Math 623, W 2007: Homework 5.

For full credit, your solutions must be clearly presented and all code included.

(1) This problem deals with the pricing of the American strangle option in homework 2 using a binomial tree.

The underlying stock price S_t follows geometric Brownian motion with volatility $\sigma = 0.2$ (in units year$^{-1/2}$) and the interest rate is $r = 3\%$ per year, continuously compounded. The stock pays a continuous dividend yield of $D = 1\%$ per year. It is currently trading at $S_0 = 75$.

The option pays $\Phi(S)$ if exercised when the stock price is S. Here

$$\Phi(S) = \begin{cases} 80 - S & \text{if } 0 \leq S \leq 80 \\ 0 & \text{if } 80 < S \leq 120 \\ S - 120 & \text{if } S > 120. \end{cases}$$

Today is $t = 0$. The option expires in $T = 6$ months. Being American, the option can be exercised at any time between $t = 0$ and $t = T$.

(a) Using the Black-Scholes formulas, find the exact value of the corresponding *European* strangle option today.

(b) Construct binomial trees with time steps $\Delta t = 2^{-1}, 2^{-2}, \dots$ (the smaller time step you are able to take, the better). Compute the parameters p_u, p_d, u, d under the following conditions.

  (i) The (noncentral) moments of $S_{t+\Delta t}/S_t$ of order 0, 1 and 2 in the tree equals the corresponding moments under geometric Brownian motion. Moreover, $ud = 1$.

  (ii) The (noncentral) moments of $S_{t+\Delta t}/S_t$ of order 0, 1 and 2 in the tree equals the corresponding moments under geometric Brownian motion. Moreover, $p_u = p_d = 1/2$.

  (iii) The (noncentral) moments of $S_{t+\Delta t}/S_t$ of order 0, 1, 2 and 3 in the tree equals the corresponding moments under geometric Brownian motion. Explain your steps carefully. (This is somewhat challenging. You will need to solve a system of equations numerically. For instance, you can have two equations for u and d. In matlab, consider using `fsolve`.)

(c) Implement your binomial tree(s) to value the American strangle option today. Report your results in a table containing Δt, the values of the four parameters, and the value of the option.

(2) Consider the following (European) basket put option written on two stocks. Four months from now, the option pays the holder the difference between the strike price $K = 60$ and the (arithmetic) average of the two stocks, but never less than zero.

Time is counted in years. The interest rate is $r = 1.5\%$ and the prices of the two stocks follow the SDE (under the risk-neutral measure Q):

$$\frac{dS_{i,t}}{S_{i,t}} = r\,dt + \sigma_i\,dW_{i,t}, \quad i = 1, 2.$$

Here $W_{i,t}$ are *correlated* Brownian motions: $E[dW_{i,t}\,dW_{j,t}] = \rho_{ij}\,dt$, where the correlation matrix $\rho = (\rho_{ij})$ is given by

$$\rho = \begin{pmatrix} 1.0 & 0.4 \\ 0.4 & 1.0 \end{pmatrix}$$

Moreover, $\sigma_1 = 0.25$ and $\sigma_2 = 0.2$. Today's stock prices are $S_{1,0} = 52$ and $S_{2,0} = 56$.

(a) Use Cholesky factorization to write the stock price dynamics as

$$\frac{dS_{i,t}}{S_{i,t}} = r\,dt + \sum_{j=1}^{2} \tilde{\sigma}_{ij}\,dZ_{j,t}, \quad i = 1, 2.$$

where $Z_{i,t}$ are *independent* Brownian motions under Q and $\tilde{\sigma}_{ij}$ are constants.

(b) Find constants a_{ij}, $i, j = 1, 2$ such that the two processes

$$X_{1,t} = a_{11}\log S_{1,t} + a_{12}\log S_{2,t} \quad \text{and} \quad X_{2,t} = a_{21}\log S_{1,t} + a_{22}\log S_{2,t}$$

satisfy the SDE's

$$dX_{i,t} = \mu_i\,dt + dZ_{i,t}, \; i = 1, 2.$$

What are the values of the constants μ_1, μ_2?

(c) Build a product tree for the two-dimensional process $(X_{1,t}, X_{2,t})$ in (b). Explain your steps.

(d) Use this tree to compute the value of the basket option today. Also compute the rho (ρ) of the option, i.e. the sensitivity to the interest rate.