Math 623, W 2007: Homework 4.

For full credit, your solutions must be clearly presented and all code included.

Time is counted in years and the interest rate is $r = 2\%$, continuously compounded.

(1) In this problem you are asked to value an out-of-the-money bear spread option using Monte Carlo simulations. The option expires at $T = 1$ (today is $t = 0$) and has payoff $\Phi(S_T)$, where

$$\Phi(S) = \begin{cases} 10 & \text{if } S \le 50 \\ 60 - S & \text{if } 50 \le S \le 60 \\ 0 & \text{if } S \ge 60. \end{cases}$$

the interest rate is $r = 2\%$, continuously compounded, and the volatility of the stock is $\sigma = 0.2$. The stock pays no dividends and is currently trading at $S_0 = 100$.

(a) Write the value of the stock price S_T at time T as a function of a standard normal variate ξ (under the risk-neutral measure).

(b) Using the Black-Scholes formula, find the exact value of the bear spread option.

(c) Use "vanilla" Monte Carlo to compute the price of the option by generating samples of the standard normal variate ξ in (a). Do not use any variance reduction techniques. Report the number of paths used (the more, the better...), the final Monte Carlo estimate, the standard error, and a convergence diagram.

(d) Repeat the simulation in (c), but now using antithetic variables. Use the same number of samples and report the result as in (c).

(e) Repeat the simulation in (c), now using moment matching (match the first two moments). Use the same number of samples and report the result as in (c).

(f) Use importance sampling as outlined in class in order to generate samples of S_T that have expected value 55 under a suitable equivalent measure. Use the same number of samples and report the result as in (c). Do not use the variance reduction techniques in (d) and (e). Explain your steps.

(g) Repeat the simulation in (f), now with antithetic variables as in (d). Use the same number of samples and report the result as in (c).

(h) Repeat the simulation in (f), now with moment matching as in (e). Use the same number of samples and report the result as in (c).

(i) Write a summarizing table with the Monte Carlo estimates $\hat{V}_N$ and standard errors ϵ_N from (c)-(h).

(2) Consider three stocks with the following dynamics under the risk-neutral measure Q:

$$\frac{dS_{i,t}}{S_{i,t}} = r\,dt + \sigma_i\,dW_{i,t}, \quad i = 1, 2, 3, \quad 0 \le t \le T = 1/4.$$

Here $W_{i,t}$ are *correlated* Brownian motions: $E[dW_{i,t}\,dW_{j,t}] = \rho_{ij}\,dt$, where the correlation matrix $\rho = (\rho_{ij})$ is reported by the middle-office wizards as being

$$\rho = \begin{bmatrix} 1.0 & -0.7 & 0.1 \\ -0.7 & 1.0 & -0.7 \\ 0.1 & -0.7 & 1.0 \end{bmatrix}$$

Moreover, we have

$$\begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.22 & 0.24 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} S_{1,0} \\ S_{2,0} \\ S_{3,0} \end{bmatrix} = \begin{bmatrix} 42 \\ 40 \\ 35 \end{bmatrix}$$

(a) Use Cholesky factorization to write the stock price dynamics as

$$\frac{dS_{i,t}}{S_{i,t}} = r\,dt + \sum_{j=1}^{3} \tilde{\sigma}_{ij}\,dZ_{j,t}, \quad i = 1, 2, 3.$$

where $Z_{i,t}$ are *independent* Brownian motions under Q and $\tilde{\sigma}_{ij}$ are constants.

(b) Using (a), write the terminal stock prices $S_{i,T}$ in terms of three independent standard normal variates $\xi_j \sim N(0,1)$, $j = 1, 2, 3$ and the constants $\tilde{\sigma}_{ij}$.

(c) Using your answer to (b), write the *geometric* average $(S_{1,T}S_{2,T}S_{3,T})^{1/3}$ of the terminal stock prices in terms of:

(i) the three independent standard normal variates ξ_j;

(ii) a single standard normal variates $\xi \sim N(0,1)$.

(d) Using your answer to (c), explain why the geometric average of the three terminal stock prices behaves like the terminal stock price of a *single* stock under the risk-neutral measure. What is the initial stock price and volatility of this fictitious stock? (The interest rate and the terminal time don't change).

(3) Now consider a basket option on the three stocks in Problem 2. The option is a bull spread on the *arithmetic* average of the three stocks:

$$V_T = \begin{cases} 0 & \text{if } X_T \leq 40 \\ X_T - 40 & \text{if } 40 \leq X_T \leq 60 \\ 20 & \text{if } X_T \geq 60 \end{cases} \qquad \text{where} \quad X_T = \frac{1}{3}(S_{1,T} + S_{2,T} + S_{3,T}).$$

at time $T = 1/4$.

(a) Use your answer to Problem 2 to compute a Monte Carlo estimate for the price V of the basket option at time $t = 0$. Explain your steps. Don't use any variance reduction techniques.

(b) Repeat the computation in (a), now using antithetic variables. Explain how you do this.

(c) Now consider the same bull-spread option, but on the *geometric* average of the stocks. This has payoff

$$C_T = \begin{cases} 0 & \text{if } Y_T \leq 40 \\ Y_T - 40 & \text{if } 40 \leq Y_T \leq 60 \\ 20 & \text{if } Y_T \geq 60 \end{cases} \qquad \text{where} \quad Y_T = (S_{1,T}\, S_{2,T}\, S_{3,T})^{1/3},$$

still at time $T = 1/4$. Using your answer to Problem 2, what is the exact price C at time $t = 0$ of this option.

(d) Explain why this option should be cheaper than the option with payoff V_T, i.e. why should $C < V$.

(e) Redo the Monte Carlo simulations in (a), now using the geometrically averaged basket option as a control variate. Don't use any other variance reduction techniques. Explain your steps.

(f) Repeat the simulation in (e), now also using antithetic variables. Use the same number of samples and report the result as in (c).

(g) Write a summarizing table with the Monte Carlo estimates $\hat{V}_N$ and standard errors ϵ_N from (a),(b),(e) and (f).