $x \in (a, b)$, then $f$ is constant on $(a, b)$.
b) If $f'(x) = g'(x)$ for all $x \in (a, b)$, then there exists $c \in \mathbb{R}$ so that $f(x) = g(x) + c$ for all $x \in (a, b)$.
c) If $f'(x) > 0$ for all $x \in (a, b)$, then $f$ is strictly increasing on $(a, b)$.
d) If $f'(x) \geq 0$ for all $x \in (a, b)$, then $f$ is increasing on $(a, b)$.
e) If $f'(x) < 0$ for all $x \in (a, b)$, then $f$ is strictly decreasing on $(a, b)$.
f) If $f'(x) \leq 0$ for all $x \in (a, b)$, then $f$ is decreasing on $(a, b)$.

Intermediate Value Theorem for Derivatives: If $f$ is differentiable on $(a, b)$, $x_1, x_2 \in (a, b)$, $x_1 < x_2$ and $c$ lies between $f'(x_1)$ and $f'(x_2)$, then there exists $x_0 \in (x_1, x_2)$ so that $f'(x_0) = c$.

Theorem: If $f$ is continuous and one-to-one on an open interval $I$, $J = f(I)$, $f$ is differentiable at $x_0 \in I$ and $f'(x_0) \neq 0$, then $f^{-1}$ is differentiable at $y_0 = f(x_0)$ and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$. 