Math 623, W 2007: Homework 2

For full credit, your solutions must be clearly presented and all code included.

(1) Consider the Black-Scholes model for a stock price S_t and a bond B_t. The interest rate is $r = 3\%$ per year, continuously compounded. The volatility is $\sigma = 0.2$ (in units year$^{-1/2}$) and the stock pays a continuous dividend yield of $D = 1\%$ per year.

An investor has written an *American strangle option*. This option pays $\Phi(S)$ if exercised when the stock price is S. Here

$$\Phi(S) = \begin{cases} 80 - S & \text{if } 0 \le S \le 80 \\ 0 & \text{if } 80 < S \le 120 \\ S - 120 & \text{if } S > 120. \end{cases}$$

Today is $t = 0$. The option expires in $T = 6$ months. Being American, the option can be exercised at any time between $t = 0$ and $t = T$. Denote the value of the option at time t and stock price S by $V(S, t)$.

(a) Plot the payoff as a function of S.

(b) Write the payoff as a linear combination of payoff of puts and calls, i.e. write

$$\Phi(S) = \sum_{i=1}^{m} a_i (K_i - S)^+ + \sum_{i=m+1}^{n} a_i (S - K_i)^+$$

where a_i and K_i are constants.

(c) What is the (exact or approximate) value of $V(0, t)$ and $V(300, t)$ for $0 \le t \le T$?

(d) Write down a variational formulation for the value $V(S, t)$ of the American strangle option in the region $0 < S < 300$, $0 < t < T$. Be careful to state all terminal and boundary conditions.

(e) Write down (carefully) a Crank-Nicholson finite difference scheme for the variational problem in (d). Implement the scheme in a computer language of your choosing. Experiment with different choices of Δt and ΔS.

(f) Draw the (three-dimensional) graph of $V(S, t)$ for $50 \le S \le 150$ and $0 \le t \le T$. (*Hint*: in Matlab, use `mesh`; for other languages, consider exporting the data to Excel or Matlab.)

(g) Plot the no-exercise region in the (S, t)-plane, i.e. the region where it is *not* optimal to exercise the option. Do this by checking, for each grid point in $0 \le S \le 300$, $0 \le t \le T$, whether or not your computed value of $V(S, t)$ exceeds $\Phi(S)$.

(2) This problem deals with an *continuously sampled, arithmetically averaged, average strike Asian call option* on the same stock as in Problem 1. Today is $t = 0$ and the option expires at time $T = 0.5$ years.

Write $I_t = \int_0^t S_u \, du$, where S_t is the price of the stock at time t. Denote the option price at time by $V(S, I, t) = V(S_t, I_t, t)$. The PDE for V is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial I} - rV = 0.$$

(a) What is the terminal condition of the PDE, i.e. what is $V(S, I, T)$?

(b) Exploit the homogeneity in the problem and use a similarity reduction as follows. Write $\xi = I/S$ and define the function $W(\xi, t)$ by $V(S, I, t) = SW(I/S, t)$. Write down the PDE for W.

(c) What is the terminal condition for the PDE in (b)?

(d) Truncate the domain of W to $0 \le t \le T$ and $0 \le \xi \le \xi_{\max}$. Why could $\xi_{\max} = 2$ be a reasonable value and what boundary condition for W at $\xi = \xi_{\max}$ could we use?

(e) Write down the implicit boundary condition at $\xi = 0$ resulting from the PDE with $\xi = 0$, there.

(f) Write down (carefully) the explicit finite difference scheme for the PDE in (b)-(e). What relationship between Δt and $\Delta\xi$ does the "rule of thumb" indicate that we should use?

(g) Implement the finite difference scheme in (f) in your choice of computer language. Experiment with different choices of Δt and $\Delta\xi$. Draw the (three-dimensional) graph of $W(\xi, t)$ for $0 \le \xi \le \xi_{\max}$ and $0 \le t \le T$. (*Hint*: in Matlab, use `mesh`; for other languages, consider exporting the data to Excel or Matlab.)

(h) Compute the value $V(20, 0, 0)$ as well as the Delta $\Delta(20, 0, 0) = \frac{\partial V}{\partial S}(20, 0, 0)$ and the Vega $\mathcal{V} = \frac{\partial V}{\partial \sigma}(20, 0, 0)$.