

Math 623, W 2007: Homework 6.

For full credit, your solutions must be clearly presented and all code included.

In this homework you will calibrate two fixed income (short rate) models to real (but not current) data. The following table gives the yields of zero coupon bonds, computed using *annual* compounding, for various maturities. The table also gives the market implied (flat) volatilities of at-the-money caps.

Maturity	Yield	Volatility
0	0.0480	—
1	0.0454	0.1520
2	0.0456	0.1620
3	0.0462	0.1640
4	0.0471	0.1630
5	0.0481	0.1605
7	0.0502	0.1555
10	0.0526	0.1475
15	0.0556	0.1350
20	0.0575	0.1260

Each cap can be described as follows. Consider dates $0 < T_0 < T_1 < \dots < T_n = T$ with $T_i = 3(i+1)$ months. The cap pays $(T_i - T_{i-1})(L(T_{i-1}, T_i) - R)^+$ at time T_i for $i = 1, \dots, n$, where $R = R(T)$ is the swap rate and $L(T_{i-1}, T_i)$ the LIBOR rate. The terminal time T is the maturity listed in the table. The implied cap volatilities come from Black's formula.

- (1) (a) Compute the price $p(0, T)$ today of the zero coupon bond maturing at T for $0 \leq T \leq 20$. Use linear interpolation *on the yields* to do this. Display your result in a plot of $p(0, T)$ vs T .
- (b) Compute the swap rates $R(T)$ for the maturity times given in the table (except for $T = 0$). Display your result in a table as well as in a plot.
- (c) Compute the LIBOR forward rates $F(0; T_{i-1}, T_i)$ for $1 \leq i \leq 80$. Display the result in a plot. Use the zero coupon bond prices computed in (a).
- (d) Compute the cap prices from Black's formula.

- (2) Now consider the Black-Derman-Toy tree for the short rate. Use the term structure of interest rates and volatilities provided in the table above. (The cap volatilities above are not exactly the volatilities that should go into the model, but ignore this fact for the moment.)
- (a) Implement the Black-Derman-Toy tree with $\Delta t = 0.25$ calibrated to the information above. Use linear interpolation/extrapolation to find yields/volatilities that are not given in the table. Outline your steps.
 - (b) Make a 3-dimensional plot of the short rates r in the tree (use `mesh` in matlab).
 - (c) Make a 3-dimensional plot of the prices Q at time zero of the Arrow-Debreu securities in the tree.
 - (d) Compute the price of a zero-coupon bond maturing in 20 years at each node in the tree. Display the result as in (c).
 - (e) Compute the price of a European bond put option. The option may be exercised 5 years from now (and only then). When exercised, the holder of the option may sell a zero coupon bond maturing at time $T = 20$ and with face value \$1,000,000, and receive \$500,000 in cash.
 - (f) Compute the price today of a Bermudan put option on the same bond. The features are the same as in (e) but the bond may be exercised at any of the times 1,2,3,4 and 5 years.
 - (g) Compute the prices in the model of the caps above.
 - (h) Compute the implied cap volatilities in the model. Plot them, and compare with those in the table.

- (3) Now use the Black-Karasinski model instead: $r_t = \exp(x_t)$, where

$$dx_t = (\theta(t) - ax_t)dt + \sigma dW_t.$$

To begin with, use $a = 0.15$ and $\sigma = 0.25$.

- (a) Implement the Black-Karasinski tree with $\Delta t = 0.25$ calibrated to the yields above (do not use the volatilities). Outline your steps.
- (b) Make a 3-dimensional plot of the short rates r in the tree (use `mesh` in matlab).
- (c) Make a 3-dimensional plot of the prices Q at time zero of the Arrow-Debreu securities in the tree.
- (d) Compute the price of a zero-coupon bond maturing in 20 years at each node in the tree. Display the result as in (c).
- (e) Compute the price of a European bond put option. The option may be exercised 5 years from now (and only then). When exercised, the holder of the option may sell a zero coupon bond maturing at time $T = 20$ and with face value \$1,000,000, and receive \$500,000 in cash.
- (f) Compute the price today of a Bermudan put option on the same bond. The features are the same as in (e) but the bond may be exercised at any of the times 1,2,3,4 and 5 years.
- (g) Compute the prices in the model of the caps above.
- (h) Compute the implied cap volatilities in the model. Compare with those in the table.
- (i) *Extra credit:* Redo steps (a),(g),(h) for different values of the parameters a and σ in the range $0 \leq a \leq 0.2$, $0.1 \leq \sigma \leq 0.4$. Report the least-squares fit, i.e. the sum of the squares of the difference between the observed and model implied cap volatilities. Find the optimal (a, σ) and plot the optimal model implied cap volatilities together with the observed ones. For these parameters, redo (e) and (f).