Math 626 Problem Set III

(1) A stochastically controlled process has dynamic equation given by

\[ x(i + 1) = ax(i) + \xi(i)u(i) , \]

where the sequence of random variables \( \xi(i) \) are drawn from a common probability distribution having non-zero mean \( \beta \) and variance \( \sigma^2 \). Let the cost function be given by \( J \),

\[ J = E_x\left[ \sum_{i=1}^{N} x^2(i) \right] , \]

where \( x(1) = x \). Show that the optimal value of \( J \) is \( J = V_N x^2 \), where

\[ V_N = \frac{a^2 \sigma^2}{\beta^2 + \sigma^2} V_{N-1} + 1 . \]

State the condition which must be satisfied if there is to be a limiting value of \( V_N \) as \( N \to \infty \) and find the limiting value.

(2) Let \( f(x,t) \) be the solution of the problem,

\[ -\frac{\partial f}{\partial t} = x^2 - \frac{\omega^2}{4} \left[ \frac{\partial f}{\partial x} \right]^2 + \frac{1}{2\varepsilon} \frac{\partial^2 f}{\partial x^2} , \quad t < 0 , \]

\[ f(x,0) = ax^2 + bx + c , \] where \( a, b, c \) are constants. Obtain a formula for \( f(x,t) \) by looking for solutions to the equation of the form, 

\[ f(x,t) = a(t)x^2 + b(t)x + c(t) . \]

(3) Consider the stochastic control problem given by

\[ \frac{dx}{dt} = ax + bu + \sqrt{\varepsilon}W(t) , \]
where $W(t)$ is white noise, with cost function,

$$f_\varepsilon(x, t) = \min_u E_x \left[ \int_t^T x^2(s) + qu^2(s) ds \right].$$

(a) Show that the corresponding values of the control parameter $u(x, t)$ are independent of $\varepsilon$ and hence are the same as for the classical control problem ($\varepsilon = 0$).

(b) Prove that $\lim_{\varepsilon \to 0} f_\varepsilon(x, t) = f(x, t)$, where $f(x, t)$ is the performance criterion for the classical control problem.

(4) Consider the stochastic control problem given by

$$\frac{dx}{dt} = ax + bu + (cx + d)W(t),$$

where $W(t)$ is white noise, with cost function,

$$f(x, t) = \min_u E_x \left[ \int_t^T x^2(s) + qu^2(s) ds \right].$$

(a) Show that $f(x, t)$ can be written in the form,

$$f(x, t) = A(t)x^2 + B(t)x + C(t),$$

where the functions $A(t)$, $B(t)$, $C(t)$ satisfy first order differential equations.

(b) Show that the optimal feedback $u$ is linear in $x$.

(5) Let $u(x, t)$ be the solution of the initial value problem,

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}(1 - x^2)u, \quad t > 0,$$

$$u(x, 0) = g(x).$$

Then

$$u(x, t) = \int_{-\infty}^{\infty} Q_t(x, y)g(y)dy,$$
where the Green’s function $Q_t(x,y)$ is given by Mehler’s formula,

$$Q_t(x,y) = \pi^{-1/2}[1-e^{-2t}]^{-1/2} \exp\left\{ -\frac{[(x^2 + y^2)(1 + e^{-2t}) - 4e^{-t}xy]}{2[1 - e^{-2t}]} \right\}.$$ 

Derive Mehler’s formula as follows:

(a) Solve the initial value problem above when $g(x) = \exp[\alpha x]$ with constant $\alpha$ by using an inverse Hopf-Cole transformation and the method in problem 2.

(b) Compute $Q_t$ from (a) by obtaining the inverse Laplace transform.