Math 623, W 2007: Homework 1.

For full credit, your solutions must be clearly presented and all code included.

(1) Consider the following initial value problem for the function $u = u(x)$ defined for $0 \le x \le 1$.

$$u_{xx} + xu_x + u = 0 \quad \text{and} \quad u(0) = 1, u_x(0) = 0.$$

(a) What is the exact solution $u(x)$? *Hint*: it is of the form $u(x) = e^{\phi(x)}$ for a polynomial $\phi(x)$.
(b) Write down the finite difference scheme for the ODE above, using a forward difference for u_x and a symmetric difference for u_{xx}.
(c) Same question as in (b) but use a backward difference for u_x.
(d) Same question as in (b) but use a central difference for u_x.
(e) Let ϵ_n be the error at grid point n, i.e. $\epsilon_n = u_n - u(n\Delta x)$. Using your answers to (a)-(d), compute the values of ϵ_N ($N = 1/\Delta x$) for $\Delta x = 2^{-1}, 2^{-2}, 2^{-3}, \ldots$ (until the computations become too slow for your computer). Do this for all three schemes in (b)-(d). Plot $-\log|\epsilon_N|$ as a function of $-\log \Delta x$. What do you observe?

(2) Consider the following PDE:

$$\begin{cases} u_t = (1+x^2)u_{xx}, \ -1 < x < 1, 0 < t < 1 \\ u(x,0) = x^4, \ -1 \le x \le 1 \\ u(-1,t) = u(1,t) = 1, \ 0 \le t \le 1 \end{cases} \tag{1}$$

(a) Write down (carefully) the explicit finite difference scheme for this PDE.
(b) Implement the scheme for $\alpha = 2^{-1}$, $\alpha = 2^{-2}$ and $\alpha = 2^{-3}$ where $\alpha = \Delta t/(\Delta x)^2$. For each value of α use $\Delta x = 2^{-1}, 2^{-2}, 2^{-3}, \ldots$ (until the computations become too slow for your computer). Report the values

$$u(-1,1), u(-0.9,1), \ldots, u(0.9,1), u(1,1)$$

for each such choice of α and Δt. Use at least 6 significant digits (*format long* in MATLAB). (You may have to use linear interpolation. The function *interp1* in MATLAB can help here).

(3) Prove that the scheme in (2) is convergent as $\Delta t \to 0$ if $0 < \alpha \le 0.25$.

(4) Consider the following PDE:

$$\begin{cases} u_t = 3y^2 u_{yy} - 3y u_y, & 1 < y < e, 0 < t < 1 \\ u(y,0) = 0, & 1 \leq y \leq e \\ u(1,t) = t, & 0 \leq t \leq 1 \\ u(e,t) = t^2 & 0 \leq t \leq 1 \end{cases} \tag{2}$$

(a) Transform the PDE to a constant coefficient PDE for a function $v(x,t)$, $0 < x < 1$, using the transformation $y = e^x$. State the new initial and boundary conditions carefully.

(b) Write down (carefully) the fully implicit finite difference scheme for this transformed PDE.

(c) Implement the scheme in (b) for $\alpha = 0.1$, $\alpha = 0.25$ and $\alpha = 2$, where $\alpha = \Delta t/(\Delta x)^2$. For each value of α use $\Delta x = 2^{-1}, 2^{-2}, 2^{-3}, \ldots$ (until the computations become too slow for your computer). In going from time step m to time step $m+1$ use the SOR algorithm with $\omega = 1, 1.1, 1.2$ and 1.5.
Report the values $v(0,1), v(.1,1), ...v(.9,1), v(1,1)$ as in Problem (2).