Math 623, W 2007: Homework 3.

For full credit, your solutions must be clearly presented and all code included.

All the problems in this homework set deal with the following (Heston) stochastic volatility model for the price S_t of a stock

$$\begin{cases} dY_t &= 2(0.04 - Y_t)dt + 0.2\sqrt{Y_t}\, dW_t \\ \frac{dS_t}{S_t} &= 0.01\, dt + \sqrt{Y_t}\,(-0.7\, dW_t + \sqrt{0.51}\, dZ_t) \end{cases} \tag{1}$$

for $0 \leq t \leq 0.25$. Here W_t and Z_t are independent (uncorrelated) Brownian motions under the risk neutral measure Q. Time is counted in years unless otherwise stated. By convention, one year is 360 days. Today is $t = 0$ and $Y_0 = 0.04$. The stock pays no dividends.

(1) First some general questions.
   (a) What is the risk-free interest rate r?
   (b) The current stock price S_0 is such that the $E^Q[S_{0.25}] = 100$. What is S_0?
   (c) What is (formally) the expected value and variance of the relative stock return dS_t/S_t (conditioned on S_t and Y_t)?
   (d) What is (formally) the expected value and variance of dY_t (conditioned on S_t and Y_t)?
   (e) Why is it reasonable to view Y_t as the square of the volatility level at time t?
   (f) Give two reasons why a volatility level of 20% plays a special role.
   (g) What is (formally) the correlation ρ between dY_t and dS_t/S_t (conditioned on S_t and Y_t)?
   (h) Explain (briefly) why the sign of ρ could make sense, financially.

(2) Now let us generate paths of the 2-dimensional process (Y_t, S_t).
   (a) Write down (carefully) the Euler discretization of the SDE (1) with a time step equal to three days.
   (b) Implement the Euler scheme in your favorite computer language. The more paths, the better. You only need to save the generated values at the terminal time, i.e. $(Y_{0.25}, S_{0.25})$.
   (c) Draw careful histograms over the generated values of $Y_{0.25}$ and $S_{0.25}$. *Hint:* in matlab, use "hist" with a well chosen number of bars.
   (d) Compute the successive Monte Carlo estimates of $E^Q[S_{0.25}]$ and $E^Q[Y_{0.25}]$. Report the number of paths used, the final Monte Carlo estimates, the standard errors and plot the two convergence diagrams.
   (e) Now generate the same number of paths as before, but using antithetic variables. Explain how this works.
   (f) Redo steps (c) and (d) with the new paths.

(3) Next let us use the simulations in problem 2 to compute the value of a European call option with strike price $K = 100$ and expiring three months from now.
   (a) What is the Black-Scholes price of this option if we use (constant) volatility $\sigma = 0.2$?
   (b) Use the first set of generated values of $S_{0.25}$ to estimate the value of the option (in the stochastic volatility model). Explain your steps. As usual, report the number of paths used, the final Monte Carlo estimates and the standard error. Plot the convergence diagram.
   (c) Do the same thing using the values of $S_{0.25}$ generated with the help of antithetic variables.
   (d) Compute the implied volatilities corresponding to the values in (b) and (c). *Hint:* in matlab, use "bslpinv" if available, otherwise compute the implied volatility graphically by plotting the Black-Scholes price of the option as a function of the parameter σ.

(4) *The following problem is optional but could yield a non-negligible amount of extra credit.*
   (a) Write down the PDE for the price $C(S, y, t)$ of the call option at time t, when $S_t = S$ and $Y_t = y$. Include the terminal condition, but not the boundary conditions.
   (b) Why could $0 < S < 400$, $0 < y < 0.2$, $0 < t < 0.25$ be a reasonable choice of truncated domain when trying to solve the PDE numerically? What boundary conditions could we use?
   (c) Write down (carefully) an explicit finite difference scheme for solving the PDE. Include all boundary conditions (explicit or implicit).
   (d) Implement the scheme in (c) in your favorite computer language.
   (e) Plot the estimated value of $C(S, y, 0)$ as a function of S and y for $0 < S < 400$ and $0 < y < 0.2$.
   (f) Plot the estimated value of $C(S, 0.04, 0)$ as a function of S. On the same graph, plot the Black-Scholes price of the option as a function of S at time 0 with constant volatility $\sigma = 0.2$. *Hint*: in matlab use the commands "clf" and "hold".
   (g) Plot the estimated implied volatility $I(S, 0.04, 0)$ for $0 < S < 400$.